CHAPTER 10
Conics, Parametric Equations, and Polar Coordinates

Section 10.1 Conics and Calculus

1. $y^2 = 4x$ Parabola

Vertex: $(0, 0)$

$p = 1 > 0$

Opens to the right

Matches (h)

2. $(x + 4)^2 = 2(y + 2)$ Parabola

Vertex: $(-4, -2)$

Opens upward

Matches (a)

3. $(x + 4)^2 = -2(y - 2)$ Parabola

Vertex: $(-4, 2)$

Opens downward

Matches (e)

4. $\dfrac{(x - 2)^2}{16} + \dfrac{(y + 1)^2}{4} = 1$ Ellipse

Center: $(2, -1)$

Matches (b)

5. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ Ellipse

Center: $(0, 0)$

Vertices: $(0, \pm 3)$

Matches (f)

6. $\dfrac{x^2}{16} + \dfrac{y^2}{16} = 1$ Circle

Matches (g)

7. $\dfrac{y^2}{16} - \dfrac{x^2}{1} = 1$ Hyperbola

Vertices: $(0, \pm 4)$

Matches (c)

8. $\dfrac{(x - 2)^2}{9} - \dfrac{y^2}{4} = 1$ Hyperbola

Vertices: $(5, 0), (-1, 0)$

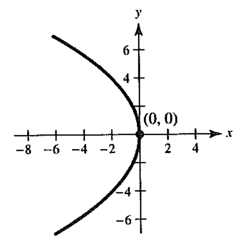
Matches (d)

9. $y^2 = -8x = 4(-2)x$

Vertex: $(0, 0)$

Focus: $(-2, 0)$

Directrix: $x = 2$



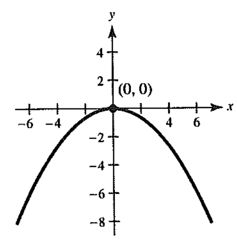
10. $x^2 + 6y = 0$

$$x^2 = -6y = 4\left(-\frac{3}{2}\right)y$$

Vertex: $(0, 0)$

Focus: $\left(0, -\dfrac{3}{2}\right)$

Directrix: $y = \dfrac{3}{2}$



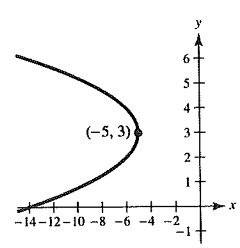
11. $(x + 5) + (y - 3)^2 = 0$

$$(y - 3)^2 = -(x + 5) = 4\left(-\tfrac{1}{4}\right)(x + 5)$$

Vertex: $(-5, 3)$

Focus: $\left(-\tfrac{21}{4}, 3\right)$

Directrix: $x = -\tfrac{19}{4}$



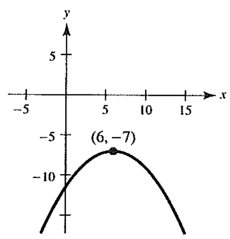
12. $(x - 6)^2 + 8(y + 7) = 0$

$$(x - 6)^2 = -8(y + 7) = 4(-2)(y + 7)$$

Vertex: $(6, -7)$

Focus: $(6, -9)$

Directrix: $y = -5$



13. $y^2 - 4y - 4x = 0$

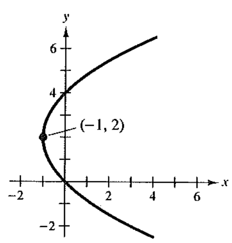
$$y^2 - 4y + 4 = 4x + 4$$

$$(y - 2)^2 = 4(1)(x + 1)$$

Vertex: $(-1, 2)$

Focus: $(0, 2)$

Directrix: $x = -2$



14. $y^2 + 6y + 8x + 25 = 0$

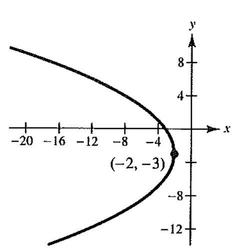
$$y^2 + 6y + 9 = -8x - 25 + 9$$

$$(y + 3)^2 = 4(-2)(x + 2)$$

Vertex: $(-2, -3)$

Focus: $(-4, -3)$

Directrix: $x = 0$



15. $x^2 + 4x + 4y - 4 = 0$

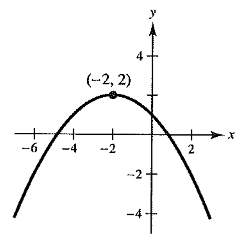
$$x^2 + 4x + 4 = -4y + 4 + 4$$

$$(x + 2)^2 = 4(-1)(y - 2)$$

Vertex: $(-2, 2)$

Focus: $(-2, 1)$

Directrix: $y = 3$



16. $y^2 + 4y + 8x - 12 = 0$

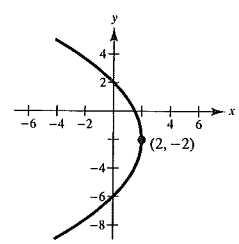
$$y^2 + 4y + 4 = -8x + 12 + 4$$

$$(y + 2)^2 = 4(-2)(x - 2)$$

Vertex: $(2, -2)$

Focus: $(0, -2)$

Directrix: $x = 4$



17. $y^2 + x + y = 0$

$$y^2 + y + \frac{1}{4} = -x + \frac{1}{4}$$

$$\left(y + \frac{1}{2}\right)^2 = 4\left(-\frac{1}{4}\right)\left(x - \frac{1}{4}\right)$$

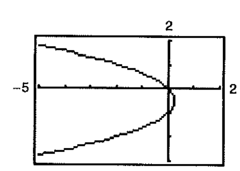
Vertex: $\left(\frac{1}{4}, -\frac{1}{2}\right)$

Focus: $\left(0, -\frac{1}{2}\right)$

Directrix: $x = \frac{1}{2}$

$$y_1 = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - x}$$

$$y_2 = -\frac{1}{2} - \sqrt{\frac{1}{4} - x}$$



18. $y = -\frac{1}{6}(x^2 - 8x + 6) = -\frac{1}{6}(x^2 - 8x + 16 - 10)$

$-6y = (x - 4)^2 - 10$

$-6y + 10 = (x - 4)^2$

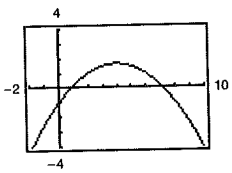
$(x - 4)^2 = -6\left(y - \frac{5}{3}\right)$

$(x - 4)^2 = 4\left(-\frac{3}{2}\right)\left(y - \frac{5}{3}\right)$

Vertex: $\left(4, \frac{5}{3}\right)$

Focus: $\left(4, \frac{1}{6}\right)$

Directrix: $y = \frac{19}{6}$



19. $y^2 - 4x - 4 = 0$

$y^2 = 4x + 4$

$= 4(1)(x + 1)$

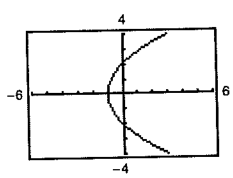
Vertex: $(-1, 0)$

Focus: $(0, 0)$

Directrix: $x = -2$

$y_1 = 2\sqrt{x + 1}$

$y_2 = -2\sqrt{x + 1}$



20. $x^2 - 2x + 8y + 9 = 0$

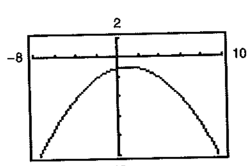
$x^2 - 2x + 1 = -8y - 9 + 1$

$(x - 1)^2 = 4(-2)(y + 1)$

Vertex: $(1, -1)$

Focus: $(1, -3)$

Directrix: $y = 1$



21. $(y - 4)^2 = 4(-2)(x - 5)$

$y^2 - 8y + 16 = -8x + 40$

$y^2 - 8y + 8x - 24 = 0$

22. $(x + 2)^2 = 4(-2)(y - 1)$

$x^2 + 4x + 8y - 4 = 0$

23. $(x - 0)^2 = 4(8)(y - 5)$

$x^2 = 4(8)(y - 5)$

$x^2 - 32y + 160 = 0$

24. Vertex: $(0, 2)$

$(y - 2)^2 = 4(2)(x - 0)$

$y^2 - 8x - 4y + 4 = 0$

25. $y = 4 - x^2$

$x^2 + y - 4 = 0$

26. $y = 4 - (x - 2)^2 = 4x - x^2$

$x^2 - 4x + y = 0$

27. Because the axis of the parabola is vertical, the form of the equation is $y = ax^2 + bx + c$. Now, substituting the values of the given coordinates into this equation, you obtain

$3 = c, 4 = 9a + 3b + c, 11 = 16a + 4b + c.$

Solving this system, you have $a = \frac{5}{3}, b = -\frac{14}{3}, c = 3.$

So,

$y = \frac{5}{3}x^2 - \frac{14}{3}x + 3$ or $5x^2 - 14x - 3y + 9 = 0.$

28. From Example 2: $4p = 8$ or $p = 2$

Vertex: $(4, 0)$

$(x - 4)^2 = 8(y - 0)$

$x^2 - 8x - 8y + 16 = 0$

29. $16x^2 + y^2 = 16$

$x^2 + \frac{y^2}{16} = 1$

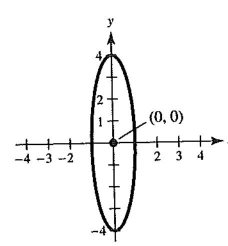
$a^2 = 16, b^2 = 1, c^2 = 16 - 1 = 15$

Center: $(0, 0)$

Foci: $\left(0, \pm\sqrt{15}\right)$

Vertices: $(0, \pm 4)$

$e = \frac{c}{a} = \frac{\sqrt{15}}{4}$



30. $3x^2 + 7y^2 = 63$

$$\frac{x^2}{21} + \frac{y^2}{9} = 1$$

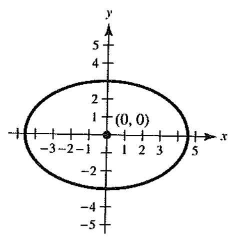
$a^2 = 21, b^2 = 9, c^2 = 21 - 9 = 12$

Center: $(0, 0)$

Foci: $\left(\pm 2\sqrt{3}, 0\right)$

Vertices: $\left(\pm\sqrt{21}, 0\right)$

$$e = \frac{c}{a} = \frac{2\sqrt{3}}{\sqrt{21}} = \frac{2\sqrt{7}}{7}$$



31. $\dfrac{(x-3)^2}{16} + \dfrac{(y-1)^2}{25} = 1$

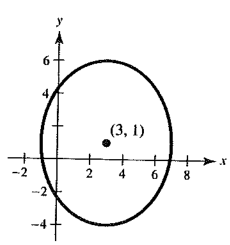
$a^2 = 25, b^2 = 16, c^2 = 25 - 16 = 9$

Center: $(3, 1)$

Foci: $(3, 1 + 3) = (3, 4), (3, 1 - 3) = (3, -2)$

Vertices: $(3, 6), (3, -4)$

$$e = \frac{c}{a} = \frac{3}{5}$$



32. $(x + 4)^2 + \dfrac{(y+6)^2}{1/4} = 1$

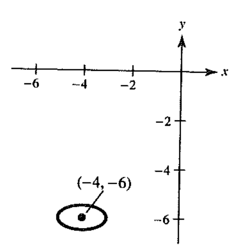
$a^2 = 1, b^2 = \dfrac{1}{4}, c^2 = 1 - \dfrac{1}{4} = \dfrac{3}{4}$

Center: $(-4, -6)$

Foci: $\left(-4 \pm \dfrac{\sqrt{3}}{2}, -6\right)$

Vertices: $(-5, -6), (-3, -6)$

$$e = \frac{c}{a} = \frac{\sqrt{3}}{2}$$



33. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

$$9(x^2 + 4x + 4) + 4(y^2 - 6y + 9) = -36 + 36 + 36$$
$$= 36$$

$$\frac{(x+2)^2}{4} + \frac{(y-3)^2}{9} = 1$$

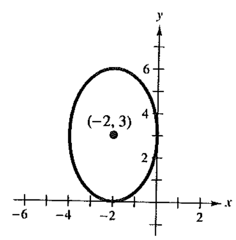
$a^2 = 9, b^2 = 4, c^2 = 5$

Center: $(-2, 3)$

Foci: $\left(-2, 3 \pm \sqrt{5}\right)$

Vertices: $(-2, 6), (-2, 0)$

$$e = \frac{\sqrt{5}}{3}$$



34. $16x^2 + 25y^2 - 64x + 150y + 279 = 0$

$$16(x^2 - 4x + 4) + 25(y^2 + 6y + 0) = -279 + 64 + 225$$
$$= 10$$

$$\frac{(x-2)^2}{(5/8)} + \frac{(y+3)^2}{(2/5)} = 1$$

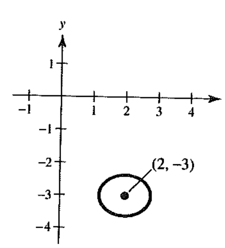
$a^2 = \dfrac{5}{8}, b^2 = \dfrac{2}{5}, c^2 = a^2 - b^2 = \dfrac{9}{40}$

Center: $(2, -3)$

Foci: $\left(2 \pm \dfrac{3\sqrt{10}}{20}, -3\right)$

Vertices: $\left(2 \pm \dfrac{\sqrt{10}}{4}, -3\right)$

$$e = \frac{c}{a} = \frac{3}{5}$$



35. $12x^2 + 20y^2 - 12x + 40y - 37 = 0$

$$12\left(x^2 - x + \frac{1}{4}\right) + 20\left(y^2 + 2y + 1\right) = 37 + 3 + 20 = 60$$

$$\frac{[x - (1/2)]^2}{5} + \frac{(y + 1)^2}{3} = 1$$

$a^2 = 5, b^2 = 3, c^2 = 2$

Center: $\left(\frac{1}{2}, -1\right)$

Foci: $\left(\frac{1}{2} \pm \sqrt{2}, -1\right)$

Vertices: $\left(\frac{1}{2} \pm \sqrt{5}, -1\right)$

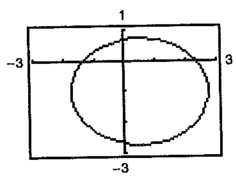
Solve for y:

$$20\left(y^2 + 2y + 1\right) = -12x^2 + 12x + 37 + 20$$

$$(y + 1)^2 = \frac{57 + 12x - 12x^2}{20}$$

$$y = -1 \pm \sqrt{\frac{57 + 12x - 12x^2}{20}}$$

(Graph each of these separately.)



36. $36x^2 + 9y^2 + 48x - 36y + 43 = 0$

$$36\left(x^2 + \frac{4}{3}x + \frac{4}{9}\right) + 9\left(y^2 - 4y + 4\right) = -43 + 16 + 36$$
$$= 9$$

$$\frac{[x + (2/3)]^2}{1/4} + \frac{(y - 2)^2}{1} = 1$$

$a^2 = 1, b^2 = \frac{1}{4}, c^2 = \frac{3}{4}$

Center: $\left(-\frac{2}{3}, 2\right)$

Foci: $\left(-\frac{2}{3}, 2 \pm \frac{\sqrt{3}}{2}\right)$

Vertices: $\left(-\frac{2}{3}, 3\right), \left(-\frac{2}{3}, 1\right)$

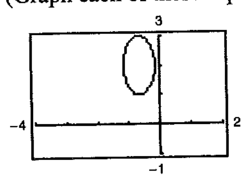
Solve for y:

$$9\left(y^2 - 4y + 4\right) = -36x^2 - 48x - 43 + 36$$

$$(y - 2)^2 = \frac{-\left(36x^2 + 48x + 7\right)}{9}$$

$$y = 2 \pm \frac{1}{3}\sqrt{-\left(36x^2 + 48x + 7\right)}$$

(Graph each of these separately.)



37. $x^2 + 2y^2 - 3x + 4y + 0.25 = 0$

$$\left(x^2 - 3x + \frac{9}{4}\right) + 2\left(y^2 + 2y + 1\right) = -\frac{1}{4} + \frac{9}{4} + 2 = 4$$

$$\frac{[x - (3/2)]^2}{4} + \frac{(y + 1)^2}{2} = 1$$

$a^2 = 4, b^2 = 2, c^2 = 2$

Center: $\left(\frac{3}{2}, -1\right)$

Foci: $\left(\frac{3}{2} \pm \sqrt{2}, -1\right)$

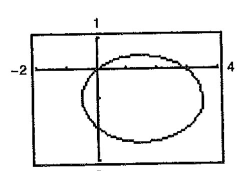
Vertices: $\left(-\frac{1}{2}, -1\right), \left(\frac{7}{2}, -1\right)$

Solve for y: $2\left(y^2 + 2y + 1\right) = -x^2 + 3x - \frac{1}{4} + 2$

$$(y + 1)^2 = \frac{1}{2}\left(\frac{7}{4} + 3x - x^2\right)$$

$$y = -1 \pm \sqrt{\frac{7 + 12x - 4x^2}{8}}$$

(Graph each of these separately.)



38.

$$2x^2 + y^2 + 4.8x - 6.4y + 3.12 = 0$$

$$50x^2 + 25y^2 + 120x - 160y + 78 = 0$$

$$50\left(x^2 + \frac{12}{5}x + \frac{36}{25}\right) + 25\left(y^2 - \frac{32}{5}y + \frac{256}{25}\right) = -78 + 72 + 256 = 250$$

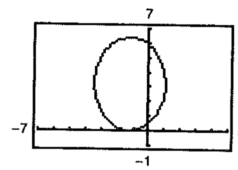
$$\frac{\left[x + (6/5)\right]^2}{5} + \frac{\left[y - (16/5)\right]^2}{10} = 1$$

$a^2 = 10, b^2 = 5, c^2 = 5$

Center: $\left(-\frac{6}{5}, \frac{16}{5}\right)$

Foci: $\left(-\frac{6}{5}, \frac{16}{5} \pm \sqrt{5}\right)$

Vertices: $\left(-\frac{6}{5}, \frac{16}{5} \pm \sqrt{10}\right)$



Solve for y: $\left(y^2 - 6.4y + 10.24\right) = -2x^2 - 4.8x - 3.12 + 10.24$

$$\left(y - 3.2\right)^2 = 7.12 - 4x - 2x^2$$

$$y = 3.2 \pm \sqrt{7.12 - 4x - 2x^2}$$ (Graph each of these separately.)

39. Center: $(0, 0)$

Focus: $(5, 0)$

Vertex: $(6, 0)$

Horizontal major axis

$a = 6, c = 5 \Rightarrow b = \sqrt{a^2 - c^2} = \sqrt{11}$

$$\frac{x^2}{36} + \frac{y^2}{11} = 1$$

40. Vertices: $(0, 3), (8, 3)$

Eccentricity: $\frac{3}{4}$

Horizontal major axis

Center: $(4, 3)$

$a = 4, e = \frac{c}{a} \Rightarrow c = 4\left(\frac{3}{4}\right) = 3$

$\Rightarrow b = \sqrt{16 - 9} = \sqrt{7}$

$$\frac{(x - 4)^2}{16} + \frac{(y - 3)^2}{7} = 1$$

41. Vertices: $(3, 1), (3, 9)$

Minor axis length: 6

Vertical major axis

Center: $(3, 5)$

$a = 4, b = 3$

$$\frac{(x - 3)^2}{9} + \frac{(y - 5)^2}{16} = 1$$

42. Foci: $(0, \pm 9)$

Major axis length: 22

Vertical major axis

Center: $(0, 0)$

$c = 9, a = 11 \Rightarrow b = \sqrt{40}$

$$\frac{x^2}{40} + \frac{y^2}{121} = 1$$

43. Center: $(0, 0)$

Horizontal major axis

Points on ellipse: $(3, 1), (4, 0)$

Because the major axis is horizontal,

$$\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) = 1.$$

Substituting the values of the coordinates of the given points into this equation, you have

$$\left(\frac{9}{a^2}\right) + \left(\frac{1}{b^2}\right) = 1, \text{ and } \frac{16}{a^2} = 1.$$

The solution to this system is $a^2 = 16, b^2 = \frac{16}{7}$.

