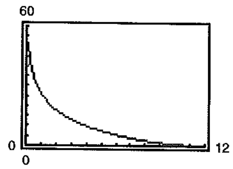
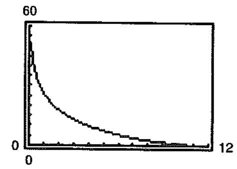
102. (a) $y = -12\ln\left(\dfrac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$

$0 < x \le 12$



(b) $x = 12 \operatorname{sech}\dfrac{t}{12},\ y = t - 12\tanh\dfrac{t}{12},\ 0 \le t$



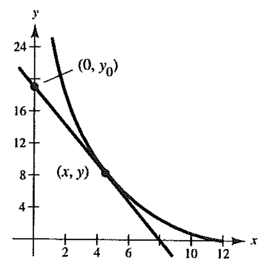
Same as the graph in (a), but has the advantage of showing the position of the object and any given time t.

(c) $\dfrac{dy}{dx} = \dfrac{1 - \operatorname{sech}^2(t/12)}{-\operatorname{sech}(t/12)\tanh(t/12)} = -\sinh\dfrac{t}{12}$

Tangent line: $y - \left(t_0 - 12\tanh\dfrac{t_0}{12}\right) = -\sinh\dfrac{t_0}{12}\left(x - 12\operatorname{sech}\dfrac{t_0}{12}\right)$

$$y = t_0 - \left(\sinh\frac{t_0}{12}\right)x$$

y-intercept: $(0, t_0)$



Distance between $(0, t_0)$ and (x, y): $d = \sqrt{\left(12\operatorname{sech}\dfrac{t_0}{12}\right)^2 + \left(-12\tanh\dfrac{t_0}{12}\right)^2} = 12$

$d = 12$ for any $t \ge 0$.

103. False. $\dfrac{d^2y}{dx^2} = \dfrac{\dfrac{d}{dt}\left[\dfrac{g'(t)}{f'(t)}\right]}{f'(t)} = \dfrac{f'(t)g''(t) - g'(t)f''(t)}{\left[f'(t)\right]^3}$

104. False. Both dx/dt and dy/dt are zero when $t = 0$. By eliminating the parameter, you have $y = x^{2/3}$ which does not have a horizontal tangent at the origin.

105. $A = \pi(2)^2 - \pi\left(\dfrac{1}{2}\right)^2 = \left(4 - \dfrac{1}{4}\right)\pi = \dfrac{15}{4}\pi$

$L = \dfrac{\frac{15}{4}\pi}{0.001} \approx 11780.97$ in. ≈ 981.7 ft

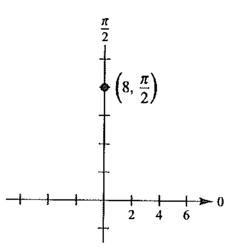
Section 10.4 Polar Coordinates and Polar Graphs

1. $\left(8, \dfrac{\pi}{2}\right)$

$x = 8\cos\dfrac{\pi}{2} = 0$

$y = 8\sin\dfrac{\pi}{2} = 8$

$(x, y) = (0, 8)$

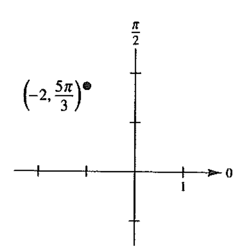


2. $\left(-2, \dfrac{5\pi}{3}\right)$

$x = -2\cos\dfrac{5\pi}{3} = -2\left(\dfrac{1}{2}\right) = -1$

$y = -2\sin\dfrac{5\pi}{3} = -2\left(\dfrac{-\sqrt{3}}{2}\right) = \sqrt{3}$

$(x, y) = \left(-1, \sqrt{3}\right)$

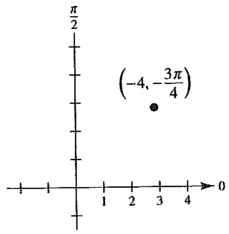


3. $\left(-4, -\dfrac{3\pi}{4}\right)$

$x = -4\cos\left(\dfrac{-3\pi}{4}\right) = -4\left(-\dfrac{\sqrt{2}}{2}\right) = 2\sqrt{2}$

$y = -4\sin\left(\dfrac{-3\pi}{4}\right) = -4\left(-\dfrac{\sqrt{2}}{2}\right) = 2\sqrt{2}$