So,

$$\frac{x^2}{16} + \frac{y^2}{16/7} = 1, \frac{x^2}{16} + \frac{7y^2}{16} = 1.$$

44. Center: $(1, 2)$

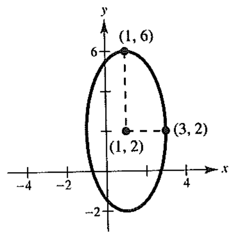
Vertical major axis

Points on ellipse: $(1, 6), (3, 2)$

From the sketch, you can see that

$h = 1, k = 2, a = 4, b = 2$

$$\frac{(x-1)^2}{4} + \frac{(y-2)^2}{16} = 1.$$



45. $y^2 - \dfrac{x^2}{9} = 1$

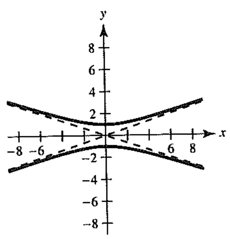
$a = 1, b = 3, c = \sqrt{10}$

Center: $(0, 0)$

Vertices: $(0, \pm 1)$

Foci: $\left(0, \pm\sqrt{10}\right)$

Asymptotes: $y = \pm\dfrac{a}{b}x = \pm\dfrac{1}{3}x$



46. $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$

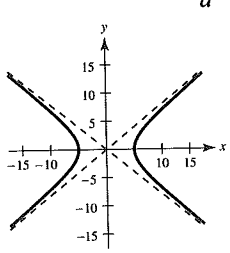
$a = 5, b = 4, c = \sqrt{25 + 16} = \sqrt{41}$

Center: $(0, 0)$

Vertices: $(\pm 5, 0)$

Foci: $\left(\pm\sqrt{41}, 0\right)$

Asymptotes: $y = \pm\dfrac{b}{a}x = \pm\dfrac{4}{5}x$



47. $\dfrac{(x-1)^2}{4} - \dfrac{(y+2)^2}{1} = 1$

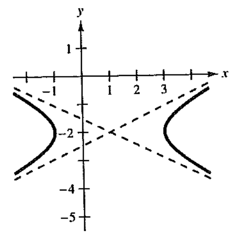
$a = 2, b = 1, c = \sqrt{5}$

Center: $(1, -2)$

Vertices: $(-1, -2), (3, -2)$

Foci: $\left(1 \pm \sqrt{5}, -2\right)$

Asymptotes: $y = -2 \pm \dfrac{1}{2}(x - 1)$



48. $\dfrac{(y+3)^2}{225} - \dfrac{(x-5)^2}{64} = 1$

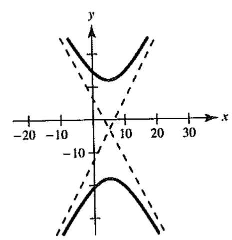
$a = 15, b = 8, c = \sqrt{225 + 64} = 17$

Center: $(5, -3)$

Vertices: $(5, 12), (5, -18)$

Foci: $(5, 14), (5, -20)$

Asymptotes: $y = k \pm \dfrac{a}{b}(x - h) = -3 \pm \dfrac{15}{8}(x - 5)$



49.
$$9x^2 - y^2 - 36x - 6y + 18 = 0$$
$$9\left(x^2 - 4x + 4\right) - \left(y^2 + 6y + 9\right) = -18 + 36 - 9$$
$$\frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1$$

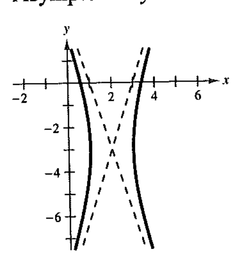
$a = 1, b = 3, c = \sqrt{10}$

Center: $(2, -3)$

Vertices: $(1, -3), (3, -3)$

Foci: $\left(2 \pm \sqrt{10}, -3\right)$

Asymptotes: $y = -3 \pm 3(x - 2)$



50. $y^2 - 16x^2 + 64x - 208 = 0$

$$y^2 - 16(x^2 - 4x + 4) = 208 - 64 = 144$$

$$\frac{y^2}{144} - \frac{(x-2)^2}{9} = 1$$

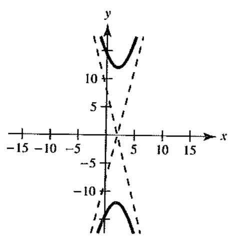
$a = 12, b = 3, c = \sqrt{144 + 9} = \sqrt{153}$

Center: $(2, 0)$

Vertices: $(2, 12), (2, -12)$

Foci: $(2, \pm\sqrt{153})$

Asymptotes: $y = \pm\frac{12}{3}(x - 2) = \pm 4(x - 2)$



51. $x^2 - 9y^2 + 2x - 54y - 80 = 0$

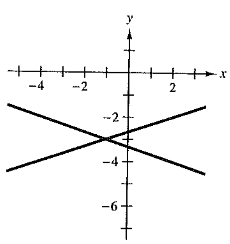
$$(x^2 + 2x + 1) - 9(y^2 + 6y + 9) = 80 + 1 - 81 = 0$$

$$(x+1)^2 - 9(y+3)^2 = 0$$

$$y + 3 = \pm\frac{1}{3}(x + 1)$$

$$y = 3 \pm \frac{1}{3}(x + 1)$$

Degenerate hyperbola is two lines intersecting at $(-1, -3)$.



52. $9(x^2 + 6x + 9) - 4(y^2 - 2y + 1) = -78 + 81 - 4 = -1$

$$9(x+3)^2 - 4(y-1)^2 = -1$$

$$\frac{(y-1)^2}{1/4} - \frac{(x+3)^2}{1/9} = 1$$

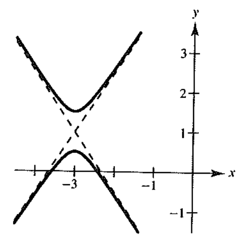
$a = \frac{1}{2}, b = \frac{1}{3}, c = \frac{\sqrt{13}}{6}$

Center: $(-3, 1)$

Vertices: $\left(-3, \frac{1}{2}\right), \left(-3, \frac{3}{2}\right)$

Foci: $\left(-3, 1 \pm \frac{1}{6}\sqrt{13}\right)$

Asymptotes: $y = 1 \pm \frac{3}{2}(x + 3)$



53. $9y^2 - x^2 + 2x + 54y + 62 = 0$

$$9(y^2 + 6y + 9) - (x^2 - 2x + 1) = -62 - 1 + 81 = 18$$

$$\frac{(y+3)^2}{2} - \frac{(x-1)^2}{18} = 1$$

$a = \sqrt{2}, b = 3\sqrt{2}, c = 2\sqrt{5}$

Center: $(1, -3)$

Vertices: $(1, -3 \pm \sqrt{2})$

Foci: $(1, -3 \pm 2\sqrt{5})$

Asymptotes: $y = \frac{1}{3}x - \frac{1}{3} - 3$

$$y = -\frac{1}{3}x + \frac{1}{3} - 3$$

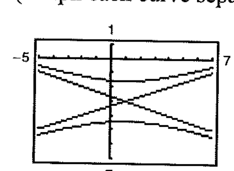
Solve for y:

$$9(y^2 + 6y + 9) = x^2 - 2x - 62 + 81$$

$$(y+3)^2 = \frac{x^2 - 2x + 19}{9}$$

$$y = -3 \pm \frac{1}{3}\sqrt{x^2 - 2x + 19}$$

(Graph each curve separately.)



54. $9x^2 - y^2 + 54x + 10y + 55 = 0$

$9(x^2 + 6x + 9) - (y^2 - 10y + 25) = -55 + 81 - 25 = 1$

$$\frac{(x+3)^2}{1/9} - \frac{(y-5)^2}{1} = 1$$

$a = \frac{1}{3}, b = 1, c = \frac{\sqrt{10}}{3}$

Center: $(-3, 5)$

Vertices: $\left(-3 \pm \frac{1}{3}, 5\right)$

Foci: $\left(-3 \pm \frac{\sqrt{10}}{3}, 5\right)$

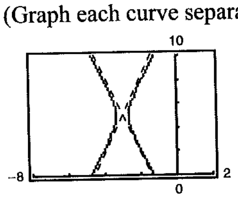
Solve for y:

$y^2 - 10y + 25 = 9x^2 + 54x + 55 + 25$

$(y - 5)^2 = 9x^2 + 54x + 80$

$y = 5 \pm \sqrt{9x^2 + 54x + 80}$

(Graph each curve separately.)



55. $3x^2 - 2y^2 - 6x - 12y - 27 = 0$

$3(x^2 - 2x + 1) - 2(y^2 + 6y + 9) = 27 + 3 - 18 = 12$

$$\frac{(x-1)^2}{4} - \frac{(y+3)^2}{6} = 1$$

$a = 2, b = \sqrt{6}, c = \sqrt{10}$

Center: $(1, -3)$

Vertices: $(-1, -3), (3, -3)$

Foci: $(1 \pm \sqrt{10}, -3)$

Asymptotes: $y = \frac{\sqrt{6}x}{2} - \frac{\sqrt{6}}{2} - 3$

$y = -\frac{\sqrt{6}x}{2} + \frac{\sqrt{6}}{2} - 3$

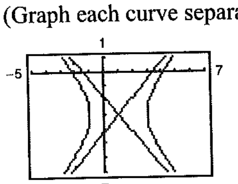
Solve for y:

$2(y^2 + 6y + 9) = 3x^2 - 6x - 27 + 18$

$(y + 3)^2 = \frac{3x^2 - 6x - 9}{2}$

$y = -3 \pm \sqrt{\frac{3(x^2 - 2x - 3)}{2}}$

(Graph each curve separately.)



56. $3y^2 - x^2 + 6x - 12y = 0$

$3(y^2 - 4y + 4) - (x^2 - 6x + 9) = 0 + 12 - 9 = 3$

$$\frac{(y-2)^2}{1} - \frac{(x-3)^2}{3} = 1$$

$a = 1, b = \sqrt{3}, c = 2$

Center: $(3, 2)$

Vertices: $(3, 1), (3, 3)$

Foci: $(3, 0), (3, 4)$

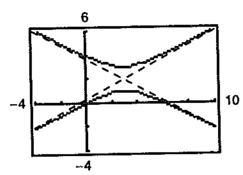
Solve for y:

$3(y^2 - 4y + 4) = x^2 - 6x + 12$

$(y - 2)^2 = \frac{x^2 - 6x + 12}{3}$

$y = 2 \pm \sqrt{\frac{x^2 - 6x + 12}{3}}$

(Graph each curve separately.)



57. Vertices: $(\pm 1, 0)$

Asymptotes: $y = \pm 5x$

Horizontal transverse axis

Center: $(0, 0)$

$a = 1, \frac{b}{a} = 5 \Rightarrow b = 5$

$$\frac{x^2}{1} - \frac{y^2}{25} = 1$$

58. Vertices: $(0, \pm 4)$

Asymptotes: $y = \pm 2x$

Vertical transverse axis

$a = 4, \frac{a}{b} = 2 \Rightarrow b = 2$

$$\frac{y^2}{16} - \frac{x^2}{4} = 1$$

59. Vertices: $(2, \pm 3)$

Point on graph: $(0, 5)$

Vertical transverse axis

Center: $(2, 0)$

$a = 3$

So, the equation is of the form

$$\frac{y^2}{9} - \frac{(x-2)^2}{b^2} = 1.$$

Substituting the coordinates of the point $(0, 5)$, you have

$$\frac{25}{9} - \frac{4}{b^2} = 1 \quad \text{or} \quad b^2 = \frac{9}{4}.$$

So, the equation is $\dfrac{y^2}{9} - \dfrac{(x-2)^2}{9/4} = 1.$

60. Vertices: $(2, \pm 3)$

Foci: $(2, \pm 5)$

Vertical transverse axis

Center: $(2, 0)$

$a = 3, c = 5, b^2 = c^2 - a^2 = 16$

So, $\dfrac{y^2}{9} - \dfrac{(x-2)^2}{16} = 1.$

61. Center: $(0, 0)$

Vertex: $(0, 2)$

Focus: $(0, 4)$

Vertical transverse axis

$a = 2, c = 4, b^2 = c^2 - a^2 = 12$

So, $\dfrac{y^2}{4} - \dfrac{x^2}{12} = 1.$

62. Center: $(0, 0)$

Vertex: $(6, 0)$

Focus: $(10, 0)$

Horizontal transverse axis

$a = 6, c = 10, b^2 = c^2 - a^2 = 100 - 36 = 64$

$$\frac{x^2}{36} - \frac{y^2}{64} = 1$$

63. Vertices: $(0, 2), (6, 2)$

Asymptotes: $y = \frac{2}{3}x, y = 4 - \frac{2}{3}x$

Horizontal transverse axis

Center: $(3, 2)$

$a = 3$

Slopes of asymptotes: $\pm\dfrac{b}{a} = \pm\dfrac{2}{3}$

So, $b = 2$. Therefore,

$$\frac{(x-3)^2}{9} - \frac{(y-2)^2}{4} = 1.$$

64. Focus: $(20, 0)$

Asymptotes: $y = \pm\frac{3}{4}x$

Horizontal transverse axis

Center: $(0, 0)$

$c = 20$

$\dfrac{b}{a} = \dfrac{3}{4} \Rightarrow b = \dfrac{3}{4}a$

$c^2 = 400 = a^2 + b^2 = a^2 + \dfrac{9}{16}a^2 = \dfrac{25}{16}a^2$

$\Rightarrow a^2 = 256 \quad \text{and} \quad b^2 = 144$

$$\frac{x^2}{256} - \frac{y^2}{144} = 1$$

65. (a) $\dfrac{x^2}{9} - y^2 = 1, \dfrac{2x}{9} - 2yy' = 0, \dfrac{x}{9y} = y'$

At $x = 6$: $y = \pm\sqrt{3}, y' = \dfrac{\pm 6}{9\sqrt{3}} = \dfrac{\pm 2\sqrt{3}}{9}$

At $(6, \sqrt{3})$: $y - \sqrt{3} = \dfrac{2\sqrt{3}}{9}(x - 6)$

or $2x - 3\sqrt{3}y - 3 = 0$

At $(6, -\sqrt{3})$: $y + \sqrt{3} = \dfrac{-2\sqrt{3}}{9}(x - 6)$

or $2x + 3\sqrt{3}y - 3 = 0$

(b) From part (a) you know that the slopes of the normal lines must be $\mp 9/(2\sqrt{3})$.

At $(6, \sqrt{3})$: $y - \sqrt{3} = -\dfrac{9}{2\sqrt{3}}(x - 6)$

or $9x + 2\sqrt{3}y - 60 = 0$

At $(6, -\sqrt{3})$: $y + \sqrt{3} = \dfrac{9}{2\sqrt{3}}(x - 6)$

or $9x - 2\sqrt{3}y - 60 = 0$

66. (a) $\frac{y^2}{4} - \frac{x^2}{2} = 1, y^2 - 2x^2 = 4, 2yy' - 4x = 0,$

$y' = \frac{4x}{2y} = \frac{2x}{y}$

At $x = 4$: $y = \pm 6, y' = \frac{\pm 2(4)}{6} = \pm\frac{4}{3}$

At $(4, 6)$: $y - 6 = -\frac{4}{3}(x - 4)$ or $4x + 3y - 34 = 0$

At $(4, -6)$: $y + 6 = -\frac{4}{3}(x - 4)$ or $4x + 3y + 2 = 0$

(b) From part (a) you know that the slopes of the normal lines must be $\mp 3/4$.

At $(4, 6)$: $y - 6 = -\frac{3}{4}(x - 4)$ or $3x + 4y - 36 = 0$

At $(4, -6)$: $y + 6 = \frac{3}{4}(x - 4)$ or $3x - 4y - 36 = 0$

67. $x^2 + 4y^2 - 6x + 16y + 21 = 0$

$(x^2 - 6x + 9) + 4(y^2 + 4y + 4) = -21 + 9 + 16$

$(x - 3)^2 + 4(y + 2)^2 = 4$

Ellipse

68. $4x^2 - y^2 - 4x - 3 = 0$

$4\left(x^2 - x + \frac{1}{4}\right) - y^2 = 3 + 1$

$4\left(x - \frac{1}{2}\right)^2 - y^2 = 4$

Hyperbola

69. $y^2 - 8y - 8x = 0$

$y^2 - 8y + 16 = 8x + 16$

$(y - 4)^2 = 4(2)(x + 2)$

Parabola

70. $25x^2 - 10x - 200y - 119 = 0$

$25\left(x^2 - \frac{2}{5}x + \frac{1}{25}\right) = 200y + 119 + 1$

$25\left(x - \frac{1}{5}\right)^2 = 200(y + 1)$

Parabola

71. $4x^2 + 4y^2 - 16y + 15 = 0$

$4x^2 + 4(y^2 - 4y + 4) = -15 + 16$

$4x^2 + 4(y - 2)^2 = 1$

Circle (Ellipse)

72. $y^2 - 4y = x + 5$

$y^2 - 4y + 4 = x + 5 + 4$

$(y - 2)^2 = x + 9$

Parabola

73. $9x^2 + 9y^2 - 36x + 6y + 34 = 0$

$9(x^2 - 4x + 4) + 9\left(y^2 + \frac{2}{3}y + \frac{1}{9}\right) = -34 + 36 + 1$

$9(x - 2)^2 + 9\left(y + \frac{1}{3}\right)^2 = 3$

Circle (Ellipse)

74. $2x(x - y) = y(3 - y - 2x)$

$2x^2 - 2xy = 3y - y^2 - 2xy$

$2x^2 + y^2 - 3y = 0$

$2x^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{4}$

Ellipse

75. $3(x - 1)^2 = 6 + 2(y + 1)^2$

$3(x - 1)^2 - 2(y + 1)^2 = 6$

$\frac{(x - 1)^2}{2} - \frac{(y + 1)^2}{3} = 1$

Hyperbola

76. $9(x + 3)^2 = 36 - 4(y - 2)^2$

$9(x + 3)^2 + 4(y - 2)^2 = 36$

$\frac{(x + 3)^2}{4} + \frac{(y - 2)^2}{9} = 1$

Ellipse

77. (a) A parabola is the set of all points (x, y) that are equidistant from a fixed line (directrix) and a fixed point (focus) not on the line.

(b) For directrix $y = k - p$: $(x - h)^2 = 4p(y - k)$

For directrix $x = h - p$: $(y - k)^2 = 4p(x - h)$

(c) If P is a point on a parabola, then the tangent line to the parabola at P makes equal angles with the line passing through P and the focus, and with the line passing through P parallel to the axis of the parabola.

78. (a) An ellipse is the set of all points (x, y), the sum of whose distance from two distinct fixed points (foci) is constant.

(b) $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$ or $\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$

79. (a) A hyperbola is the set of all points (x, y) for which the absolute value of the difference between the distances from two distinct fixed points (foci) is constant.

(b) Transverse axis is horizontal:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Transverse axis is vertical:

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

(c) Transverse axis is horizontal:

$y = k + (b/a)(x - h)$ and $y = k - (b/a)(x - h)$

Transverse axis is vertical:

$y = k + (a/b)(x - h)$ and $y = k - (a/b)(x - h)$

80. $e = \dfrac{c}{a}, c = \sqrt{a^2 - b^2}, \quad 0 < e < 1$

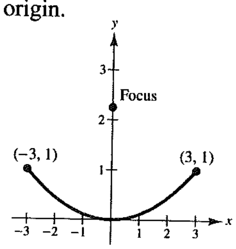
For $e \approx 0$, the ellipse is nearly circular.

For $e \approx 1$, the ellipse is elongated.

81. Assume that the vertex is at the origin.

$$x^2 = 4py$$
$$(3)^2 = 4p(1)$$
$$\frac{9}{4} = p$$

The pipe is located $\frac{9}{4}$ meters from the vertex.



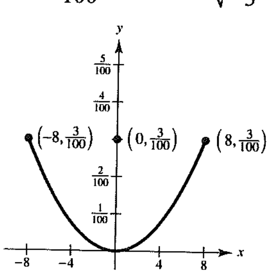
82. Assume that the vertex is at the origin.

(a) $x^2 = 4py$

$$8^2 = 4p\left(\frac{3}{100}\right)$$
$$\frac{1600}{3} = p$$
$$x^2 = 4\left(\frac{1600}{3}\right)y = \frac{6400}{3}y$$

(b) The deflection is 1 cm when

$$y = \frac{2}{100} \Rightarrow x = \pm\sqrt{\frac{128}{3}} \approx \pm 6.53 \text{ meters.}$$



83. $y = ax^2$

$y' = 2ax$

The equation of the tangent line is

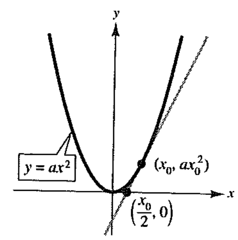
$y - ax_0^2 = 2ax_0(x - x_0)$

or $y = 2ax_0x - ax_0^2$.

Let $y = 0$. Then:

$$-ax_0^2 = 2ax_0x - 2ax_0^2$$
$$ax_0^2 = 2ax_0x$$
$$x = \frac{x_0}{2}$$

So, $\left(\dfrac{x_0}{2}, 0\right)$ is the x-intercept.



84. (a) Without loss of generality, place the coordinate system so that the equation of the parabola is $x^2 = 4py$ and, so,

$$y' = \left(\frac{1}{2p}\right)x.$$

So, for distinct tangent lines, the slopes are unequal and the lines intersect.

(b) $x^2 - 4x - 4y = 0$

$$2x - 4 - 4\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{1}{2}x - 1$$

At $(0, 0)$, the slope is -1: $y = -x$. At $(6, 3)$, the slope is 2: $y = 2x - 9$. Solving for x,

$$-x = 2x - 9$$
$$-3x = -9$$
$$x = 3$$
$$y = -3.$$

Point of intersection: $(3, -3)$

85. (a) Consider the parabola $x^2 = 4py$. Let m_0 be the slope of the one tangent line at (x_1, y_1) and so, $-\frac{1}{m_0}$ is the slope of the second at (x_2, y_2). Differentiating, $2x = 4py'$ or $y' = \frac{x}{2p}$, and you have:

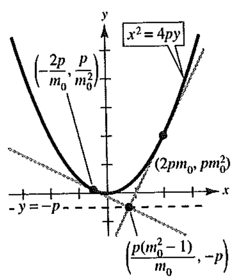
$$m_0 = \frac{1}{2p}x_1 \quad \text{or} \quad x_1 = 2pm_0$$

$$\frac{-1}{m_0} = \frac{1}{2p}x_2 \quad \text{or} \quad x_2 = \frac{-2p}{m_0}.$$

Substituting these values of x into the equation $x^2 = 4py$, we have the coordinates of the points of tangency $(2pm_0, pm_0^2)$ and $(-2p/m_0, p/m_0^2)$ and the equations of the tangent lines are

$$(y - pm_0^2) = m_0(x - 2pm_0) \quad \text{and} \quad \left(y - \frac{p}{m_0^2}\right) = \frac{-1}{m_0}\left(x + \frac{2p}{m_0}\right).$$

The point of intersection of these lines is $\left(\frac{p(m_0^2 - 1)}{m_0}, -p\right)$ and is on the directrix, $y = -p$.



(b) $x^2 - 4x - 4y + 8 = 0$

$$(x - 2)^2 = 4(y - 1)$$

Vertex: $(2, 1)$

$$2x - 4 - 4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{1}{2}x - 1$$

At $(-2, 5)$, $\frac{dy}{dx} = -2$. At $\left(3, \frac{5}{4}\right)$, $\frac{dy}{dx} = \frac{1}{2}$.

Tangent line at $(-2, 5)$: $y - 5 = -2(x + 2) \Rightarrow 2x + y - 1 = 0$.

Tangent line at $\left(3, \frac{5}{4}\right)$: $y - \frac{5}{4} = \frac{1}{2}(x - 3) \Rightarrow 2x - 4y - 1 = 0$.

Because $m_1 m_2 = (-2)\left(\frac{1}{2}\right) = -1$, the lines are perpendicular.

Point of intersection: $-2x + 1 = \frac{1}{2}x - \frac{1}{4}$

$$-\frac{5}{2}x = -\frac{5}{4}$$

$$x = \frac{1}{2}$$

$$y = 0$$

Directrix: $y = 0$ and the point of intersection $\left(\frac{1}{2}, 0\right)$ lies on this line.

86. The focus of $x^2 = 8y = 4(2)y$ is $(0, 2)$. The distance from a point on the parabola, $(x, x^2/8)$, and the focus, $(0, 2)$, is

$$d = \sqrt{(x-0)^2 + \left(\frac{x^2}{8} - 2\right)^2}.$$

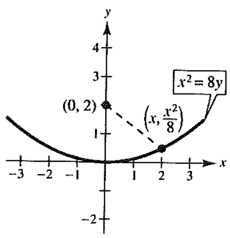
Because d is minimized when d^2 is minimized, it is sufficient to minimize the function

$$f(x) = x^2 + \left(\frac{x^2}{8} - 2\right)^2.$$

$$f'(x) = 2x + 2\left(\frac{x^2}{8} - 2\right)\left(\frac{x}{4}\right) = \frac{x^3}{16} + x.$$

$$f'(x) = 0 \text{ implies that } \frac{x^3}{16} + x = x\left(\frac{x^2}{16} + 1\right) = 0 \Rightarrow x = 0.$$

This is a minimum by the First Derivative Test. So, the closest point to the focus is the vertex, $(0, 0)$.



87. $y = x - x^2$

$$\frac{dy}{dx} = 1 - 2x$$

At the point of tangency (x_1, y_1) on the mountain, $m = 1 - 2x_1$. Also, $m = \dfrac{y_1 - 1}{x_1 + 1}$.

$$\frac{y_1 - 1}{x_1 + 1} = 1 - 2x_1$$

$$(x_1 - x_1^2) - 1 = (1 - 2x_1)(x_1 + 1)$$

$$-x_1^2 + x_1 - 1 = -2x_1^2 - x_1 + 1$$

$$x_1^2 + 2x_1 - 2 = 0$$

$$x_1 = \frac{-2 \pm\sqrt{2^2 - 4(1)(-2)}}{2(1)} = \frac{-2 \pm 2\sqrt{3}}{2} = -1 \pm\sqrt{3}$$

Choosing the positive value for x_1, we have $x_1 = -1 + \sqrt{3}$.

$$m = 1 - 2\left(-1 + \sqrt{3}\right) = 3 - 2\sqrt{3}$$

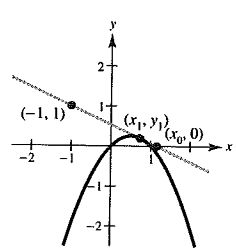
$$m = \frac{0 - 1}{x_0 + 1} = -\frac{1}{x_0 + 1}$$

So, $$-\frac{1}{x_0 + 1} = 3 - 2\sqrt{3}$$

$$\frac{-1}{3 - 2\sqrt{3}} = x_0 + 1$$

$$\frac{3 + 2\sqrt{3}}{3} - 1 = x_0$$

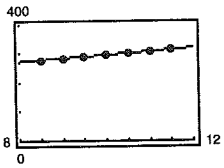
$$\frac{2\sqrt{3}}{3} = x_0.$$



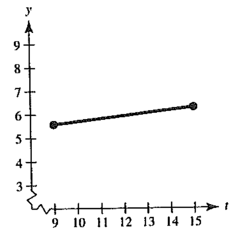
The closest the receiver can be to the hill is $\left(2\sqrt{3}/3\right) - 1 \approx 0.155$.

88. (a) $A = 0.06t^2 + 4.5t + 234 \; (t = 9 \leftrightarrow 1999)$

(b)



(c) $\frac{dA}{dt} = 0.12t + 4.5$



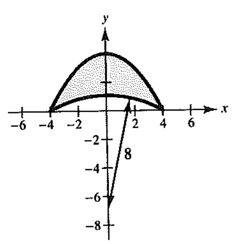
The average change per year is linear.

$\frac{dA}{dt}$ is increasing.

89. Parabola

Vertex: $(0, 4)$

$$x^2 = 4p(y - 4)$$
$$4^2 = 4p(0 - 4)$$
$$p = -1$$
$$x^2 = -4(y - 4)$$
$$y = 4 - \frac{x^2}{4}$$



Circle

Center: $(0, k)$

Radius: 8

$$x^2 + (y - k)^2 = 64$$
$$4^2 + (0 - k)^2 = 64$$
$$k^2 = 48$$
$$k = -4\sqrt{3} \quad \text{(Center is on the negative } y\text{-axis.)}$$

$$x^2 + \left(y + 4\sqrt{3}\right)^2 = 64$$
$$y = -4\sqrt{3} \pm \sqrt{64 - x^2}$$

Because the y-value is positive when $x = 0$, we have $y = -4\sqrt{3} + \sqrt{64 - x^2}$.

$$A = 2\int_0^4 \left[\left(4 - \frac{x^2}{4}\right) - \left(-4\sqrt{3} + \sqrt{64 - x^2}\right)\right] dx$$
$$= 2\left[4x - \frac{x^3}{12} + 4\sqrt{3}x - \frac{1}{2}\left(x\sqrt{64 - x^2} + 64 \arcsin\frac{x}{8}\right)\right]_0^4$$
$$= 2\left(16 - \frac{64}{12} + 16\sqrt{3} - 2\sqrt{48} - 32 \arcsin\frac{1}{2}\right) = \frac{16\left(4 + 3\sqrt{3} - 2\pi\right)}{3} \approx 15.536 \text{ square feet}$$

90.

$$x = \frac{1}{4}y^2$$

$$x' = \frac{1}{2}y$$

$$1 + (x')^2 = 1 + \frac{y^2}{4}$$

$$s = \int_0^4 \sqrt{1 + \left(\frac{y^2}{4}\right)}\, dy = \frac{1}{2}\int_0^4 \sqrt{4 + y^2}\, dy$$

$$= \frac{1}{4}\left[y\sqrt{4 + y^2} + 4\ln\left|y + \sqrt{4 + y^2}\right|\right]_0^4$$

$$= \frac{1}{4}\left(4\sqrt{20} + 4\ln\left|4 + \sqrt{20}\right| - 4\ln 2\right) = 2\sqrt{5} + \ln\left(2 + \sqrt{5}\right) \approx 5.916$$

91. (a) Assume that $y = ax^2$.

$$20 = a(60)^2 \Rightarrow a = \frac{2}{360} = \frac{1}{180} \Rightarrow y = \frac{1}{180}x^2$$

(b) $f(x) = \frac{1}{180}x^2,\ f'(x) = \frac{1}{90}x$

$$S = 2\int_0^{60} \sqrt{1 + \left(\frac{1}{90}x\right)^2}\, dx = \frac{2}{90}\int_0^{60} \sqrt{90^2 + x^2}\, dx$$

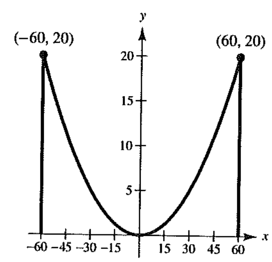
$$= \frac{2}{90}\frac{1}{2}\left[x\sqrt{90^2 + x^2} + 90^2\ln\left|x + \sqrt{90^2 + x^2}\right|\right]_0^{60} \quad \text{(Formula 26)}$$

$$= \frac{1}{90}\left[60\sqrt{11{,}700} + 90^2\ln\left(60 + \sqrt{11{,}700}\right) - 90^2\ln 90\right]$$

$$= \frac{1}{90}\left[1800\sqrt{13} + 90^2\ln\left(60 + 30\sqrt{13}\right) - 90^2\ln 90\right]$$

$$= 20\sqrt{13} + 90\ln\left(\frac{60 + 30\sqrt{13}}{90}\right)$$

$$= 10\left[2\sqrt{13} + 9\ln\left(\frac{2 + \sqrt{13}}{3}\right)\right] \approx 128.4 \text{ m}$$



92. $x^2 = 20y$

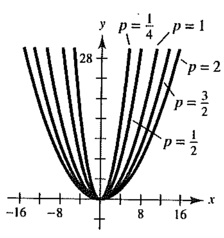
$$y = \frac{x^2}{20}$$

$$y' = \frac{x}{10}$$

$$S = 2\pi\int_0^r x\sqrt{1 + \left(\frac{x}{10}\right)^2}\, dx = 2\pi\int_0^r \frac{x\sqrt{100 + x^2}}{10}\, dx = \left[\frac{\pi}{10}\cdot\frac{2}{3}\left(100 + x^2\right)^{3/2}\right]_0^r = \frac{\pi}{15}\left[\left(100 + r^2\right)^{3/2} - 1000\right]$$

93. $x^2 = 4py,\ p = \frac{1}{4}, \frac{1}{2}, 1, \frac{3}{2}, 2$

As p increases, the graph becomes wider.

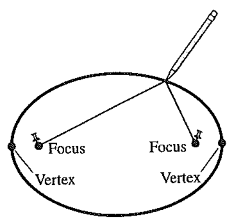


94. $A = 2\int_0^h \sqrt{4py}\, dy$

$= 4\sqrt{p}\int_0^h y^{1/2}\, dy = \left[4\sqrt{p}\left(\frac{2}{3}\right)y^{3/2}\right]_0^h = \frac{8}{3}\sqrt{p}h^{3/2}$

95. (a) At the vertices you notice that the string is horizontal and has a length of $2a$.

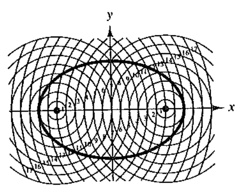
(b) The thumbtacks are located at the foci and the length of string is the constant sum of the distances from the foci.



96. $a = \frac{5}{2}, b = 2, c = \sqrt{\left(\frac{5}{2}\right)^2 - (2)^2} = \frac{3}{2}$

The tacks should be placed 1.5 feet from the center. The string should be $2a = 5$ feet long.

97.



98. $e = \frac{c}{a}$

$0.0167 = \frac{c}{149{,}598{,}000}$

$c \approx 2{,}498{,}286.6$

Least distance: $a - c = 147{,}099{,}713.4$ km

Greatest distance: $a + c = 152{,}096{,}286.6$ km

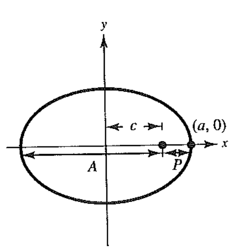
99. $e = \frac{c}{a}$

$A + P = 2a$

$a = \frac{A + P}{2}$

$c = a - P = \frac{A + P}{2} - P = \frac{A - P}{2}$

$e = \frac{c}{a} = \frac{(A - P)/2}{(A + P)/2} = \frac{A - P}{A + P}$



100. $e = \frac{A - P}{A + P}$

$= \frac{(123{,}000 + 4000) - (119 + 4000)}{(123{,}000 + 4000) + (119 + 4000)}$

$= \frac{122{,}881}{131{,}119} \approx 0.9372$

101. $e = \frac{A - P}{A + P}$ (Exercise 99)

$= \frac{(1865 + 4000) - (96 + 4000)}{(1865 + 4000) + (96 + 4000)} = \frac{1769}{9961} \approx 0.1776$

102. $9x^2 + 4y^2 - 36x - 24y - 36 = 0$

(a) $9(x^2 - 4x + 4) + 4(y^2 - 6y + 9) = 36 + 36 + 36$

$9(x - 2)^2 + 4(y - 3)^2 = 108$

$\frac{(x - 2)^2}{12} + \frac{(y - 3)^2}{27} = 1$

Ellipse

(b) $9x^2 - 4y^2 - 36x - 24y - 36 = 0$

$9(x^2 - 4x + 4) - 4(y^2 + 6y + 9) = 36 + 36 - 36$

$\frac{(x - 2)^2}{4} - \frac{(y + 3)^2}{9} = 1$

Hyperbola

(c) $4x^2 + 4y^2 - 36x - 24y - 36 = 0$

$4\left(x^2 - 9x + \frac{81}{4}\right) + 4(y^2 - 6y + 9) = 36 + 81 + 36$

$\left(x - \frac{9}{2}\right)^2 + (y - 3)^2 = \frac{153}{4}$

Circle

(d) Sample answer: Eliminate the y^2-term

103. $e = \frac{A - P}{A + P} = \frac{35.29 - 0.59}{35.29 + 0.59} = 0.9671$

104. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{x^2}{a^2} + \frac{y^2}{a^2(b^2/a^2)} = 1$

$\frac{x^2}{a^2} + \frac{y^2}{a^2(a^2 - c^2)/a^2} = 1$

$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$

As $e \to 0, 1 - e^2 \to 1$ and you have

$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$ or the circle $x^2 + y^2 = a^2$.

105. $\dfrac{x^2}{10^2} + \dfrac{y^2}{5^2} = 1$

$\dfrac{2x}{10^2} + \dfrac{2yy'}{5^2} = 0$

$y' = \dfrac{-5^2 x}{10^2 y} = \dfrac{-x}{4y}$

At $(-8, 3)$: $y' = \dfrac{8}{12} = \dfrac{2}{3}$

The equation of the tangent line is $y - 3 = \dfrac{2}{3}(x + 8)$. It will cross the y-axis when $x = 0$ and $y = \dfrac{2}{3}(8) + 3 = \dfrac{25}{3}$.

106. $\dfrac{x^2}{(4.5)^2} + \dfrac{y^2}{(2.5)^2} = 1$

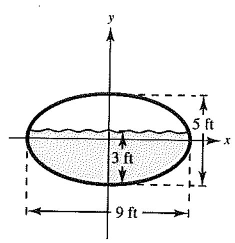
$x^2 = (4.5)^2\left[1 - \dfrac{y^2}{(2.5)^2}\right]$

$x = \pm\dfrac{9}{5}\sqrt{(2.5)^2 - y^2}$

$V = (\text{Area of bottom})(\text{Length}) + (\text{Area of top})(\text{Length})$

$V = \left[\dfrac{\pi(4.5)(2.5)}{2}\right](16) + 16\int_0^{0.5} 2\dfrac{9}{5}\sqrt{(2.5)^2 - y^2}\, dy$ (Recall: Area of ellipse is πab.)

$= 90\pi + \dfrac{144}{5}\cdot\left[y\sqrt{(2.5)^2 - y^2} + (2.5)^2 \arcsin\dfrac{y}{2.5}\right]_0^{0.5} = 90\pi + \dfrac{144}{5}\left[0.5\sqrt{6} + (2.5)^2 \arcsin\dfrac{1}{5}\right] \approx 354.3 \text{ ft}^3$



107. $16x^2 + 9y^2 + 96x + 36y + 36 = 0$

$32x + 18yy' + 96 + 36y' = 0$

$y'(18y + 36) = -(32x + 96)$

$y' = \dfrac{-(32x + 96)}{18y + 36}$

$y' = 0$ when $x = -3$. y' is undefined when $y = -2$.

At $x = -3$, $y = 2$ or -6.

Endpoints of major axis: $(-3, 2), (-3, -6)$

At $y = -2$, $x = 0$ or -6.

Endpoints of minor axis: $(0, -2), (-6, -2)$

Note: Equation of ellipse is $\dfrac{(x + 3)^2}{9} + \dfrac{(y + 2)^2}{16} = 1$

108. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

$18x + 8yy' + 36 - 24y' = 0$

$(8y - 24)y' = -(18x + 36)$

$y' = \dfrac{-(18x + 36)}{8y - 24}$

$y' = 0$ when $x = -2$. y' undefined when $y = 3$.

At $x = -2$, $y = 0$ or 6.

Endpoints of major axis: $(-2, 0), (-2, 6)$

At $y = 3$, $x = 0$ or -4.

Endpoints of minor axis: $(0, 3), (-4, 3)$

Note: Equation of ellipse is $\dfrac{(x + 2)^2}{4} + \dfrac{(y - 3)^2}{9} = 1$

109. (a) $A = 4\int_0^2 \frac{1}{2}\sqrt{4 - x^2}\,dx = \left[x\sqrt{4 - x^2} + 4\arcsin\left(\frac{x}{2}\right)\right]_0^2 = 2\pi \quad \left[\text{or, } A = \pi ab = \pi(2)(1) = 2\pi\right]$

(b) **Disk:** $V = 2\pi\int_0^2 \frac{1}{4}\left(4 - x^2\right)dx = \frac{1}{2}\pi\left[4x - \frac{1}{3}x^3\right]_0^2 = \frac{8\pi}{3}$

$$y = \frac{1}{2}\sqrt{4 - x^2}$$

$$y' = \frac{-x}{2\sqrt{4 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{16 - 4x^2}} = \sqrt{\frac{16 - 3x^2}{4y}}$$

$$S = 2(2\pi)\int_0^2 y\left(\frac{\sqrt{16 - 3x^2}}{4y}\right)dx = \pi\int_0^2 \sqrt{16 - 3x^2}\,dx$$

$$= \frac{\pi}{2\sqrt{3}}\left[\sqrt{3}x\sqrt{16 - 3x^2} + 16\arcsin\left(\frac{\sqrt{3}x}{4}\right)\right]_0^2 = \frac{2\pi}{9}\left(9 + 4\sqrt{3}\pi\right) \approx 21.48$$

(c) **Shell:** $V = 2\pi\int_0^2 x\sqrt{4 - x^2}\,dx = -\pi\int_0^2 -2x\left(4 - x^2\right)^{1/2}dx = -\frac{2\pi}{3}\left[\left(4 - x^2\right)^{3/2}\right]_0^2 = \frac{16\pi}{3}$

$$x = 2\sqrt{1 - y^2}$$

$$x' = \frac{-2y}{\sqrt{1 - y^2}}$$

$$\sqrt{1 + (x')^2} = \sqrt{1 + \frac{4y^2}{1 - y^2}} = \frac{\sqrt{1 + 3y^2}}{\sqrt{1 - y^2}}$$

$$S = 2(2\pi)\int_0^1 2\sqrt{1 - y^2}\frac{\sqrt{1 + 3y^2}}{\sqrt{1 - y^2}}\,dy = 8\pi\int_0^1 \sqrt{1 + 3y^2}\,dy$$

$$= \frac{8\pi}{2\sqrt{3}}\left[\sqrt{3}y\sqrt{1 + 3y^2} + \ln\left|\sqrt{3}y + \sqrt{1 + 3y^2}\right|\right]_0^1$$

$$= \frac{4\pi}{3}\left|6 + \sqrt{3}\ln\left(2 + \sqrt{3}\right)\right| \approx 34.69$$

110. (a) $A = 4\int_0^4 \frac{3}{4}\sqrt{16 - x^2}\,dx = \frac{3}{2}\left[x\sqrt{16 - x^2} + 16\arcsin\frac{x}{4}\right]_0^4 = 12\pi$

(b) **Disk:** $V = 2\pi\int_0^4 \frac{9}{16}\left(16 - x^2\right)dx = \frac{9\pi}{8}\left[\left(16x - \frac{1}{3}x^3\right]_0^4 = 48\pi$

$$y = \frac{3}{4}\sqrt{16 - x^2}$$

$$y' = \frac{-3x}{4\sqrt{16 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{9x^2}{16\left(16 - x^2\right)}}$$

$$S = 2(2\pi)\int_0^4 \frac{3}{4}\sqrt{16 - x^2}\sqrt{\frac{16\left(16 - x^2\right) + 9x^2}{16\left(16 - x^2\right)}}\,dx = 4\pi\int_0^4 \frac{3}{4}\sqrt{16 - x^2}\frac{\sqrt{256 - 7x^2}}{4\sqrt{16 - x^2}}\,dx = \frac{3\pi}{4}\int_0^4 \sqrt{256 - 7x^2}\,dx$$

$$= \frac{3\pi}{8\sqrt{7}}\left[\sqrt{7}x\sqrt{256 - 7x^2} + 256\arcsin\frac{\sqrt{7}x}{16}\right]_0^4 = \frac{3\pi}{8\sqrt{7}}\left(48\sqrt{7} + 256\arcsin\frac{\sqrt{7}}{4}\right) \approx 138.93$$

(c) **Shell:** $V = 4\pi\int_0^4 x\left(\frac{3}{4}\sqrt{16 - x^2}\right) dx = 3\pi\left[\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(16 - x^2\right)^{3/2}\right]_0^4 = 64\pi$

$$x = \frac{4}{3}\sqrt{9 - y^2}$$

$$x' = \frac{-4y}{3\sqrt{9 - y^2}}$$

$$\sqrt{1 + (x')^2} = \sqrt{1 + \frac{16y^2}{9(9 - y^2)}}$$

$$S = 2(2\pi)\int_0^3 \frac{4}{3}\sqrt{9 - y^2}\sqrt{\frac{9(9 - y^2) + 16y^2}{9(9 - y^2)}}\, dy$$

$$= 4\pi\int_0^3 \frac{4}{9}\sqrt{81 + 7y^2}\, dy$$

$$= \frac{16}{9}\left(\frac{\pi}{2\sqrt{7}}\right)\left[\sqrt{7}y\sqrt{81 + 7y^2} + 81\ln\left|\sqrt{7}y + \sqrt{81 + 7y^2}\right|\right]_0^3$$

$$= \frac{8\pi}{9\sqrt{7}}3\sqrt{7}(12) + 81\ln\left(3\sqrt{7} + 12\right) - 81\ln 9 \approx 168.53$$

111. From Example 5,

$$C = 4a\int_0^{\pi/2}\sqrt{1 - e^2\sin^2\theta}\, d\theta$$

For $\frac{x^2}{25} + \frac{y^2}{49} = 1$, you have

$a = 7, b = 5, c = \sqrt{49 - 25} = 2\sqrt{6}, e = \frac{c}{a} = \frac{2\sqrt{6}}{7}.$

$$C = 4(7)\int_0^{\pi/2}\sqrt{1 - \frac{24}{49}\sin^2\theta}\, d\theta \approx 28(1.3558) \approx 37.96$$

112. (1) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$$

$$y' = -\frac{xb^2}{ya^2}$$

At P, $y' = -\frac{b^2}{a^2}\cdot\frac{x_0}{y_0} = m.$

(2) Slope of line through $(-c, 0)$ and (x_0, y_0): $m_1 = \frac{y_0}{x_0 + c}$

Slope of line through $(c, 0)$ and (x_0, y_0): $m_2 = \frac{y_0}{x_0 - c}$

$$\text{(3)}\quad \tan\alpha = \frac{m_2 - m}{1 + m_2 m} = \frac{\dfrac{y_0}{x_0 - c} - \left(-\dfrac{b^2x_0}{a^2y_0}\right)}{1 + \left(\dfrac{y_0}{x_0 - c}\right)\left(-\dfrac{b^2x_0}{a^2y_0}\right)} = \frac{a^2y_0^2 + b^2x_0(x_0 - c)}{a^2y_0(x_0 - c) - b^2x_0y_0}$$

$$= \frac{a^2y_0^2 + b^2x_0^2 - b^2x_0c}{x_0y_0(a^2 - b^2) - a^2y_0c} = \frac{a^2b^2 - b^2x_0c}{x_0y_0c^2 - a^2y_0c} = \frac{b^2(a^2 - x_0c)}{y_0c(x_0c - a^2)} = -\frac{b^2}{y_0c}$$

$$\alpha = \arctan\left(-\frac{b^2}{y_0c}\right) = -\arctan\left(\frac{b^2}{y_0c}\right)$$

$$\tan\beta = \frac{m_1 - m}{1 + m_1 m} = \frac{\dfrac{y_0}{x_0 + c} - \left(-\dfrac{b^2x_0}{a^2y_0}\right)}{1 + \left(\dfrac{y_0}{x_0 + c}\right)\left(-\dfrac{b^2x_0}{a^2y_0}\right)} = \frac{a^2y_0^2 + b^2x_0(x_0 + c)}{a^2y_0(x_0 + c) - b^2x_0y_0}$$

$$= \frac{a^2y_0^2 + b^2x_0^2 + b^2x_0c}{a^2x_0y_0 + a^2cy_0 - b^2x_0y_0} = \frac{a^2b^2 + b^2x_0c}{x_0y_0(a^2 - b^2) + a^2cy_0} = \frac{b^2(a^2 + x_0c)}{y_0c(x_0c + a^2)} = \frac{b^2}{y_0c}$$

$$\beta = \arctan\left(\frac{b^2}{y_0c}\right)$$

Because $|\alpha| = |\beta|$, the tangent line to an ellipse at a point P makes equal angles with the line through P and the foci.