$(x, y) = \left(2\sqrt{2}, 2\sqrt{2}\right)$

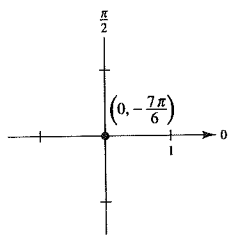


4. $\left(0, -\dfrac{7\pi}{6}\right)$

$x = 0\cos\left(-\dfrac{7\pi}{6}\right) = 0$

$y = 0\sin\left(-\dfrac{7\pi}{6}\right) = 0$

$(x, y) = (0, 0)$

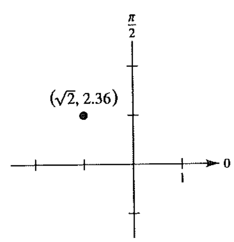


5. $\left(\sqrt{2}, 2.36\right)$

$x = \sqrt{2}\cos(2.36) \approx -1.004$

$y = \sqrt{2}\sin(2.36) \approx 0.996$

$(x, y) = (-1.004, 0.996)$

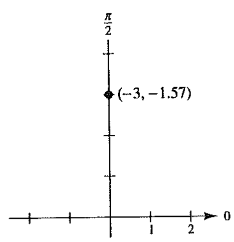


6. $(-3, -1.57)$

$x = -3\cos(-1.57) \approx -0.0024$

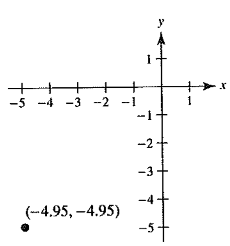
$y = -3\sin(-1.57) \approx 3$

$(x, y) = (-0.0024, 3)$



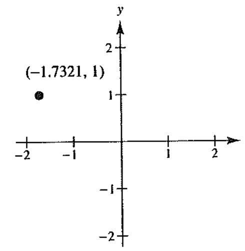
7. $(r, \theta) = \left(7, \dfrac{5\pi}{4}\right)$

$(x, y) = (-4.9497, -4.9497)$



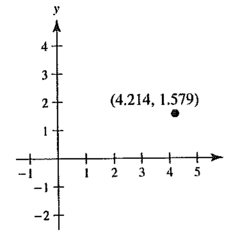
8. $(r, \theta) = \left(-2, \dfrac{11\pi}{6}\right)$

$(x, y) = (-1.7321, 1)$



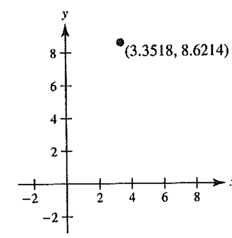
9. $(r, \theta) = (-4.5, 3.5)$

$(x, y) = (4.2141, 1.5785)$



10. $(r, \theta) = (9.25, 1.2)$

$(x, y) = (3.3518, 8.6214)$



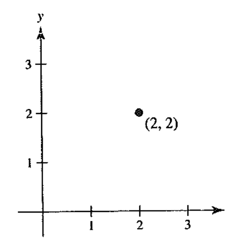
11. $(x, y) = (2, 2)$

$r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

$\tan\theta = \dfrac{2}{2} = 1$

$\theta = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

$\left(2\sqrt{2}, \dfrac{\pi}{4}\right), \left(-2\sqrt{2}, \dfrac{5\pi}{4}\right)$



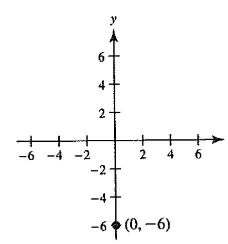
12. $(x, y) = (0, -6)$

$r = \pm 6$

$\tan\theta$ undefined

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

$\left(6, \dfrac{3\pi}{2}\right), \left(-6, \dfrac{\pi}{2}\right)$

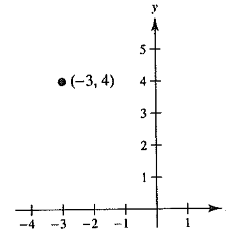


13. $(x, y) = (-3, 4)$

$r = \pm\sqrt{9 + 16} = \pm 5$

$\tan\theta = -\frac{4}{3}$

$\theta \approx 2.214, 5.356, (5, 2.214),$
$(-5, 5.356)$



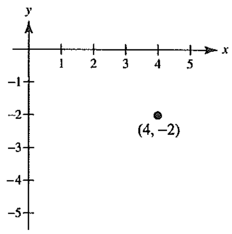
14. $(x, y) = (4, -2)$

$r = \pm\sqrt{16 + 4} = \pm 2\sqrt{5}$

$\tan\theta = -\frac{2}{4} = -\frac{1}{2}$

$\theta \approx -0.464$

$\left(2\sqrt{5}, -0.464\right), \left(-2\sqrt{5}, 2.678\right)$



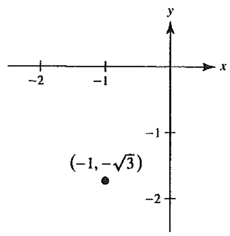
15. $(x, y) = \left(-1, -\sqrt{3}\right)$