113. Area circle $= \pi r^2 = 100\pi$

Area ellipse $= \pi ab = \pi a(10)$

$2(100\pi) = 10\pi a \Rightarrow a = 20$

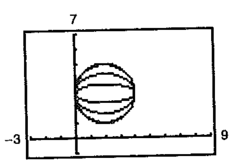
So, the length of the major axis is $2a = 40$.

114. (a) $e = \dfrac{c}{a} = \dfrac{\sqrt{a^2 + b^2}}{a} \Rightarrow (ea)^2 - a^2 = b^2$. So,

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{a^2(1 - e^2)} = 1.$$

(b) $\dfrac{(x - 2)^2}{4} + \dfrac{(y - 3)^2}{4(1 - e^2)} = 1$



(c) As e approaches 0, the ellipse approaches a circle.

115. The transverse axis is horizontal since $(2, 2)$ and $(10, 2)$ are the foci (see definition of hyperbola).

Center: $(6, 2)$

$c = 4, 2a = 6, b^2 = c^2 - a^2 = 7$

So, the equation is $\dfrac{(x - 6)^2}{9} - \dfrac{(y - 2)^2}{7} = 1.$

116. The transverse axis is vertical because $(-3, 0)$ and $(-3, 3)$ are the foci.

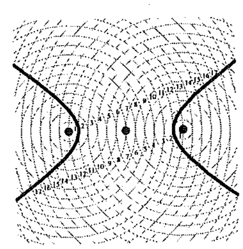
Center: $\left(-3, \dfrac{3}{2}\right)$

$c = \dfrac{3}{2}, 2a = 2, b^2 = c^2 - a^2 = \dfrac{5}{4}$

So, the equation is $\dfrac{[y - (3/2)]^2}{1} - \dfrac{(x + 3)^2}{5/4} = 1.$

117. $2a = 10 \Rightarrow a = 5$

$c = 6 \Rightarrow b = \sqrt{11}$



118. Center: $(0, 0)$

Horizontal transverse axis

Foci: $(\pm c, 0)$

Vertices: $(\pm a, 0)$

The difference of the distances from any point on the hyperbola is constant. At a vertex, this constant difference is $(a + c) - (c - a) = 2a$.

Now, for any point (x, y) on the hyperbola, the difference of the distances between (x, y) and the two foci must also be $2a$.

$$\sqrt{(x - c)^2 + (y - 0)^2} - \sqrt{(x + c)^2 + (y - 0)^2} = 2a$$

$$\sqrt{(x - c)^2 + y^2} = 2a + \sqrt{(x + c)^2 + y^2}$$

$$(x - c)^2 + y^2 = 4a^2 + 4a\sqrt{(x + c)^2 + y^2} + (x + c)^2 + y^2$$

$$-4xc - 4a^2 = 4a\sqrt{(x + c)^2 + y^2}$$

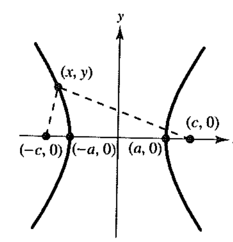
$$-(xc + a^2) = a\sqrt{(x + c)^2 + y^2}$$

$$x^2c^2 + 2a^2cx + a^4 = a^2\left[x^2 + 2cx + c^2 + y^2\right]$$

$$x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2)$$

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

Because $a^2 + b^2 = c^2$, we have $\left(\frac{x^2}{a^2}\right) - \left(\frac{y^2}{b^2}\right) = 1.$



119. Time for sound of bullet hitting target to reach (x, y): $\frac{2c}{v_m} + \frac{\sqrt{(x - c)^2 + y^2}}{v_s}$

Time for sound of rifle to reach (x, y): $\frac{\sqrt{(x + c)^2 + y^2}}{v_s}$

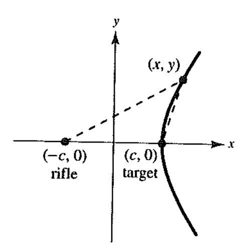
Because the times are the same, you have: $\frac{2c}{v_m} + \frac{\sqrt{(x - c)^2 + y^2}}{v_s} = \frac{\sqrt{(x + c)^2 + y^2}}{v_s}$

$$\frac{4c^2}{v_m^2} + \frac{4c}{v_m v_s}\sqrt{(x - c)^2 + y^2} + \frac{(x - c)^2 + y^2}{v_s^2} = \frac{(x + c)^2 + y^2}{v_s^2}$$

$$\sqrt{(x - c)^2 + y^2} = \frac{v_m^2 x - v_s^2 c}{v_s v_m}$$

$$\left(1 - \frac{v_m^2}{v_s^2}\right)x^2 + y^2 = \left(\frac{v_s^2}{v_m^2} - 1\right)c^2$$

$$\frac{x^2}{c^2 v_s^2 / v_m^2} - \frac{y^2}{c^2\left(v_m^2 - v_s^2\right)/v_m^2} = 1$$



120. $c = 150, 2a = 0.001(186{,}000), a = 93,$

$b = \sqrt{150^2 - 93^2} = \sqrt{13{,}851}$

$$\frac{x^2}{93^2} - \frac{y^2}{13{,}851} = 1$$

When $y = 75$, you have

$$x^2 = 93^2\left(1 + \frac{75^2}{13{,}851}\right)$$

$x \approx 110.3$ mi.

121. The point (x, y) lies on the line between $(0, 10)$ and $(10, 0)$. So, $y = 10 - x$. The point also lies on the hyperbola $(x^2/36) - (y^2/64) = 1$. Using substitution, you have:

$$\frac{x^2}{36} - \frac{(10-x)^2}{64} = 1$$

$$16x^2 - 9(10-x)^2 = 576$$

$$7x^2 + 180x - 1476 = 0$$

$$x = \frac{-180 \pm \sqrt{180^2 - 4(7)(-1476)}}{2(7)}$$

$$= \frac{-180 \pm 192\sqrt{2}}{14} = \frac{-90 \pm 96\sqrt{2}}{7}$$

Choosing the positive value for x we have:

$$x = \frac{-90 + 96\sqrt{2}}{7} \approx 6.538 \text{ and}$$

$$y = \frac{160 - 96\sqrt{2}}{7} \approx 3.462$$

122.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \text{ or } y' = \frac{b^2x}{a^2y}$$

$$y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0)$$

$$a^2y_0y - a^2y_0^2 = b^2x_0x - b^2x_0^2$$

$$b^2x_0^2 - a^2y_0^2 = b^2x_0x - a^2y_0y$$

$$a^2b^2 = b^2x_0x - a^2y_0y$$

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$$

123.

$$\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \Rightarrow \frac{2y^2}{b^2} = 1 - \frac{x^2}{a^2}. \text{ Let } c^2 = a^2 - b^2.$$

$$\frac{x^2}{a^2 - b^2} - \frac{2y^2}{b^2} = 1 \Rightarrow \frac{2y^2}{b^2} = \frac{x^2}{a^2 - b^2} - 1$$

$$1 - \frac{x^2}{a^2} = \frac{x^2}{a^2 - b^2} - 1 \Rightarrow 2 = x^2\left(\frac{1}{a^2} + \frac{1}{a^2 - b^2}\right)$$

$$x^2 = \frac{2a^2(a^2 - b^2)}{2a^2 - b^2} \Rightarrow x = \pm\frac{\sqrt{2}a\sqrt{a^2 - b^2}}{\sqrt{2a^2 - b^2}} = \pm\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}$$

$$\frac{2y^2}{b^2} = 1 - \frac{1}{a^2}\left(\frac{2a^2c^2}{2a^2 - b^2}\right) \Rightarrow \frac{2y^2}{b^2} = \frac{b^2}{2a^2 - b^2}$$

$$y^2 = \frac{b^4}{2(2a^2 - b^2)} \Rightarrow y = \pm\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}$$

There are four points of intersection: $\left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \pm\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)$, $\left(-\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \pm\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)$

$$\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} + \frac{4yy'}{b^2} = 0 \Rightarrow y_e' = -\frac{b^2x}{2a^2y}$$

$$\frac{x^2}{a^2 - b^2} - \frac{2y^2}{b^2} = 1 \Rightarrow \frac{2x}{c^2} - \frac{4yy'}{b^2} = 0 \Rightarrow y_h' = \frac{b^2x}{2c^2y}$$

At $\left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)$, the slopes of the tangent lines are:

$$y_e' = \frac{-b^2\left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}\right)}{2a^2\left(\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)} = -\frac{c}{a} \quad \text{and} \quad y_h' = \frac{b^2\left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}\right)}{2c^2\left(\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)} = \frac{a}{c}.$$

Because the slopes are negative reciprocals, the tangent lines are perpendicular. Similarly, the curves are perpendicular at the other three points of intersection.

124. $$Ax^2 + Cy^2 + Dx + Ey + F = 0 \quad \text{(Assume } A \neq 0 \text{ and } C \neq 0\text{; see (b) below)}$$

$$A\left(x^2 + \frac{D}{A}x\right) + C\left(y^2 + \frac{E}{C}y\right) = -F$$

$$A\left(x^2 + \frac{D}{A}x + \frac{D^2}{4A^2}\right) + C\left(y^2 + \frac{E}{C}y + \frac{E^2}{4C^2}\right) = -F + \frac{D^2}{4A} + \frac{E^2}{4C} = R$$

$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{C} + \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{A} = \frac{R}{AC}$$

(a) If $A = C$, you have

$$\left(x + \frac{D}{2A}\right)^2 + \left(y + \frac{E}{2C}\right)^2 = \frac{R}{A}$$

which is the standard equation of a circle.

(b) If $C = 0$, you have

$$A\left(x + \frac{D}{2A}\right)^2 = -F - Ey + \frac{D^2}{4A}.$$

If $A = 0$, you have

$$C\left(y + \frac{E}{2C}\right)^2 = -F - Dx + \frac{E^2}{4C}.$$

These are the equations of parabolas.

(c) If $AC < 0$, you have

$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{\left|\frac{R}{A}\right|} + \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{\left|\frac{R}{C}\right|} = 1$$

which is the equation of an ellipse.

(d) If $AC < 0$, you have

$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{\left|\frac{R}{A}\right|} - \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{\left|\frac{R}{C}\right|} = \pm 1$$

which is the equation of a hyperbola.

125. False. See the definition of a parabola.

126. True

127. True

128. False. $y^2 - x^2 + 2x + 2y = 0$ yields two intersecting lines: $y + 1 = \pm(x - 1)$

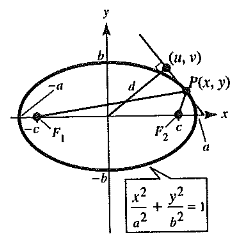
129. True

130. True

131. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the equation of the ellipse with $a > b > 0$. Let $(\pm c, 0)$ be the foci, $c^2 = a^2 - b^2$. Let (u, v) be a point on the tangent line at $P(x, y)$, as indicated in the figure.

$$x^2b^2 + y^2a^2 = a^2b^2$$
$$2xb^2 + 2yy'a^2 = 0$$
$$y' = -\frac{b^2x}{a^2y} \quad \text{Slope at } P(x, y)$$

Now,
$$\frac{y-v}{x-u} = -\frac{b^2x}{a^2y}$$
$$y^2a^2 - a^2vy = -b^2x^2 + b^2xu$$
$$y^2a^2 + x^2b^2 = a^2vy + b^2ux$$
$$a^2b^2 = a^2vy + b^2ux$$



Because there is a right angle at (u, v),

$$\frac{v}{u} = \frac{a^2y}{b^2x}$$
$$vb^2x = a^2uy.$$

You have two equations:

$$a^2vy + b^2ux = a^2b^2$$
$$a^2uy - b^2vx = 0.$$

Multiplying the first by v and the second by u, and adding,

$$a^2v^2y + a^2u^2y = a^2b^2v$$
$$y\left[u^2 + v^2\right] = b^2v$$
$$yd^2 = b^2v$$
$$v = \frac{yd^2}{b^2}.$$

Similarly, $u = \frac{xd^2}{a^2}$.

From the figure, $u = d\cos\theta$ and $v = d\sin\theta$. So, $\cos\theta = \frac{xd}{a^2}$ and $\sin\theta = \frac{yd}{b^2}$.

$$\cos^2\theta + \sin^2\theta = \frac{x^2d^2}{a^4} + \frac{y^2d^2}{b^4} = 1$$
$$x^2b^4d^2 + y^2a^4d^2 = a^4b^4$$
$$d^2 = \frac{a^4b^4}{x^2b^4 + y^2a^4}$$

Let $r_1 = PF_1$ and $r_2 = PF_2$, $r_1 + r_2 = 2a$.

$$r_1r_2 = \frac{1}{2}\left[(r_1 + r_2)^2 - r_1^2 - r_2^2\right] = \frac{1}{2}\left[4a^2 - (x + c)^2 - y^2 - (x - c)^2 - y^2\right] = 2a^2 - x^2 - y^2 - c^2 = a^2 + b^2 - x^2 - y^2$$

Finally,
$$d^2r_1r_2 = \frac{a^4b^4}{x^2b^4 + y^2a^4}\cdot\left[a^2 + b^2 - x^2 - y^2\right]$$
$$= \frac{a^4b^4}{b^2(b^2x^2) + a^2(a^2y^2)}\cdot\left[a^2 + b^2 - x^2 - y^2\right]$$
$$= \frac{a^4b^4}{b^2(a^2b^2 - a^2y^2) + a^2(a^2b^2 - b^2x^2)}\cdot\left[a^2 + b^2 - x^2 - y^2\right]$$
$$= \frac{a^4b^4}{a^2b^2\left[a^2 + b^2 - x^2 - y^2\right]}\cdot\left[a^2 + b^2 - x^2 - y^2\right] = a^2b^2, \text{ a constant!}$$

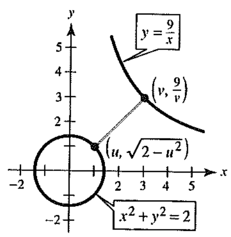
132. Consider circle $x^2 + y^2 = 2$ and hyperbola $y = \dfrac{9}{x}$.

Let $\left(u, \sqrt{2 - u^2}\right)$ and $\left(v, \dfrac{9}{v}\right)$ be points on the circle and hyperbola, respectively. We need to minimize the distance between these 2 points:

$$(\text{Distance})^2 = f(u, v) = (u - v)^2 + \left(\sqrt{2 + u^2} - \frac{9}{v}\right)^2.$$

The tangent lines at $(1, 1)$ and $(3, 3)$ are both perpendicular to $y = x$, and so they are parallel.

The minimum value is $(3 - 1)^2 + (3 - 1)^2 = 8$.



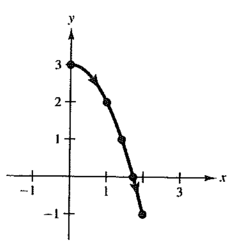
Section 10.2 Plane Curves and Parametric Equations

1. $x = \sqrt{t}, y = 3 - t$

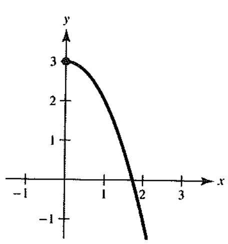
(a)

t	0	1	2	3	4
x	0	1	$\sqrt{2}$	$\sqrt{3}$	2
y	3	2	1	0	−1

(b), (c)



(d) $x^2 = t, y = 3 - x^2, x \ge 0$



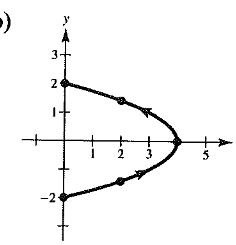
2. $x = 4\cos^2\theta \quad y = 2\sin\theta$

$0 \le x \le 4 \quad -2 \le y \le 2$

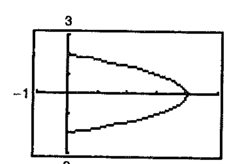
(a)

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
x	0	2	4	2	0
y	−2	$-\sqrt{2}$	0	$\sqrt{2}$	2

(b)



(c)



(d)
$$\frac{x}{4} = \cos^2\theta$$
$$\frac{y^2}{4} = \sin^2\theta$$
$$\frac{x}{4} + \frac{y^2}{4} = 1$$
$$x = 4 - y^2, -2 \le y \le 2$$

(e) The graph would be oriented in the opposite direction.

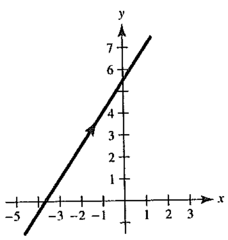
3. $x = 2t - 3$

$y = 3t + 1$

$t = \dfrac{x+3}{2}$

$y = 3\left(\dfrac{x+3}{2}\right) + 1 = \dfrac{3}{2}x + \dfrac{11}{2}$

$3x - 2y + 11 = 0$

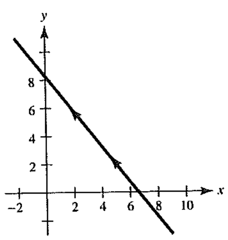


4. $x = 5 - 4t$

$y = 2 + 5t$

$t = \dfrac{5-x}{4}$

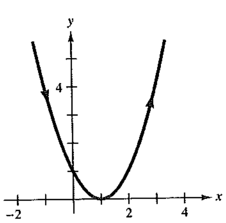
$y = 2 + 5\left(\dfrac{5-x}{4}\right) = -\dfrac{5}{4}x + \dfrac{33}{4}$



5. $x = t + 1$

$y = t^2$

$y = (x-1)^2$



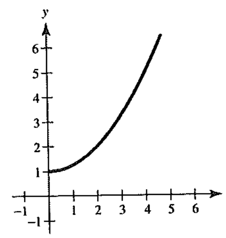
6. $x = 2t^2$

$y = t^4 + 1$

$y = \left(\dfrac{x}{2}\right)^2 + 1 = \dfrac{x^2}{4} + 1, x \ge 0$

For $t < 0$, the orientation is right to left.

For $t > 0$, the orientation is left to right.

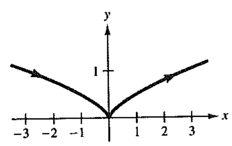


7. $x = t^3$

$y = \frac{1}{2}t^2$

$y = t^3$ implies $t = x^{1/3}$

$y = \frac{1}{2}x^{2/3}$



8. $x = t^2 + t, y = t^2 - t$

Subtracting the second equation from the first, you have

$x - y = 2t$ or $t = \dfrac{x-y}{2}$.

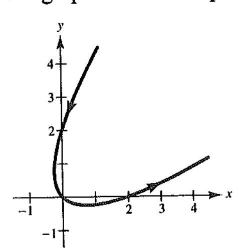
$y = \dfrac{(x-y)^2}{4} - \dfrac{x-y}{2}$

t	-2	-1	0	1	2
x	2	0	0	2	6
y	6	2	0	0	2

Because the discriminant is

$B^2 - 4AC = (-2)^2 - 4(1)(1) = 0,$

the graph is a rotated parabola.

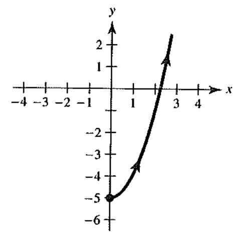


9. $x = \sqrt{t}$

$y = t - 5$

$x^2 = t$

$y = x^2 - 5, x \geq 0$

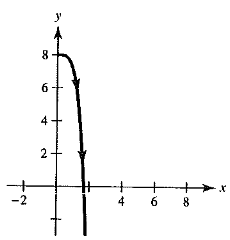


10. $x = \sqrt[4]{t}$

$y = 8 - t$

$x^4 = t$

$y = 8 - x^4, x \geq 0$

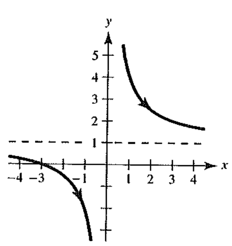


11. $x = t - 3$

$y = \dfrac{t}{t - 3}$

$t = x + 3$

$y = \dfrac{x + 3}{(x + 3) - 3} = 1 + \dfrac{3}{x} = \dfrac{x + 3}{x}$

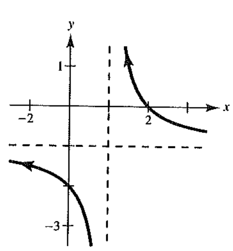


12. $x = 1 + \dfrac{1}{t}$

$y = t - 1$

$x = 1 + \dfrac{1}{t}$ implies $t = \dfrac{1}{x - 1}$

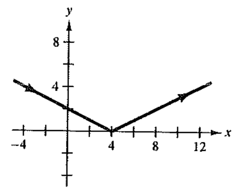
$y = \dfrac{1}{x - 1} - 1$



13. $x = 2t$

$y = |t - 2|$

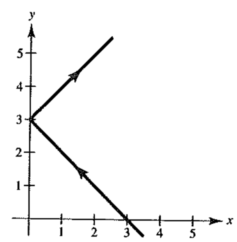
$y = \left|\dfrac{x}{2} - 2\right| = \dfrac{|x - 4|}{2}$



14. $x = |t - 1|$

$y = t + 2$

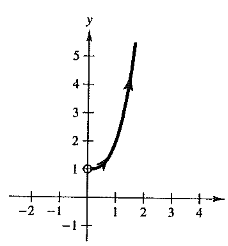
$x = |(y - 2) - 1| = |y - 3|$



15. $x = e^t, x > 0$

$y = e^{3t} + 1$

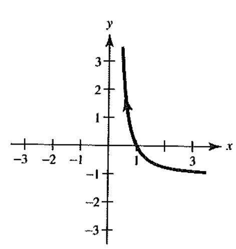
$y = x^3 + 1, x > 0$



16. $x = e^{-t}, x > 0$

$y = e^{2t} - 1$

$y = x^{-2} - 1 = \dfrac{1}{x^2} - 1, x > 0$



17. $x = \sec \theta$

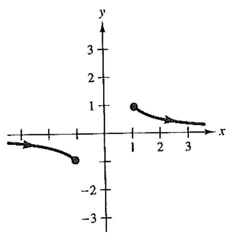
$y = \cos \theta$

$0 \le \theta < \frac{\pi}{2}, \frac{\pi}{2} < \theta \le \pi$

$xy = 1$

$y = \frac{1}{x}$

$|x| \ge 1, |y| \le 1$



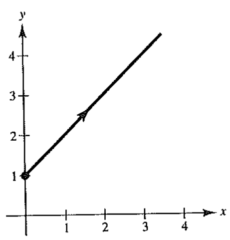
18. $x = \tan^2 \theta$

$y = \sec^2 \theta$

$\sec^2 \theta = \tan^2 \theta + 1$

$y = x + 1$

$x \ge 0$

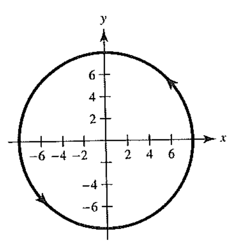


19. $x = 8 \cos \theta$

$y = 8 \sin \theta$

$x^2 + y^2 = 64 \cos^2 \theta + 64 \sin^2 \theta = 64(1) = 64$

Circle



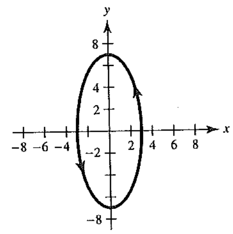
20. $x = 3 \cos \theta$

$y = 7 \sin \theta$

$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{7}\right)^2 = \cos^2 \theta + \sin^2 \theta = 1$

$\frac{x^2}{9} + \frac{y^2}{49} = 1$

Ellipse



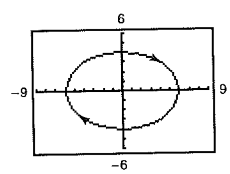
21. $x = 6 \sin 2\theta$

$y = 4 \cos 2\theta$

$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{4}\right)^2 = \sin^2 2\theta + \cos^2 2\theta = 1$

$\frac{x^2}{36} + \frac{y^2}{16} = 1$

Ellipse



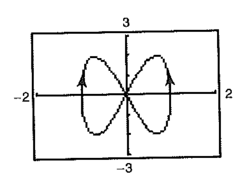
22. $x = \cos \theta$

$y = 2 \sin 2\theta$

$y = 4 \sin \theta \cos \theta$

$1 - x^2 = \sin^2 \theta$

$y = \pm 4x\sqrt{1 - x^2}$



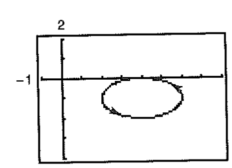
23. $x = 4 + 2 \cos \theta$

$y = -1 + \sin \theta$

$\frac{(x - 4)^2}{4} = \cos^2 \theta$

$\frac{(y + 1)^2}{1} = \sin^2 \theta$

$\frac{(x - 4)^2}{4} + \frac{(y + 1)^2}{1} = 1$



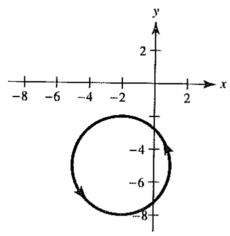
24. $x = -2 + 3\cos\theta$

$y = -5 + 3\sin\theta$

$(x+2)^2 + (y+5)^2 = 9\cos^2\theta + 9\sin^2\theta = 9$

$(x+2)^2 + (y+5)^2 = 9$

Circle



25. $x = -3 + 4\cos\theta$

$y = 2 + 5\sin\theta$

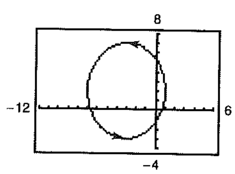
$x + 3 = 4\cos\theta$

$y - 2 = 5\sin\theta$

$$\left(\frac{x+3}{4}\right)^2 + \left(\frac{y-2}{5}\right)^2 = \cos^2\theta + \sin^2\theta = 1$$

$$\frac{(x+3)^2}{16} + \frac{(y-2)^2}{25} = 1$$

Ellipse



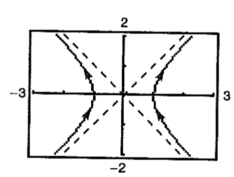
26. $x = \sec\theta$

$y = \tan\theta$

$x^2 = \sec^2\theta$

$y^2 = \tan^2\theta$

$x^2 - y^2 = 1$



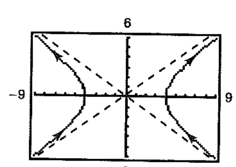
27. $x = 4\sec\theta$

$y = 3\tan\theta$

$\frac{x^2}{16} = \sec^2\theta$

$\frac{y^2}{9} = \tan^2\theta$

$\frac{x^2}{16} - \frac{y^2}{9} = 1$



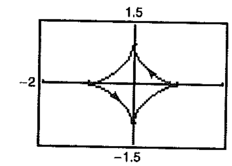
28. $x = \cos^3\theta$

$y = \sin^3\theta$

$x^{2/3} = \cos^2\theta$

$y^{2/3} = \sin^2\theta$

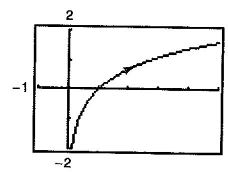
$x^{2/3} + y^{2/3} = 1$



29. $x = t^3$

$y = 3\ln t$

$y = 3\ln\sqrt[3]{x} = \ln x$

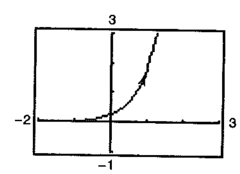


30. $x = \ln 2t$

$y = t^2$

$t = \frac{e^x}{2}$

$y = \frac{e^{2x}}{r} = \frac{1}{4}e^{2x}$



31. $x = e^{-t}$

$y = e^{3t}$

$e^t = \frac{1}{x}$

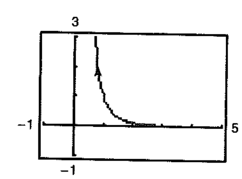
$e^t = \sqrt[3]{y}$

$\sqrt[3]{y} = \frac{1}{x}$

$y = \frac{1}{x^3}$

$x > 0$

$y > 0$



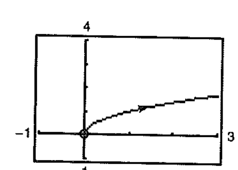
32. $x = e^{2t}$

$y = e^t$

$y^2 = x$

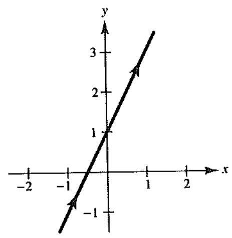
$y > 0$

$y = \sqrt{x},\ x > 0$



33. By eliminating the parameters in (a) – (d), you get $y = 2x + 1$. They differ from each other in orientation and in restricted domains. These curves are all smooth except for (b).

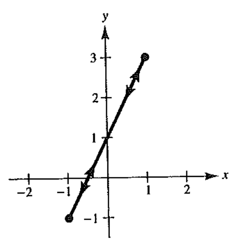
(a) $x = t, y = 2t + 1$



(b) $x = \cos\theta$ $\quad$ $y = 2\cos\theta + 1$

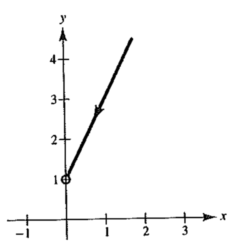
$-1 \le x \le 1$ $\quad$ $-1 \le y \le 3$

$\frac{dx}{d\theta} = \frac{dy}{d\theta} = 0$ when $\theta = 0, \pm\pi, \pm 2\pi, \ldots$.



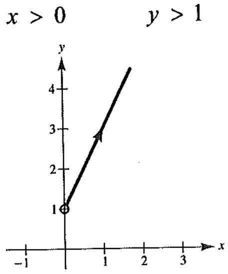
(c) $x = e^{-t}$ $\quad$ $y = 2e^{-t} + 1$

$x > 0$ $\quad$ $y > 1$



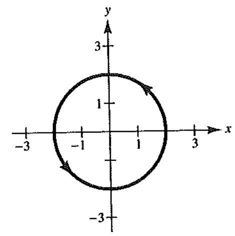
(d) $x = e^{t}$ $\quad$ $y = 2e^{t} + 1$

$x > 0$ $\quad$ $y > 1$



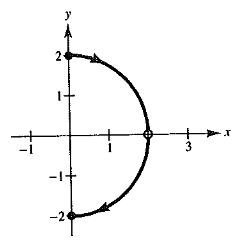
34. By eliminating the parameters in (a) – (d), you get $x^2 + y^2 = 4$. They differ from each other in orientation and in restricted domains. These curves are all smooth.

(a) $x = 2\cos\theta, y = 2\sin\theta$



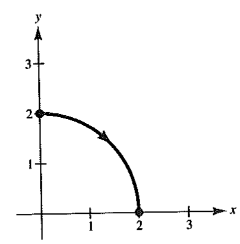
(b) $x = \frac{\sqrt{4t^2 - 1}}{|t|} = \sqrt{4 - \frac{1}{t^2}}$ $\quad$ $y = \frac{1}{t}$

$x \ge 0, x \ne 2$ $\quad$ $y \ne 0$



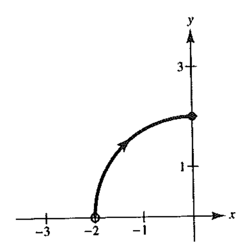
(c) $x = \sqrt{t}$ $\quad$ $y = \sqrt{4 - t}$

$x \ge 0$ $\quad$ $y \ge 0$



(d) $x = -\sqrt{4 - e^{2t}}$ $\quad$ $y = e^{t}$

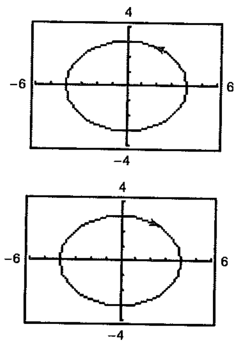
$-2 < x \le 0$ $\quad$ $y > 0$



35. The curves are identical on $0 < \theta < \pi$. They are both smooth. They represent $y = 2(1 - x^2)$ for $-1 \le x \le 1$. The orientation is from right to left in part (a) and in part (b)

36. The orientations are reversed. The graphs are the same. They are both smooth.

37. (a)



(b) The orientation of the second curve is reversed.

(c) The orientation will be reversed.

(d) Answers will vary. For example,

$x = 2 \sec t \qquad x = 2 \sec(-t)$

$y = 5 \sin t \qquad y = 5 \sin(-t)$

have the same graphs, but their orientations are reversed.

38. The set of points (x, y) corresponding to the rectangular equation of a set of parametric equations does not show the orientation of the curve nor any restriction on the domain of the original parametric equations.

39.
$$x = x_1 + t(x_2 - x_1)$$
$$y = y_1 + t(y_2 - y_1)$$
$$\frac{x - x_1}{x_2 - x_1} = t$$
$$y = y_1 + \left(\frac{x - x_1}{x_2 - x_1}\right)(y_2 - y_1)$$
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$
$$y - y_1 = m(x - x_1)$$

40.
$$x = h + r \cos \theta$$
$$y = k + r \sin \theta$$
$$\cos \theta = \frac{x - h}{r}$$
$$\sin \theta = \frac{y - k}{r}$$
$$\cos^2 \theta + \sin^2 \theta = \frac{(x - h)^2}{r^2} + \frac{(y - k)^2}{r^2} = 1$$
$$(x - h)^2 + (y - k)^2 = r^2$$

41.
$$x = h + a \cos \theta$$
$$y = k + b \sin \theta$$
$$\frac{x - h}{a} = \cos \theta$$
$$\frac{y - k}{b} = \sin \theta$$
$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

42.
$$x = h + a \sec \theta$$
$$y = k + b \tan \theta$$
$$\frac{x - h}{a} = \sec \theta$$
$$\frac{y - k}{b} = \tan \theta$$
$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

43. From Exercise 39 you have

$x = 4t$

$y = -7t$

Solution not unique

44. From Exercise 39 you have

$x = 1 + 4t$

$y = 4 - 6t.$

Solution not unique

45. From Exercise 40 you have

$x = 3 + 2 \cos \theta$

$y = 1 + 2 \sin \theta$

Solution not unique

46. From Exercise 40 you have

$x = -6 + 4 \cos \theta$

$y = 2 + 4 \sin \theta$

47. From Exercise 41 you have

$a = 10, c = 8 \Rightarrow b = 6$

$x = 10 \cos \theta$

$y = 6 \sin \theta$

Center: $(0, 0)$

Solution not unique

48. From Exercise 41 you have

$a = 5, c = 3 \Rightarrow b = 4$

$x = 4 + 5 \cos$

$y = 2 + 4 \sin \theta.$

Center: $(4, 2)$

Solution not unique

49. From Exercise 42 you have

$a = 4, c = 5 \Rightarrow b = 3$

$x = 4 \sec \theta$

$y = 3 \tan \theta.$

Center: $(0, 0)$

Solution not unique

50. From Exercise 42 you have

$a = 1, c = 2 \Rightarrow b = \sqrt{3}$

$x = \sqrt{3} \tan \theta$

$y = \sec \theta.$

Center: $(0, 0)$

Solution not unique

The transverse axis is vertical, so, x and y are interchanged.

51. $y = 6x - 5$

Examples:

$x = t, y = 6t - 5$

$x = t + 1, y = 6t + 1$

52. $y = \dfrac{4}{x - 1}$

Examples:

$x = t, y = \dfrac{4}{t - 1}$

$x = t + 1, y = \dfrac{4}{t}$

53. $y = x^3$

<u>Example</u>

$x = t,\quad y = t^3$

$x = \sqrt[3]{t},\quad y = t$

$x = \tan t,\quad y = \tan^3 t$

54. $y = x^2$

<u>Example</u>

$x = t,\quad y = t^2$

$x = t^3,\quad y = t^6$

55. $y = 2x - 5$

At $(3, 1), t = 0$: $x = 3 - t$

$y = 2(3 - t) - 5 = -2t + 1$

or, $x = t + 3$

$y = 2t + 1$

56. $y = 4x + 1$

At $(-2, -7), t = -1$: $x = -1 + t$

$y = 4(-1 + t) + 1 = 4t - 3$

57. $y = x^2$

$t = 4$ at $(4, 16)$: $x = t$

$y = t^2$

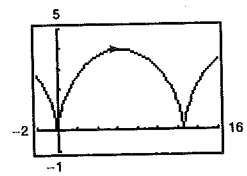
58. $y = 4 - x^2$

$t = 1$ at $(1, 3)$: $x = t$

$y = 4 - t^2$

59. $x = 2(\theta - \sin \theta)$

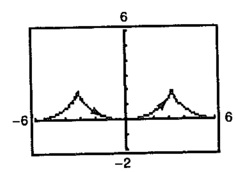
$y = 2(1 - \cos \theta)$



Not smooth at $\theta = 2n\pi$

60. $x = \theta + \sin \theta$

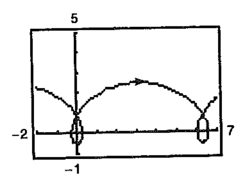
$y = 1 - \cos \theta$



Not smooth at $x = (2n - 1)\pi$

61. $x = \theta - \frac{3}{2} \sin \theta$

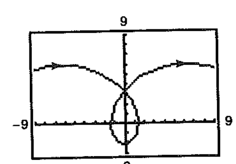
$y = 1 - \frac{3}{2} \cos \theta$



Smooth Everywhere

62. $x = 2\theta - 4 \sin \theta$

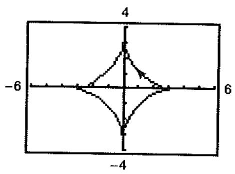
$y = 2 - 4 \cos \theta$



Smooth everywhere

63. $x = 3\cos^3\theta$

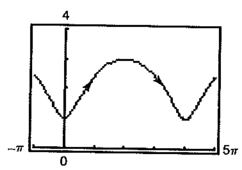
$y = 3\sin^3\theta$



Not smooth at $(x, y) = (\pm 3, 0)$ and $(0, \pm 3)$, or $\theta = \frac{1}{2}n\pi$.

64. $x = 2\theta - \sin\theta$

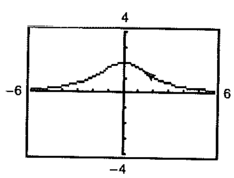
$y = 2 - \cos\theta$



Smooth everywhere

65. $x = 2\cot\theta$

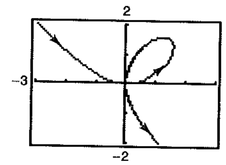
$y = 2\sin^2\theta$



Smooth everywhere

66. $x = \dfrac{3t}{1 + t^3}$

$y = \dfrac{3t^2}{1 + t^3}$



Smooth everywhere

67. Each point (x, y) in the plane is determined by the plane curve $x = f(t)$, $y = g(t)$. For each t, plot (x, y). As t increases, the curve is traced out in a specific direction called the orientation of the curve.

68. (a) Matches (iv) because $(0, 2)$ is on the graph.

(b) Matches (v) because $(1, 0)$ is on the graph.

(c) Matches (ii) because $-1 \le x \le 0$ and $1 \le y \le 3$.

(d) Matches (iii) because $(4, 0)$ is on the graph.

(e) Matches (vi) because undefined at $\theta = 0$.

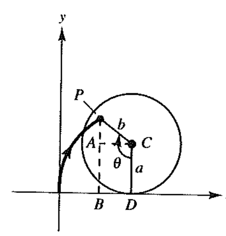
(f) Matches (i) because $x = (y - 2)^2 - 1$ for all y.

69. When the circle has rolled θ radians, you know that the center is at $(a\theta, a)$.

$$\sin\theta = \sin(180° - \theta) = \frac{|AC|}{b} = \frac{|BD|}{b} \text{ or } |BD| = b\sin\theta$$

$$\cos\theta = -\cos(180° - \theta) = \frac{|AP|}{-b} \text{ or } |AP| = -b\cos\theta$$

So, $x = a\theta - b\sin\theta$ and $y = a - b\cos\theta$.



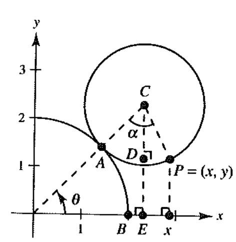
70. Let the circle of radius 1 be centered at C. A is the point of tangency on the line OC. $OA = 2$, $AC = 1$, $OC = 3$. $P = (x, y)$ is the point on the curve being traced out as the angle θ changes $\widehat{AB} = \widehat{AP}$, $\widehat{AB} = 2\theta$ and $\widehat{AP} = \alpha \Rightarrow \alpha = 2\theta$. Form the right triangle ΔCDP. The angle $OCE = (\pi/2) - \theta$ and

$$\angle DCP = \alpha - \left(\frac{\pi}{2} - \theta\right) = \alpha + \theta - \left(\frac{\pi}{2}\right) = 3\theta - \left(\frac{\pi}{2}\right).$$

$$x = OE + Ex = 3\sin\left(\frac{\pi}{2} - \theta\right) + \sin\left(3\theta - \frac{\pi}{2}\right) = 3\cos\theta - \cos 3\theta$$

$$y = EC - CD = 3\sin\theta - \cos\left(3\theta - \frac{\pi}{2}\right) = 3\sin\theta - \sin 3\theta$$

So, $x = 3\cos\theta - \cos 3\theta$, $y = 3\sin\theta - \sin 3\theta$.



71. False

$x = t^2 \Rightarrow x \geq 0$

$y = t^2 \Rightarrow y \geq 0$

The graph of the parametric equations is only a portion of the line $y = x$ when $x \geq 0$.

72. False. Let $x = t^2$ and $y = t$. Then $x = y^2$ and y is not a function of x.

73. True. $y = \cos x$

74. $x = 8\cos t,\ y = 8\sin t$

(a) $\left(\frac{x}{8}\right)^2 + \left(\frac{y}{8}\right)^2 = \cos^2 t + \sin^2 t = 1$

$x^2 + y^2 = 64$ Circle radius 8,
Center: $(0, 0)$ Oriented counterclockwise

(b) Circle of radius 8, but Center: $(3, 6)$

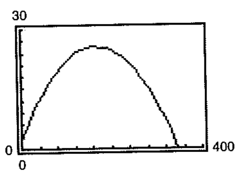
(c) The orientation is reversed.

75. (a) $100 \text{ mi/hr} = \frac{(100)(5280)}{3600} = \frac{440}{3}$ ft/sec

$$x = (v_0 \cos\theta)t = \left(\frac{440}{3}\cos\theta\right)t$$

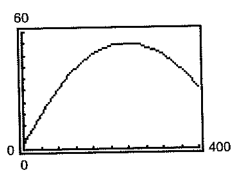
$$y = h + (v_0 \sin\theta)t - 16t^2 = 3 + \left(\frac{440}{3}\sin\theta\right)t - 16t^2$$

(b)



It is not a home run—when $x = 400$, $y < 10$.

(c)



Yes, it's a home run when $x = 400$, $y > 10$.

(d) You need to find the angle θ (and time t) such that

$$x = \left(\frac{440}{3}\cos\theta\right)t = 400$$

$$y = 3 + \left(\frac{440}{3}\sin\theta\right)t - 16t^2 = 10.$$

From the first equation $t = 1200/440\cos\theta$. Substituting into the second equation,

$$10 = 3 + \left(\frac{440}{3}\sin\theta\right)\left(\frac{1200}{440\cos\theta}\right) - 16\left(\frac{1200}{440\cos\theta}\right)^2$$

$$7 = 400\tan\theta - 16\left(\frac{120}{44}\right)^2 \sec^2\theta = 400\tan\theta - 16\left(\frac{120}{44}\right)^2(\tan^2\theta + 1).$$

You now solve the quadratic for $\tan\theta$:

$$16\left(\frac{120}{44}\right)^2 \tan^2\theta - 400\tan\theta + 7 + 16\left(\frac{120}{44}\right)^2 = 0.$$

$\tan\theta \approx 0.35185 \Rightarrow \theta \approx 19.4°$

76. (a) $x = (v_0 \cos\theta)t$

$y = h + (v_0 \sin\theta)t - 16t^2$

$$t = \frac{x}{v_0 \cos\theta} \Rightarrow y = h + (v_0 \sin\theta)\frac{x}{v_0\cos\theta} - 16\left(\frac{x}{v_0\cos\theta}\right)^2$$

$$y = h + (\tan\theta)x - \frac{16\sec^2\theta}{v_0^2}x^2$$

(b) $y = 5 + x - 0.005x^2 = h + (\tan\theta)x - \dfrac{16\sec^2\theta}{v_0^2}x^2$

$h = 5$, $\tan\theta = 1 \Rightarrow \theta = \dfrac{\pi}{4}$, and

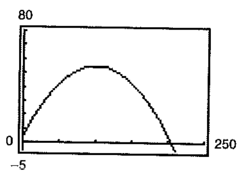
$$0.005 = \frac{16\sec^2(\pi/4)}{v_0^2} = \frac{16}{v_0^2}(2)$$

$$v_0^2 = \frac{32}{0.005} = 6400 \Rightarrow v_0 = 80.$$

So, $x = (80\cos(45°))t$

$y = 5 + (80\sin(45°))t - 16t^2.$

(c)



(d) Maximum height: $y = 55$ (at $x = 100$)

Range: 204.88

Section 10.3 Parametric Equations and Calculus

1. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-6}{2t} = -\dfrac{3}{t}$

2. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-1}{(1/3)t^{-2/3}} = -3t^{2/3}$

3. $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{-2\cos\theta\sin\theta}{2\sin\theta\cos\theta} = -1$

$\left[\text{Note: } x + y = 1 \Rightarrow y = 1 - x \text{ and } \dfrac{dy}{d\theta} = -1\right]$

4. $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{(-1/2)e^{-\theta/2}}{2e^{\theta}} = -\dfrac{1}{4}e^{-3\theta/2} = \dfrac{-1}{4e^{3\theta/2}}$

5. $x = 4t$, $y = 3t - 2$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3}{4}$

$\dfrac{d^2y}{dx^2} = 0$

At $t = 3$, slope is $\dfrac{3}{4}$. (Line)

Neither concave upward nor downward.

6. $x = \sqrt{t}$, $y = 3t - 1$

$\dfrac{dy}{dx} = \dfrac{3}{1/(2\sqrt{t})} = 6\sqrt{t} = 6$ when $t = 1$.

$\dfrac{d^2y}{dx^2} = \dfrac{3/\sqrt{t}}{1/(2\sqrt{t})} = 6$ concave upward

7. $x = t + 1$, $y = t^2 + 3t$

$\dfrac{dy}{dx} = \dfrac{2t + 3}{1} = 1$ when $t = -1$.

$\dfrac{d^2y}{dx^2} = 2$ concave upward

8. $x = t^2 + 5t + 4,\ y = 4t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4}{2t + 5}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{4}{2t+5}\right]}{dx/dt} = \frac{\frac{-8}{(2t+5)^2}}{2t+5} = \frac{-8}{(2t+5)^3}$$

At $t = 0, \frac{dy}{dx} = \frac{4}{5}$.

At $t = 0, \frac{d^2y}{dx^2} = -\frac{8}{125}$

concave downward

9. $x = 4\cos\theta,\ y = 4\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{4\cos\theta}{-4\sin\theta} = \frac{-\cos\theta}{\sin\theta} = -\cot\theta$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}[-\cot\theta]}{dx/d\theta} = \frac{\csc^2\theta}{-4\sin\theta} = \frac{-1}{4\sin^3\theta} = -\frac{1}{4}\csc^3\theta$$

At $\theta = \frac{\pi}{4}, \frac{dy}{dx} = -1$.

$$\frac{d^2y}{dx^2} = \frac{-1}{4\left(\sqrt{2}/2\right)^3} = \frac{-\sqrt{2}}{2}$$

concave downward

10. $x = \cos\theta,\ y = 3\sin\theta$

$\frac{dy}{dx} = \frac{3\cos\theta}{-\sin\theta} = -3\cot\theta \cdot \frac{dy}{dx}$ is undefined when $\theta = 0$.

$\frac{d^2y}{dx^2} = \frac{3\csc^2\theta}{-\sin\theta} = \frac{-3}{\sin^3\theta} \cdot \frac{d^2y}{dx^2}$ is undefined when $\theta = 0$.

11. $x = 2 + \sec\theta,\ y = 1 + 2\tan\theta$

$$\frac{dy}{dx} = \frac{2\sec^2\theta}{\sec\theta\tan\theta} = \frac{2\sec\theta}{\tan\theta} = 2\csc\theta = 4 \text{ when } \theta = \frac{\pi}{6}.$$

$$\frac{d^2y}{dx^2} = \frac{d\left[\frac{dy}{dx}\right]\Big/\frac{dx}{d\theta}}{} = \frac{-2\csc\theta\cot\theta}{\sec\theta\tan\theta}$$

$$= -2\cot^3\theta = -6\sqrt{3} \text{ when } \theta = \frac{\pi}{6}.$$

concave downward

12. $x = \sqrt{t},\ y = \sqrt{t-1}$

$$\frac{dy}{dx} = \frac{1/\left(2\sqrt{t-1}\right)}{1/\left(2\sqrt{t}\right)} = \frac{\sqrt{t}}{\sqrt{t-1}} = \sqrt{2} \text{ when } t = 2.$$

$$\frac{d^2y}{dx^2} = \frac{\left[\sqrt{t-1}/\left(2\sqrt{t}\right) - \sqrt{t}\left(1/2\sqrt{t-1}\right)\right]/(t-1)}{1/\left(2\sqrt{t}\right)}$$

$$= \frac{-1}{(t-1)^{3/2}} = -1 \text{ when } t = 2.$$

concave downward

13. $x = \cos^3\theta,\ y = \sin^3\theta$

$$\frac{dy}{dx} = \frac{3\sin^2\theta\cos\theta}{-3\cos^2\theta\sin\theta} = -\tan\theta = -1 \text{ when } \theta = \frac{\pi}{4}.$$

$$\frac{d^2y}{dx^2} = \frac{-\sec^2\theta}{-3\cos^2\theta\sin\theta} = \frac{1}{3\cos^4\theta\sin\theta}$$

$$= \frac{\sec^4\theta\csc\theta}{3} = \frac{4\sqrt{2}}{3} \text{ when } \theta = \frac{\pi}{4}.$$

concave upward

14. $x = \theta - \sin\theta,\ y = 1 - \cos\theta$

$$\frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta} = 0 \text{ when } \theta = \pi.$$

$$\frac{d^2y}{dx^2} = \frac{\left[\frac{(1-\cos\theta)\cos\theta - \sin^2\theta}{(1-\cos\theta)^2}\right]}{(1-\cos\theta)}$$

$$= \frac{-1}{(1-\cos\theta)^2} = -\frac{1}{4} \text{ when } \theta = \pi.$$

concave downward

15. $x = 2\cot\theta,\ y = 2\sin^2\theta$

$$\frac{dy}{dx} = \frac{4\sin\theta\cos\theta}{-2\csc^2\theta} = -2\sin^3\theta\cos\theta$$

At $\left(-\frac{2}{\sqrt{3}}, \frac{3}{2}\right)$, $\theta = \frac{2\pi}{3}$, and $\frac{dy}{dx} = \frac{3\sqrt{3}}{8}$.

Tangent line: $y - \frac{3}{2} = \frac{3\sqrt{3}}{8}\left(x + \frac{2}{\sqrt{3}}\right)$

$$3\sqrt{3}x - 8y + 18 = 0$$

At $(0, 2)$, $\theta = \frac{\pi}{2}$, and $\frac{dy}{dx} = 0$.

Tangent line: $y - 2 = 0$

At $\left(2\sqrt{3}, \frac{1}{2}\right)$, $\theta = \frac{\pi}{6}$, and $\frac{dy}{dx} = -\frac{\sqrt{3}}{8}$.

Tangent line: $y - \frac{1}{2} = -\frac{\sqrt{3}}{8}\left(x - 2\sqrt{3}\right)$

$$\sqrt{3}x + 8y - 10 = 0$$

16. $x = 2 - 3\cos\theta,\ y = 3 + 2\sin\theta$

$$\frac{dy}{dx} = \frac{2\cos\theta}{3\sin\theta} = \frac{2}{3}\cot\theta$$

At $(-1, 3)$, $\theta = 0$, and $\dfrac{dy}{dx}$ is undefined.

Tangent line: $x = -1$

At $(2, 5)$, $\theta = \dfrac{\pi}{2}$, and $\dfrac{dy}{dx} = 0$.

Tangent line: $y = 5$

At $\left(\dfrac{4 + 3\sqrt{3}}{2}, 2\right)$, $\theta = \dfrac{7\pi}{6}$, and $\dfrac{dy}{dx} = \dfrac{2\sqrt{3}}{3}$.

Tangent line:

$$y - 2 = \frac{2\sqrt{3}}{3}\left(x - \frac{4 + 3\sqrt{3}}{2}\right)$$

$$2\sqrt{3}x - 3y - 4\sqrt{3} - 3 = 0$$

17. $x = t^2 - 4$

$y = t^2 - 2t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t - 2}{2t}$$

At $(0, 0)$, $t = 2$, $\dfrac{dy}{dx} = \dfrac{1}{2}$.

Tangent line: $y = \dfrac{1}{2}x$

$2y - x = 0$

At $(-3, -1)$, $t = 1$, $\dfrac{dy}{dx} = 0$.

Tangent line: $y = -1$

$y + 1 = 0$

At $(-3, 3)$, $t = -1$, $\dfrac{dy}{dx} = 2$.

Tangent line: $y - 3 = 2(x + 3)$

$2x - y + 9 = 0$

18. $x = t^4 + 2$

$y = t^3 + t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 + 1}{4t^3}$$

At $(2, 0)$, $t = 0$, $\dfrac{dy}{dx}$ undefined.

Tangent line: $x = 2$ (vertical tangent)

At $(3, -2)$, $t = -1$, $\dfrac{dy}{dx} = -1$.

Tangent line: $y + 2 = -(x - 3)$

$y = -x + 1$

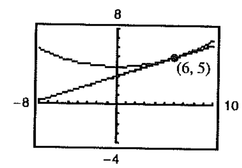
At $(18, 10)$, $t = 2$, $\dfrac{dy}{dx} = \dfrac{13}{32}$.

Tangent line: $y - 10 = \dfrac{13}{32}(x - 18)$

$$y = \frac{13}{32}x + \frac{43}{16}$$

19. $x = 6t,\ y = t^2 + 4,\ t = 1$

(a), (d)



(b) At $t = 1$, $(x, y) = (6, 5)$

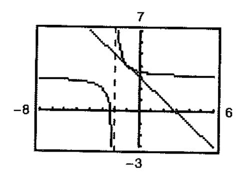
$$\frac{dx}{dt} = 6,\ \frac{dy}{dt} = 2,\ \frac{dy}{dx} = \frac{1}{3}.$$

(c) $y - 5 = \dfrac{1}{3}(x - 6)$

$$y = \frac{1}{3}x + 3$$

20. $x = t - 2,\ y = \dfrac{1}{t} + 3,\ t = 1$

(a), (d)



(b) At $t = 1$, $(x, y) = (-1, 4)$

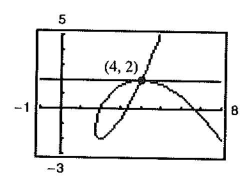
$$\frac{dx}{dt} = 1,\ \frac{dy}{dt} = -1,\ \frac{dy}{dx} = -1.$$

(c) $y - 4 = -(x + 1)$

$$y = -x + 3$$

21. $x = t^2 - t + 2,\ y = t^3 - 3t,\ t = -1$

(a), (d)



(b) At $t = -1$, $(x, y) = (4, 2)$, and

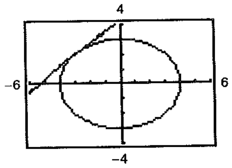
$$\frac{dx}{dt} = -3,\ \frac{dy}{dt} = 0,\ \frac{dy}{dx} = 0.$$

(c) $\dfrac{dy}{dx} = 0$. At $(4, 2)$, $y - 2 = 0(x - 4)$

$$y = 2.$$

22. $x = 4\cos\theta,\ y = 3\sin\theta,\ \theta = \frac{3\pi}{4}$

(a), (d)



(b) At $\theta = \frac{3\pi}{4}$, $(x, y) = \left(\frac{-4}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$, and

$$\frac{dx}{dt} = -2\sqrt{2},\ \frac{dy}{dt} = -\frac{3\sqrt{2}}{2},\ \frac{dy}{dx} = \frac{3}{4}.$$

(c) $\frac{dy}{dx} = \frac{3}{4}$. At $\left(\frac{-4}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$,

$$y - \frac{3}{\sqrt{2}} = \frac{3}{4}\left(x + \frac{4}{\sqrt{2}}\right)$$

$$y = \frac{3}{4}x + 3\sqrt{2}.$$

23. $x = 2\sin 2t,\ y = 3\sin t$ crosses itself at the origin, $(x, y) = (0, 0)$.

At this point, $t = 0$ or $t = \pi$.

$$\frac{dy}{dx} = \frac{3\cos t}{4\cos 2t}$$

At $t = 0$: $\frac{dy}{dx} = \frac{3}{4}$ and $y = \frac{3}{4}x$. Tangent Line

At $t = \pi$, $\frac{dy}{dx} = -\frac{3}{4}$ and $y = -\frac{3}{4}x$. Tangent Line

24. $x = 2 - \pi\cos t,\ y = 2t - \pi\sin t$ crosses itself at a point on the x-axis: $(2, 0)$. The corresponding t-values are $t = \pm\pi/2$.

$$\frac{dy}{dt} = 2 - \pi\cos t,\ \frac{dx}{dt} = \pi\sin t,\ \frac{dy}{dx} = \frac{2 - \pi\cos t}{\pi\sin t}$$

At $t = \frac{\pi}{2}$: $\frac{dy}{dx} = \frac{2}{\pi}$.

Tangent line: $y - 0 = \frac{2}{\pi}(x - 2)$

$$y = \frac{2}{\pi}x - \frac{4}{\pi}$$

At $t = -\frac{\pi}{2}$: $\frac{dy}{dx} = -\frac{2}{\pi}$.

Tangent line: $y - 0 = -\frac{2}{\pi}(x - 2)$

$$y = -\frac{2}{\pi}x + \frac{4}{\pi}$$

25. $x = t^2 - t,\ y = t^3 - 3t - 1$ crosses itself at the point $(x, y) = (2, 1)$.

At this point, $t = -1$ or $t = 2$.

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t - 1}$$

At $t = -1$, $\frac{dy}{dx} = 0$ and $y = 1$. Tangent Line

At $t = 2$, $\frac{dy}{dt} = \frac{9}{3} = 3$ and $y - 1 = 3(x - 2)$ or $y = 3x - 5$.

Tangent Line

26. $x = t^3 - 6t,\ y = t^2$ crosses itself at $(0, 6)$. The corresponding t-values are $t = \pm\sqrt{6}$.

$$\frac{dy}{dx} = \frac{2t}{3t^2 - 6}$$

At $t = \sqrt{6}$, $\frac{dy}{dx} = \frac{2\sqrt{6}}{12} = \frac{\sqrt{6}}{6}$.

Tangent line: $y - 6 = \frac{\sqrt{6}}{6}(x - 0)$

$$y = \frac{\sqrt{6}}{6}x + 6$$

At $t = -\sqrt{6}$, $\frac{dy}{dx} = -\frac{2\sqrt{6}}{12} = -\frac{\sqrt{6}}{6}$.

Tangent line: $y = -\frac{\sqrt{6}}{6}x + 6$

27. $x = \cos\theta + \theta\sin\theta,\ y = \sin\theta - \theta\cos\theta$

Horizontal tangents: $\frac{dy}{d\theta} = \theta\sin\theta = 0$ when

$\theta = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

Points: $(-1, [2n - 1]\pi), (1, 2n\pi)$ where n is an integer.

Points shown: $(1, 0), (-1, \pi), (1, -2\pi)$

Vertical tangents: $\frac{dx}{d\theta} = \theta\cos\theta = 0$ when

$\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

Note: $\theta = 0$ corresponds to the cusp at $(x, y) = (1, 0)$.

$$\frac{dy}{dx} = \frac{\theta\sin\theta}{\theta\cos\theta} = \tan\theta = 0 \text{ at } \theta = 0$$

Points: $\left(\frac{(-1)^{n+1}(2n - 1)\pi}{2}, (-1)^{n+1}\right)$

Points shown: $\left(\frac{\pi}{2}, 1\right), \left(-\frac{3\pi}{2}, -1\right), \left(\frac{5\pi}{2}, 1\right)$

28. $x = 2\theta,\ y = 2(1 - \cos\theta)$

Horizontal tangents: $\dfrac{dy}{d\theta} = 2\sin\theta = 0$ when

$\theta = 0, \pm\pi, \pm 2\pi, \ldots$

Points: $(4n\pi, 0), (2[2n - 1]\pi, 4)$ where n is an integer

Points shown: $(0, 0), (2\pi, 4), (4\pi, 0)$

Vertical tangents: $\dfrac{dx}{d\theta} = 2 \neq 0$; none

29. $x = 4 - t,\ y = t^2$

Horizontal tangents: $\dfrac{dy}{dt} = 2t = 0$ when $t = 0$.

Point: $(4, 0)$

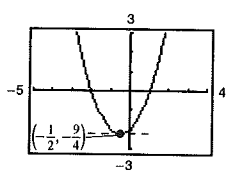
Vertical tangents: $\dfrac{dx}{dt} = -1 \neq 0$ None

30. $x = t + 1,\ y = t^2 + 3t$

Horizontal tangents: $\dfrac{dy}{dt} = 2t + 3 = 0$ when $t = -\dfrac{3}{2}$

Point: $\left(-\dfrac{1}{2}, -\dfrac{9}{4}\right)$

Vertical tangents: $\dfrac{dx}{dt} = 1 \neq 0$; none



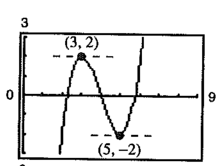
31. $x = t + 4,\ y = t^3 - 3t$

Horizontal tangents:

$\dfrac{dy}{dt} = 3t^2 - 3 = 3(t - 1)(t + 1) = 0 \Rightarrow t = \pm 1$

Points: $(5, -2), (3, 2)$

Vertical tangents: $\dfrac{dx}{dt} = 1 \neq 0$ None



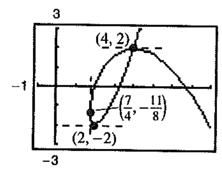
32. $x = t^2 - t + 2,\ y = t^3 - 3t$

Horizontal tangents: $\dfrac{dy}{dt} = 3t^2 - 3 = 0$ when $t = \pm 1$.

Points: $(2, -2), (4, 2)$

Vertical tangents: $\dfrac{dx}{dt} = 2t - 1 = 0$ when $t = \dfrac{1}{2}$.

Point: $\left(\dfrac{7}{4}, -\dfrac{11}{8}\right)$



33. $x = 3\cos\theta,\ y = 3\sin\theta$

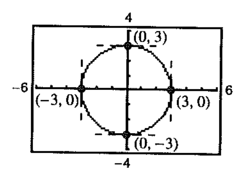
Horizontal tangents: $\dfrac{dy}{d\theta} = 3\cos\theta = 0$ when

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Points: $(0, 3), (0, -3)$

Vertical tangents: $\dfrac{dx}{d\theta} = -3\sin\theta = 0$ when $\theta = 0, \pi$.

Points: $(3, 0), (-3, 0)$



34. $x = \cos\theta,\ y = 2\sin 2\theta$

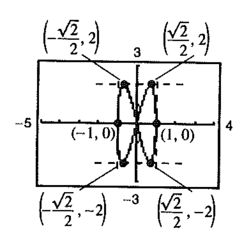
Horizontal tangents: $\dfrac{dy}{d\theta} = 4\cos 2\theta = 0$ when

$\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$.

Points: $\left(\dfrac{\sqrt{2}}{2}, 2\right), \left(-\dfrac{\sqrt{2}}{2}, -2\right), \left(-\dfrac{\sqrt{2}}{2}, 2\right), \left(\dfrac{\sqrt{2}}{2}, -2\right)$

Vertical tangents: $\dfrac{dx}{d\theta} = -\sin\theta = 0$ when $\theta = 0, \pi$.

Points: $(1, 0), (-1, 0)$



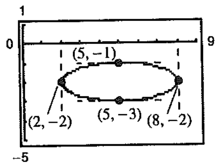
35. $x = 5 + 3\cos\theta,\ y = -2 + \sin\theta$

Horizontal tangents: $\dfrac{dy}{dt} = \cos\theta = 0 \Rightarrow \theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

Points: $(5, -1), (5, -3)$

Vertical tangents: $\dfrac{dx}{dt} = -3\sin\theta = 0 \Rightarrow \theta = 0, \pi$

Points: $(8, -2), (2, -2)$



36. $x = 4\cos^2\theta,\ y = 2\sin\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = 2\cos\theta = 0$ when

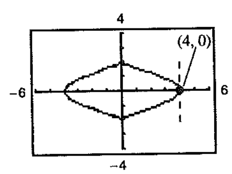
$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}.$

Because $dx/d\theta = 0$ at $\pi/2$ and $3\pi/2$, exclude them.

Vertical tangents: $\dfrac{dx}{d\theta} = -8\cos\theta\sin\theta = 0$ when

$\theta = 0, \pi.$

Point: $(4, 0)$



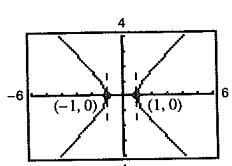
37. $x = \sec\theta,\ y = \tan\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = \sec^2\theta \neq 0$; None

Vertical tangents: $\dfrac{dx}{d\theta} = \sec\theta\tan\theta = 0$ when

$x = 0, \pi.$

Points: $(1, 0), (-1, 0)$



38. $x = \cos^2\theta,\ y = \cos\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = -\sin\theta = 0$ when $x = 0, \pi.$

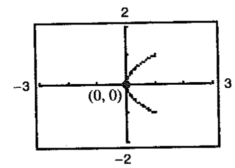
Since $dx/d\theta = 0$ at these values, exclude them.

Vertical tangents: $\dfrac{dx}{d\theta} = -2\cos\theta\sin\theta = 0$ when

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}.$

(Exclude $0, \pi$.)

Point: $(0, 0)$



39. $x = 3t^2,\ y = t^3 - t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 1}{6t} = \frac{t}{2} - \frac{1}{6t}$$

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left[\dfrac{t}{2} - \dfrac{1}{6t}\right]}{dx/dt} = \frac{\dfrac{1}{2} + \dfrac{1}{6t^2}}{6t} = \frac{6t^2 + 2}{36t^3}$$

Concave upward for $t > 0$

Concave downward for $t < 0$

40. $x = 2 + t^2,\ y = t^2 + t^3$

$$\frac{dy}{dx} = \frac{2t + 3t^2}{2t} = 1 + \frac{3}{2}t$$

$$\frac{d^2y}{dx^2} = \frac{3/2}{2t} = \frac{3}{4t}$$

Concave upward for $t > 0$

Concave downward for $t < 0$

41. $x = 2t + \ln t,\ y = 2t - \ln t,\ t > 0$

$$\frac{dy}{dx} = \frac{2 - (1/t)}{2 + (1/t)} = \frac{2t - 1}{2t + 1}$$

$$\frac{d^2y}{dx^2} = \left[\frac{(2t + 1)2 - (2t - 1)2}{(2t + 1)^2}\right] \Big/ \left(2 + \frac{1}{t}\right)$$

$$= \frac{4}{(2t + 1)^2} \cdot \frac{t}{2t + 1} = \frac{4t}{(2t + 1)^3}$$

Because $t > 0$, $\dfrac{d^2y}{dx^2} > 0$

Concave upward for $t > 0$

42. $x = t^2, y = \ln t, t > 0$

$$\frac{dy}{dx} = \frac{1/t}{2t} = \frac{1}{2t^2}$$

$$\frac{d^2y}{dx^2} = -\frac{1/t^3}{2t} = -\frac{1}{2t^4}$$

Because $t > 0, \frac{d^2y}{dx^2} < 0$

Concave downward for $t > 0$

43. $x = \sin t, y = \cos t, 0 < t < \pi$

$$\frac{dy}{dx} = -\frac{\sin t}{\cos t} = -\tan t$$

$$\frac{d^2y}{dx^2} = -\frac{\sec^2 t}{\cos t} = -\frac{1}{\cos^3 t}$$

Concave upward on $\pi/2 < t < \pi$

Concave downward on $0 < t < \pi/2$

44. $x = 4\cos t, y = 2\sin t, 0 < t < 2\pi$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-4\sin t} = -\frac{1}{2}\cot t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[-\frac{1}{2}\cot t\right]}{dx/dt} = \frac{\frac{1}{2}\csc^2 t}{-4\sin t} = \frac{-1}{8\sin^3 t}$$

Concave upward on $\pi < t < 2\pi$

Concave downward on $0 < t < \pi$

45. $x = 3t - t^2, y = 2t^{3/2}, 1 \le t \le 3$

$$\frac{dx}{dt} = 3 - 2t, \frac{dy}{dt} = 3t^{1/2}$$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_1^3 \sqrt{(3-2t)^2 + 9t}\,dt = \int_1^3 \sqrt{4t^2 - 3t + 9}\,dt$$

46. $x = \ln t, y = 4t - 3, 1 \le t \le 5$

$$\frac{dx}{dt} = \frac{1}{t}, \frac{dy}{dt} = 4$$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_1^5 \sqrt{\frac{1}{t^2} + 16}\,dt$$

47. $x = e^t + 2, y = 2t + 1, -2 \le t \le 2$

$$\frac{dx}{dt} = e^t, \frac{dy}{dt} = 2$$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_{-2}^2 \sqrt{e^{2t} + 4}\,dt$$

48. $x = t + \sin t, y = t - \cos t, 0 \le t \le \pi$

$$\frac{dx}{dt} = 1 + \cos t, \frac{dy}{dt} = 1 + \sin t$$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^\pi \sqrt{(1+\cos t)^2 + (1+\sin t)^2}\,dt$$

$$= \int_0^\pi \sqrt{3 + 2\cos t + 2\sin t}\,dt$$

49. $x = 3t + 5, y = 7 - 2t, -1 \le t \le 3$

$$\frac{dx}{dt} = 3, \frac{dy}{dt} = -2$$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_{-1}^3 \sqrt{9+4}\,dt$$

$$\left[\sqrt{13}\,t\right]_{-1}^3 = 4\sqrt{13} \approx 14.422$$

50. $x = t^2, y = 2t, 0 \le t \le 2$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 2,$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 4 = 4(t^2 + 1)$$

$$s = 2\int_0^2 \sqrt{t^2+1}\,dt$$

$$= \left[t\sqrt{t^2+1} + \ln\left|t + \sqrt{t^2+1}\right|\right]_0^2$$

$$= 2\sqrt{5} + \ln\left(2 + \sqrt{5}\right) \approx 5.916$$

51. $x = 6t^2, y = 2t^3, 1 \le t \le 4$

$$\frac{dx}{dt} = 12t, \frac{dy}{dt} = 6t^2$$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_1^4 \sqrt{144t^2 + 36t^4}\,dt$$

$$= \int_1^4 6t\sqrt{4 + t^2}\,dt$$

$$= \left[2\left(4 + t^2\right)^{3/2}\right]_1^4$$

$$= 2\left(20^{3/2} - 5^{3/2}\right)$$

$$= 70\sqrt{5} \approx 156.525$$

52. $x = t^2 + 1, y = 4t^3 + 3, -1 \le t \le 0$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 12t^2, \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 144t^4$$

$$s = \int_{-1}^{0} \sqrt{4t^2 + 144t^4}\, dt = \int_{-1}^{0} -2t\sqrt{1 + 36t^2}\, dt$$

$$= \left[\frac{-(1 + 36t^2)^{3/2}}{54}\right]_{-1}^{0}$$

$$= \frac{-1}{54}\left(1 - 37^{3/2}\right) \approx 4.149$$

53. $x = e^{-t}\cos t, y = e^{-t}\sin t, 0 \le t \le \dfrac{\pi}{2}$

$$\frac{dx}{dt} = -e^{-t}(\sin t + \cos t), \frac{dy}{dt} = e^{-t}(\cos t - \sin t)$$

$$s = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{2e^{-2t}}\, dt = -\sqrt{2}\int_0^{\pi/2} e^{-t}(-1)\, dt$$

$$= \left[-\sqrt{2}e^{-t}\right]_0^{\pi/2}$$

$$= \sqrt{2}\left(1 - e^{-\pi/2}\right) \approx 1.12$$

54. $x = \arcsin t, y = \ln\sqrt{1 - t^2}, 0 \le t \le \dfrac{1}{2}$

$$\frac{dx}{dt} = \frac{1}{\sqrt{1 - t^2}}, \frac{dy}{dt} = \frac{1}{2}\left(\frac{-2t}{1 - t^2}\right) = \frac{t}{1 - t^2}$$

$$s = \int_0^{1/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{1/2} \sqrt{\frac{1}{\left(1 - t^2\right)^2}}\, dt = \int_0^{1/2} \frac{1}{1 - t^2}\, dt$$

$$= \left[-\frac{1}{2}\ln\left|\frac{t - 1}{t + 1}\right|\right]_0^{1/2}$$

$$= -\frac{1}{2}\ln\left(\frac{1}{3}\right) = \frac{1}{2}\ln(3) \approx 0.549$$

55. $x = \sqrt{t}, y = 3t - 1, \dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t}}, \dfrac{dy}{dt} = 3$

$$s = \int_0^1 \sqrt{\frac{1}{4t} + 9}\, dt = \frac{1}{2}\int_0^1 \frac{\sqrt{1 + 36t}}{\sqrt{t}}\, dt$$

$$= \frac{1}{6}\int_0^6 \sqrt{1 + u^2}\, du$$

$$= \frac{1}{12}\left[\ln\left(\sqrt{1 + u^2} + u\right) + u\sqrt{1 + u^2}\right]_0^6$$

$$= \frac{1}{12}\left[\ln\left(\sqrt{37} + 6\right) + 6\sqrt{37}\right] \approx 3.249$$

$$u = 6\sqrt{t}, du = \frac{3}{\sqrt{t}}\, dt$$

56. $x = t, y = \dfrac{t^5}{10} + \dfrac{1}{6t^3}, \dfrac{dx}{dt} = 1, \dfrac{dy}{dt} = \dfrac{t^4}{2} - \dfrac{1}{2t^4}$

$$S = \int_1^2 \sqrt{1 + \left(\frac{t^4}{2} - \frac{1}{2t^4}\right)^2}\, dt$$

$$= \int_1^2 \sqrt{\left(\frac{t^4}{2} + \frac{1}{2t^4}\right)^2}\, dt$$

$$= \int_1^2 \left(\frac{t^4}{2} + \frac{1}{2t^4}\right) dt = \left[\frac{t^5}{10} - \frac{1}{6t^3}\right]_1^2 = \frac{779}{240}$$

57. $x = a\cos^3\theta, y = a\sin^3\theta, \dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,$

$$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$s = 4\int_0^{\pi/2} \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\, d\theta$$

$$= 12a\int_0^{\pi/2} \sin\theta\cos\theta\sqrt{\cos^2\theta + \sin^2\theta}\, d\theta$$

$$= 6a\int_0^{\pi/2} \sin 2\theta\, d\theta = \left[-3a\cos 2\theta\right]_0^{\pi/2} = 6a$$

58. $x = a\cos\theta, y = a\sin\theta, \dfrac{dx}{d\theta} = -a\sin\theta,$

$$\frac{dy}{d\theta} = a\cos\theta$$

$$S = 4\int_0^{\pi/2} \sqrt{a^2\sin^2\theta + a^2\cos^2\theta}\, d\theta$$

$$= 4a\int_0^{\pi/2} d\theta = \left[4a\theta\right]_0^{\pi/2} = 2\pi a$$

59. $x = a(\theta - \sin\theta), y = a(1 - \cos\theta),$

$$\frac{dx}{d\theta} = a(1 - \cos\theta), \frac{dy}{d\theta} = a\sin\theta$$

$$s = 2\int_0^{\pi} \sqrt{a^2(1 - \cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$

$$= 2\sqrt{2}a\int_0^{\pi} \sqrt{1 - \cos\theta}\, d\theta$$

$$= 2\sqrt{2}a\int_0^{\pi} \frac{\sin\theta}{\sqrt{1 + \cos\theta}}\, d\theta$$

$$= \left[-4\sqrt{2}a\sqrt{1 + \cos\theta}\right]_0^{\pi} = 8a$$

60. $x = \cos\theta + \theta\sin\theta, y = \sin\theta - \theta\cos\theta,$

$$\frac{dx}{d\theta} = \theta\cos\theta$$

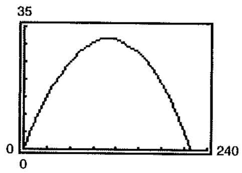
$$\frac{dy}{d\theta} = \theta\sin\theta$$

$$S = \int_0^{2\pi} \sqrt{\theta^2\cos^2\theta + \theta^2\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \theta\, d\theta = \left[\frac{\theta^2}{2}\right]_0^{2\pi} = 2\pi^2$$

61. $x = (90\cos 30°)t,\ y = (90\sin 30°)t - 16t^2$

(a)



(b) Range: 219.2 ft, $\left(t = \dfrac{45}{16}\right)$

(c) $\dfrac{dx}{dt} = 90\cos 30°,\ \dfrac{dy}{dt} = 90\sin 30° - 32t$

$y = 0$ for $t = \dfrac{45}{16}$.

$$s = \int_0^{45/16} \sqrt{(90\cos 30°)^2 + (90\sin 30° - 32t)^2}\,dt$$

≈ 230.8 ft

62. $y = 0 \Rightarrow (90\sin\theta)t = 16t^2 \Rightarrow t = 0, \dfrac{90}{16}\sin\theta$

$$x = (90\cos\theta)t = (90\cos\theta)\frac{90}{16}\sin\theta$$

$$= \frac{90^2}{16}\sin\theta\cos\theta = \frac{90^2}{32}\sin 2\theta$$

$$x'(\theta) = \frac{90^2}{32}2\cos 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}$$

By the First Derivative Test, $\theta = \dfrac{\pi}{4}(45°)$ maximizes the range $(x = 253.125$ feet$)$.

To maximize the arc length, you have

$\dfrac{dx}{dt} = 90\cos\theta,\ \dfrac{dy}{dt} = 90\sin\theta - 32t.$

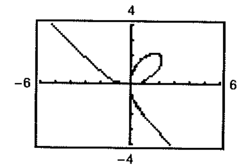
$$s = \int_0^{(90/16)\sin\theta} \sqrt{(90\cos\theta)^2 + (90\sin\theta - 32t)^2}\,dt$$

$$= \frac{2025}{8}\sin\theta + \frac{2025}{16}\cos^2\theta\ln\left[\frac{1+\sin\theta}{1-\sin\theta}\right]$$

Using a graphing utility, we see that s is a maximum of approximately 303.67 feet at $\theta \approx 0.9855(56.5°)$.

63. $x = \dfrac{4t}{1+t^3},\ y = \dfrac{4t^2}{1+t^3}$

(a) $x^3 + y^3 = 4xy$



(b) $$\frac{dy}{dt} = \frac{(1+t^3)(8t) - 4t^2(3t^2)}{(1+t^3)^2}$$

$$= \frac{4t(2-t^3)}{(1+t^3)^2} = 0 \text{ when } t = 0 \text{ or } t = \sqrt[3]{2}.$$

Points: $(0, 0), \left(\dfrac{4\sqrt[3]{2}}{3}, \dfrac{4\sqrt[3]{4}}{3}\right) \approx (1.6799, 2.1165)$

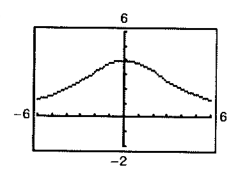
(c) $$s = 2\int_0^1 \sqrt{\left[\frac{4(1-2t^3)}{(1+t^3)^2}\right]^2 + \left[\frac{4t(2-t^3)}{(1+t^3)^2}\right]^2}\,dt$$

$$= 2\int_0^1 \sqrt{\frac{16}{(1+t^3)^4}\left[t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1\right]}\,dt$$

$$= 8\int_0^1 \frac{\sqrt{t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1}}{(1+t^3)^2}\,dt \approx 6.557$$

64. $x = 4\cot\theta = \dfrac{4}{\tan\theta},\ y = 4\sin^2\theta,\ -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

(a)



(b) $\dfrac{dy}{d\theta} = 8\sin\theta\cdot\cos\theta$

$\dfrac{dx}{d\theta} = -4\csc^2\theta$

$\dfrac{dy}{d\theta} = 0$ for $\theta = 0, \pm\dfrac{\pi}{2}$

Horizontal tangent at $(x, y) = (0, 4)\left(\theta = \pm\dfrac{\pi}{2}\right)$

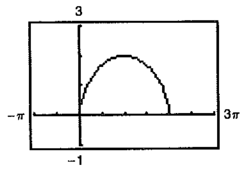
(Function is not defined at $\theta = 0$)

(c) Arc length over $\dfrac{\pi}{4} \le t \le \dfrac{\pi}{2}$: 4.5183

65. (a) $x = t - \sin t$

$y = 1 - \cos t$

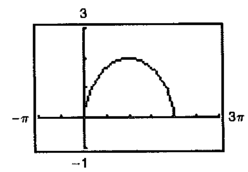
$0 \le t \le 2\pi$



$x = 2t - \sin(2t)$

$y = 1 - \cos(2t)$

$0 \le t \le \pi$



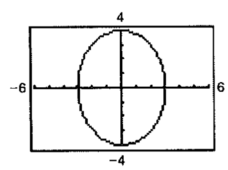
(b) The average speed of the particle on the second path is twice the average speed of a particle on the first path.

(c) $x = \frac{1}{2}t - \sin\left(\frac{1}{2}t\right)$

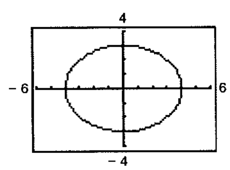
$y = 1 - \cos\left(\frac{1}{2}t\right)$

The time required for the particle to traverse the same path is $t = 4\pi$.

66. (a) First particle: $x = 3\cos t,\ y = 4\sin t,\ 0 \le t \le 2\pi$



Second particle: $x = 4\sin t,\ y = 3\cos t,$
$0 \le t \le 2\pi$



(b) There are 4 points of intersection.

(c) Suppose at time t that

$3\cos t = 4\sin t$ and $4\sin t = 3\cos t$

$\tan t = \frac{3}{4}$ and $\tan t = \frac{3}{4}$.

Yes, the particles are at the same place at the same time for $\tan t = \frac{3}{4}$. $t \approx 0.6435, 3.7851$. The intersection points are $(2.4, 2.4)$ and $(-2.4, -2.4)$

(d) The curves intersect twice, but not at the same time.

67. $x = 3t, \dfrac{dx}{dt} = 3$

$y = t + 2, \dfrac{dy}{dt} = 1$

$$S = 2\pi\int_0^4 (t+2)\sqrt{3^2 + 1^2}\,dt$$

$$= 2\pi\sqrt{10}\left[\frac{t^2}{2} + 2t\right]_0^4$$

$$= 2\pi\sqrt{10}[8 + 8] = 32\sqrt{10}\pi \approx 317.9068$$

68. $x = \dfrac{1}{4}t^2 \quad \dfrac{dx}{dt} = \dfrac{t}{2}$

$y = t + 3, \ \dfrac{dy}{dt} = 1$

$$S = 2\pi\int_0^3 (t+3)\sqrt{\left(\frac{t}{2}\right)^2 + 1}\,dt$$

$$= 2\pi\int_0^3 (t+3)\sqrt{\frac{t^2}{4} + 1}\,dt$$

$$\approx 114.1999$$

69. $x = \cos^2\theta, \dfrac{dx}{d\theta} = -2\cos\theta\sin\theta$

$y = \cos\theta, \dfrac{dy}{d\theta} = -\sin\theta$

$$S = 2\pi\int_0^{\pi/2} \cos\theta\sqrt{4\cos^2\theta\sin^2\theta + \sin^2\theta}\,d\theta$$

$$= 2\pi\int_0^{\pi/2} \cos\theta\sin\theta\sqrt{4\cos^2\theta + 1}\,d\theta$$

$$= \frac{\left(5\sqrt{5} - 1\right)\pi}{6}$$

$$\approx 5.3304$$

70. $x = \theta + \sin\theta, \dfrac{dx}{d\theta} = 1 + \cos\theta$

$y = \theta + \cos\theta, \dfrac{dy}{d\theta} = 1 - \sin\theta$

$$S = 2\pi\int_0^{\pi/2} (\theta + \cos\theta)\sqrt{(1+\cos\theta)^2 + (1-\sin\theta)^2}\,d\theta$$

$$= 2\pi\int_0^{\pi/2} (\theta + \cos\theta)\sqrt{3 + 2\cos\theta - 2\sin\theta}\,d\theta$$

$$\approx 23.2433$$

71. $x = 2t, \dfrac{dx}{dt} = 2$

$y = 3t, \dfrac{dy}{dt} = 3$

(a) $S = 2\pi\int_0^3 3t\sqrt{4+9}\,dt$

$$= 6\sqrt{13}\pi\left[\frac{t^2}{2}\right]_0^3 = 6\sqrt{13}\pi\left(\frac{9}{2}\right) = 27\sqrt{13}\pi$$

(b) $S = 2\pi\int_0^3 2t\sqrt{4+9}\,dt$

$$= 4\sqrt{13}\pi\left[\frac{t^2}{2}\right]_0^3 = 4\sqrt{13}\pi\left(\frac{9}{2}\right) = 18\sqrt{13}\pi$$

72. $x = t,\ y = 4 - 2t,\ \dfrac{dx}{dt} = 1,\ \dfrac{dy}{dt} = -2$

(a) $S = 2\pi \int_0^2 (4 - 2t)\sqrt{1 + 4}\, dt$

$= \left[2\sqrt{5}\pi\left(4t - t^2\right)\right]_0^2 = 8\pi\sqrt{5}$

(b) $S = 2\pi \int_0^2 t\sqrt{1 + 4}\, dt = \left[\sqrt{5}\pi t^2\right]_0^2 = 4\pi\sqrt{5}$

73. $x = 5\cos\theta\ \dfrac{dx}{d\theta} = -5\sin\theta$

$y = 5\sin\theta\ \dfrac{dy}{d\theta} = 5\cos\theta$

$S = 2\pi \int_0^{\pi/2} 5\cos\theta\sqrt{25\sin^2\theta + 25\cos^2\theta}\, d\theta$

$= 10\pi \int_0^{\pi/2} 5\cos\theta\, d\theta$

$= 50\pi\left[\sin\theta\right]_0^{\pi/2} = 50\pi$

[**Note:** This is the surface area of a hemisphere of radius 5]

74. $x = \dfrac{1}{3}t^3,\ y = t + 1, 1 \le t \le 2$, y-axis

$\dfrac{dx}{dt} = t^2,\ \dfrac{dy}{dt} = 1$

$S = 2\pi \int_1^2 \dfrac{1}{3}t^3\sqrt{t^4 + 1}\, dt = \dfrac{\pi}{9}\left[\left(x^4 + 1\right)^{3/2}\right]_1^2$

$= \dfrac{\pi}{9}\left(17^{3/2} - 2^{3/2}\right) \approx 23.48$

75. $x = a\cos^3\theta,\ y = a\sin^3\theta,\ \dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,\ \dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$S = 4\pi \int_0^{\pi/2} a\sin^3\theta\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\, d\theta$

$= 12a^2\pi \int_0^{\pi/2} \sin^4\theta\cos\theta\, d\theta = \dfrac{12\pi a^2}{5}\left[\sin^5\theta\right]_0^{\pi/2} = \dfrac{12}{5}\pi a^2$

76. $x = a\cos\theta,\ y = b\sin\theta,\ \dfrac{dx}{d\theta} = -a\sin\theta,\ \dfrac{dy}{d\theta} = b\cos\theta$

(a) $S = 4\pi \int_0^{\pi/2} b\sin\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$

$= 4\pi \int_0^{\pi/2} ab\sin\theta\sqrt{1 - \left(\dfrac{a^2 - b^2}{a^2}\right)\cos^2\theta}\, d\theta = \dfrac{-4ab\pi}{e}\int_0^{\pi/2} (-e\sin\theta)\sqrt{1 - e^2\cos^2\theta}\, d\theta$

$= \dfrac{-2ab\pi}{e}\left[e\cos\theta\sqrt{1 - e^2\cos^2\theta} + \arcsin(e\cos\theta)\right]_0^{\pi/2} = \dfrac{2ab\pi}{e}\left[e\sqrt{1 - e^2} + \arcsin(e)\right]$

$= 2\pi b^2 + \left(\dfrac{2\pi a^2 b}{\sqrt{a^2 - b^2}}\right)\arcsin\left(\dfrac{\sqrt{a^2 - b^2}}{a}\right) = 2\pi b^2 + 2\pi\left(\dfrac{ab}{e}\right)\arcsin(e)$

$\left(e = \dfrac{\sqrt{a^2 - b^2}}{a} = \dfrac{c}{a}\text{: eccentricity}\right)$

(b) $S = 4\pi \int_0^{\pi/2} a\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$

$= 4\pi \int_0^{\pi/2} a\cos\theta\sqrt{b^2 + c^2\sin^2\theta}\, d\theta = \dfrac{4a\pi}{c}\int_0^{\pi/2} c\cos\theta\sqrt{b^2 + c^2\sin^2\theta}\, d\theta$

$= \dfrac{2a\pi}{c}\left[c\sin\theta\sqrt{b^2 + c^2\sin^2\theta} + b^2\ln\left|c\sin\theta + \sqrt{b^2 + c^2\sin^2\theta}\right|\right]_0^{\pi/2}$

$= \dfrac{2a\pi}{c}\left[c\sqrt{b^2 + c^2} + b^2\ln\left|c + \sqrt{b^2 + c^2}\right| - b^2\ln b\right]$

$= 2\pi a^2 + \dfrac{2\pi ab^2}{\sqrt{a^2 - b^2}}\ln\left|\dfrac{a + \sqrt{a^2 - b^2}}{b}\right| = 2\pi a^2 + \left(\dfrac{\pi b^2}{e}\right)\ln\left|\dfrac{1 + e}{1 - e}\right|$

77. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

See Theorem 10.7.

78. $x = t,\ y = 3 \Rightarrow \dfrac{dy}{dx} = 0$

79. $x = t,\ y = 6t - 5 \Rightarrow \dfrac{dy}{dx} = \dfrac{6}{1} = 6$

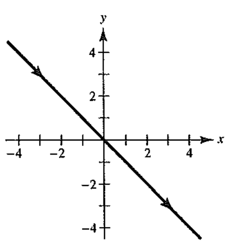
80. $s = \displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

See Theorem 10.8.

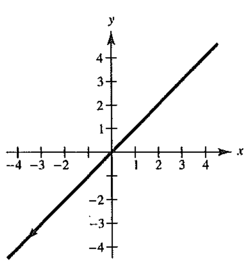
81. (a) $S = 2\pi \displaystyle\int_a^b g(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

(b) $S = 2\pi \displaystyle\int_a^b f(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

82. One possible answer is the graph given by $x = t,\ y = -t.$



One possible answer is the graph given by $x = -t,\ y = -t.$

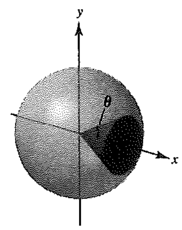


83. Let y be a continuous function of x on $a \le x \le b$. Suppose that $x = f(t)$, $y = g(t)$, and $f(t_1) = a$, $f(t_2) = b$. Then using integration by substitution, $dx = f'(t)\, dt$ and

$$\int_a^b y\, dx = \int_{t_1}^{t_2} g(t) f'(t)\, dt.$$

84. $x = r\cos\phi,\ y = r\sin\phi$

$$S = 2\pi\int_0^\theta r\sin\phi\sqrt{r^2\sin^2\phi + r^2\cos^2\phi}\, d\phi$$
$$= 2\pi r^2\int_0^\theta \sin\phi\, d\phi$$
$$= \Big[-2\pi r^2\cos\phi\Big]_0^\theta$$
$$= 2\pi r^2(1 - \cos\theta)$$

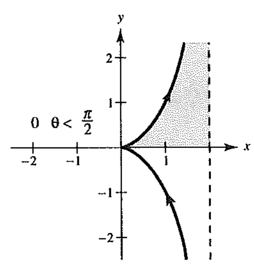


85. $x = 2\sin^2\theta$

$y = 2\sin^2\theta\tan\theta$

$\dfrac{dx}{d\theta} = 4\sin\theta\cos\theta$

$$A = \int_0^{\pi/2} 2\sin^2\theta\tan\theta(4\sin\theta\cos\theta)\, d\theta$$
$$= 8\int_0^{\pi/2}\sin^4\theta\, d\theta$$
$$= 8\left[\frac{-\sin^3\theta\cos\theta}{4} - \frac{3}{8}\sin\theta\cos\theta + \frac{3}{8}\theta\right]_0^{\pi/2} = \frac{3\pi}{2}$$



86. $x = 2\cot\theta,\ y = 2\sin^2\theta,\ \dfrac{dx}{d\theta} = -2\csc^2\theta$

$$A = 2\int_{\pi/2}^0 \left(2\sin^2\theta\right)\left(-2\csc^2\theta\right) d\theta$$
$$= -8\int_{\pi/2}^0 d\theta = \left[-8\theta\right]_{\pi/2}^0 = 4\pi$$

87. πab is area of ellipse (d).

88. $\frac{3}{8}\pi a^2$ is area of asteroid (b).

89. $6\pi a^2$ is area of cardioid (f).

90. $2\pi a^2$ is area of deltoid (c).

91. $\frac{8}{3}ab$ is area of hourglass (a).

92. $2\pi ab$ is area of teardrop (e).

93. $x = \sqrt{t}, y = 4 - t, 0 < t < 4$

$$A = \int_0^2 y\,dx = \int_0^4 (4-t)\frac{1}{2\sqrt{t}}\,dt = \frac{1}{2}\int_0^4 \left(4t^{-1/2} - t^{1/2}\right)dt = \left[\frac{1}{2}\left(8\sqrt{t} - \frac{2}{3}t\sqrt{t}\right)\right]_0^4 = \frac{16}{3}$$

$$\bar{x} = \frac{1}{A}\int_0^2 yx\,dx = \frac{3}{16}\int_0^4 (4-t)\sqrt{t}\left(\frac{1}{2\sqrt{t}}\right)dt = \frac{3}{32}\int_0^4 (4-t)\,dt = \left[\frac{3}{32}\left(4t - \frac{t^2}{2}\right)\right]_0^4 = \frac{3}{4}$$

$$\bar{y} = \frac{1}{A}\int_0^2 \frac{y^2}{2}\,dx = \frac{3}{32}\int_0^4 (4-t)^2\frac{1}{2\sqrt{t}}\,dt = \frac{3}{64}\int_0^4 \left(16t^{-1/2} - 8t^{1/2} + t^{3/2}\right)dt = \frac{3}{64}\left[32\sqrt{t} - \frac{16}{3}t\sqrt{t} + \frac{2}{5}t^2\sqrt{t}\right]_0^4 = \frac{8}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{3}{4}, \frac{8}{5}\right)$$

94. $x = \sqrt{4-t}, y = \sqrt{t}, \dfrac{dx}{dt} = -\dfrac{1}{2\sqrt{4-t}}, 0 \le t \le 4$

$$A = \int_4^0 \sqrt{t}\left(-\frac{1}{2\sqrt{4-t}}\right)dt = \int_0^2 \sqrt{4-u^2}\,du = \frac{1}{2}\left[u\sqrt{4-u^2} + 4\arcsin\frac{u}{2}\right]_0^2 = \pi$$

Let $u = \sqrt{4-t}$, then $du = -1/\left(2\sqrt{4-t}\right)dt$ and $\sqrt{t} = \sqrt{4-u^2}$.

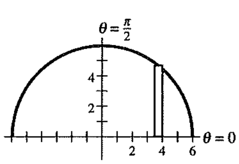
$$\bar{x} = \frac{1}{\pi}\int_4^0 \sqrt{4-t}\sqrt{t}\left(-\frac{1}{2\sqrt{4-t}}\right)dt = -\frac{1}{2\pi}\int_4^0 \sqrt{t}\,dt = \left[-\frac{1}{2\pi}\frac{2}{3}t^{3/2}\right]_4^0 = \frac{8}{3\pi}$$

$$\bar{y} = \frac{1}{2\pi}\int_4^0 \left(\sqrt{t}\right)^2\left(-\frac{1}{2\sqrt{4-t}}\right)dt = -\frac{1}{4\pi}\int_4^0 \frac{t}{\sqrt{4-t}}\,dt = -\frac{1}{4\pi}\left[\frac{-2(8+t)}{3}\sqrt{4-t}\right]_4^0 = \frac{8}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8}{3\pi}, \frac{8}{3\pi}\right)$$

95. $x = 6\cos\theta, y = 6\sin\theta, \dfrac{dx}{d\theta} = -6\sin\theta\,d\theta$

$$\begin{aligned} V &= 2\pi\int_{\pi/2}^0 (6\sin\theta)^2(-6\sin\theta)\,d\theta \\ &= -432\pi\int_{\pi/2}^0 \sin^3\theta\,d\theta \\ &= -432\pi\int_{\pi/2}^0 \left(1 - \cos^2\theta\right)\sin\theta\,d\theta \\ &= -432\pi\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_{\pi/2}^0 \\ &= -432\pi\left(-1 + \frac{1}{3}\right) = 288\pi \end{aligned}$$



Note: Volume of sphere is $\frac{4}{3}\pi\left(6^3\right) = 288\pi$.

96. $x = \cos\theta, y = 3\sin\theta, \dfrac{dx}{d\theta} = -\sin\theta$

$$\begin{aligned} V &= 2\pi\int_{\pi/2}^0 (3\sin\theta)^2(-\sin\theta)\,d\theta \\ &= -18\pi\int_{\pi/2}^0 \sin^3\theta\,d\theta \\ &= -18\pi\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_{\pi/2}^0 = 12\pi \end{aligned}$$

97. $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$

(a) $\dfrac{dy}{d\theta} = a\sin\theta,\ \dfrac{dx}{d\theta} = a(1 - \cos\theta)$

$\dfrac{dy}{dx} = \dfrac{a\sin\theta}{a(1 - \cos\theta)} = \dfrac{\sin\theta}{1 - \cos\theta}$

$\dfrac{d^2y}{dx^2} = \left[\dfrac{(1 - \cos\theta)\cos\theta - \sin\theta(\sin\theta)}{(1 - \cos\theta)^2}\right] \bigg/ \left[a(1 - \cos\theta)\right] = \dfrac{\cos\theta - 1}{a(1 - \cos\theta)^3} = \dfrac{-1}{a(\cos\theta - 1)^2}$

(b) At $\theta = \dfrac{\pi}{6},\ x = a\left(\dfrac{\pi}{6} - \dfrac{1}{2}\right),\ y = a\left(1 - \dfrac{\sqrt{3}}{2}\right),\ \dfrac{dy}{dx} = \dfrac{1/2}{1 - \sqrt{3}/2} = 2 + \sqrt{3}.$

Tangent line: $y - a\left(1 - \dfrac{\sqrt{3}}{2}\right) = \left(2 + \sqrt{3}\right)\left(x - a\left(\dfrac{\pi}{6} - \dfrac{1}{2}\right)\right)$

(c) $\dfrac{dy}{dx} = \dfrac{\sin\theta}{1 - \cos\theta} = 0 \Rightarrow \sin\theta = 0,\ 1 - \cos\theta \neq 0$

Points of horizontal tangency: $(x, y) = \left(a(2n + 1)\pi, 2a\right)$

(d) Concave downward on all open θ-intervals:

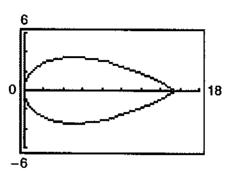
$\ldots, (-2\pi, 0), (0, 2\pi), (2\pi, 4\pi), \ldots$

(e) $s = \displaystyle\int_0^{2\pi} \sqrt{a^2\sin^2\theta + a^2(1 - \cos\theta)^2}\, d\theta$

$= a\displaystyle\int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta = a\int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta = 2a\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta = \left[-4a\cos\left(\frac{\theta}{2}\right)\right]_0^{2\pi} = 8a$

98. $x = t^2\sqrt{3},\ y = 3t - \dfrac{1}{3}t^3$

(a)



(b) $\dfrac{dx}{dt} = 2\sqrt{3}t,\ \dfrac{dy}{dt} = 3 - t^2,\ \dfrac{dy}{dx} = \dfrac{3 - t^2}{2\sqrt{3}t}$

$\dfrac{d^2y}{dx^2} = \left[\dfrac{2\sqrt{3}(t)(-2t) - (3 - t^2)2\sqrt{3}}{12t^2}\right] \bigg/ \left[2\sqrt{3}t\right] = \dfrac{-2\sqrt{3}t^2 - 6\sqrt{3}}{(12t^2)(2\sqrt{3}t)} = -\dfrac{t^2 + 3}{12t^3}$

(c) $(x, y) = \left(\sqrt{3}, \dfrac{8}{3}\right)$ at $t = 1$. $\dfrac{dy}{dx} = \dfrac{2}{2\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

$y - \dfrac{8}{3} = \dfrac{\sqrt{3}}{3}\left(x - \sqrt{3}\right)$

$y = \dfrac{\sqrt{3}}{3}x + \dfrac{5}{3}$

(d) $s = \displaystyle\int_{-3}^{3} \sqrt{12t^2 + (3 - t)^2}\, dt = \int_{-3}^{3} \sqrt{t^4 - 6t^2 + 9 + 12t^2}\, dt = \int_{-3}^{3} \sqrt{(t^2 + 3)^2}\, dt = \int_{-3}^{3} (t^3 + 3)\, dt = 36$

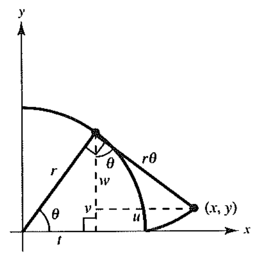
(e) $S = 2\pi\displaystyle\int_0^3 \left(3t - \frac{1}{3}t^3\right)(t^3 + 3)\, dt = 81\pi$

99. $x = t + u = r\cos\theta + r\theta\sin\theta$

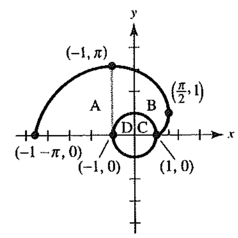
$= r(\cos\theta + \theta\sin\theta)$

$y = v - w = r\sin\theta - r\theta\cos\theta$

$= r(\sin\theta - \theta\cos\theta)$



100. Focus on the region above the x-axis. From Exercise 99, the equation of the involute from $(1, 0)$ to $(-1, \pi)$ is



$x = \cos\theta + \theta\sin\theta$

$y = \sin\theta - \theta\cos\theta$

$0 \le \theta \le \pi.$

At $(-1, \pi)$, the string is fully extended and has length π.

So, the area of region A is $\frac{1}{4}\pi(\pi^2) = \frac{1}{4}\pi^3$.

You now need to find the area of region B.

$$\frac{dx}{d\theta} = -\sin\theta + \sin\theta + \theta\cos\theta = \theta\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}. \ (\theta = 0 \text{ is cusp.})$$

So, the far right point on the involute is $(\pi/2, 1)$.

The area of the region $B + C + D$ is given by

$$\int_{\theta=\pi}^{\theta=\pi/2} y\,dx - \int_{\theta=0}^{\theta=\pi/2} y\,dx = \int_{\theta=\pi}^{\theta=0} y\,dx$$

where $y = \sin\theta - \theta\cos\theta$ and $dx = \theta\cos\theta\,d\theta$.

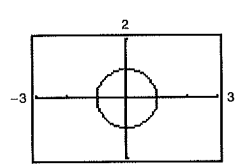
So, you can calculate

$$\int_{\pi}^{0} [\sin\theta - \theta\cos\theta]\theta\cos\theta\,d\theta = \frac{\pi}{6}(\pi^2 + 3).$$

Because the area of $C + D$ is $\pi/2$, you have

$$\text{Total area covered} = 2\left[\frac{1}{4}\pi^3 + \frac{\pi}{6}(\pi^2 + 3) - \frac{\pi}{2}\right] = \frac{5}{6}\pi^3.$$

101. (a)



(b) $x = \dfrac{1 - t^2}{1 + t^2}, y = \dfrac{2t}{1 + t^2}, \quad -20 \le t \le 20$

The graph (for $-\infty < t < \infty$) is the circle $x^2 + y^2 = 1$, except the point $(-1, 0)$.

Verify:

$$x^2 + y^2 = \left(\frac{1 - t^2}{1 + t^2}\right)^2 + \left(\frac{2t}{1 + t^2}\right)^2$$

$$= \frac{1 - 2t^2 + t^4 + 4t^2}{(1 + t^2)^2} = \frac{(1 + t^2)^2}{(1 + t^2)^2} = 1$$

(c) As t increases from -20 to 0, the speed increases, and as t increases from 0 to 20, the speed decreases.