$r = \sqrt{4} = 2$

$\tan\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3}$

$\theta = \frac{\pi}{3}, \frac{4\pi}{3}$

$\left(2, \frac{4\pi}{3}\right), \left(-2, \frac{\pi}{3}\right)$

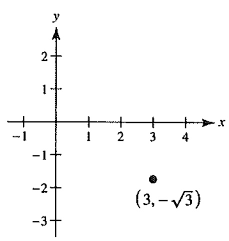


16. $(x, y) = \left(3, -\sqrt{3}\right)$

$r = \sqrt{9 + 3} = 2\sqrt{3}$

$\tan\theta = -\frac{\sqrt{3}}{3}$

$(r, \theta) = \left(2\sqrt{3}, \frac{11\pi}{6}\right) = \left(-2\sqrt{3}, \frac{5\pi}{6}\right)$



17. $(x, y) = (3, -2)$

$(r, \theta) = (3.606, -0.588)$

18. $(x, y) = \left(3\sqrt{2}, 3\sqrt{2}\right)$

$(r, \theta) = (6, 0.785)$

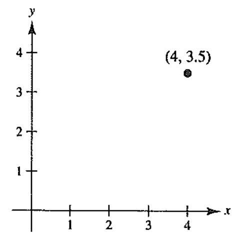
19. $(x, y) = \left(\frac{7}{4}, \frac{5}{2}\right)$

$(r, \theta) = (3.0516, 0.9601)$

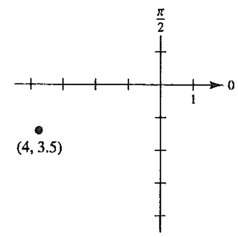
20. $(x, y) = (0, -5)$

$(r, \theta) = (5, -1.571)$

21. (a) $(x, y) = (4, 3.5)$



(b) $(r, \theta) = (4, 3.5)$



22. (a) Moving horizontally, the x-coordinate changes. Moving vertically, the y-coordinate changes.

(b) Both r and θ values change.

(c) In polar mode, horizontal (or vertical) changes result in changes in both r and θ.

23. $r = 2\sin\theta$ circle

Matches (c)

24. $r = 4\cos 2\theta$

Rose curve

Matches (b)

25. $r = 3(1 + \cos\theta)$

Cardioid

Matches (a)

26. $r = 2\sec\theta$

Line

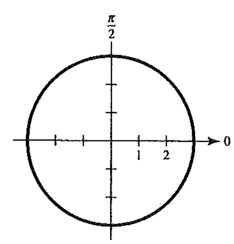
Matches (d)

27. $x^2 + y^2 = 9$

$r^2 = 9$

$r = 3$

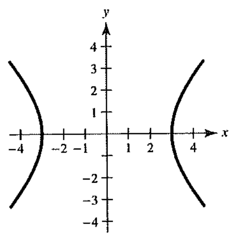
Circle



28. $x^2 - y^2 = 9$

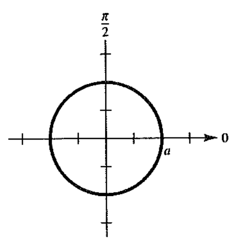
$$(r\cos\theta)^2 - (r\sin\theta)^2 = 9$$
$$r^2(\cos^2\theta - \sin^2\theta) = 9$$
$$r^2\cos 2\theta = 9$$
$$r = \frac{3}{\sqrt{\cos 2\theta}}$$

Hyperbola



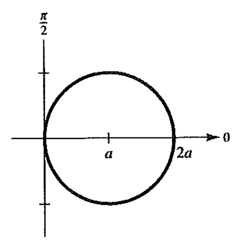
29. $x^2 + y^2 = a^2$

$$r = a$$



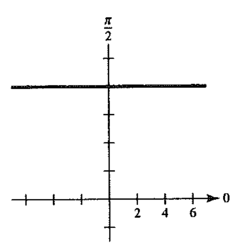
30. $x^2 + y^2 - 2ax = 0$

$$r^2 - 2ar\cos\theta = 0$$
$$r(r - 2a\cos\theta) = 0$$
$$r = 2a\cos\theta$$



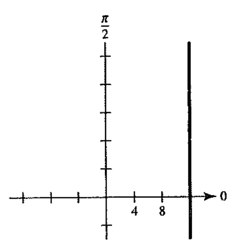
31. $y = 8$

$$r\sin\theta = 8$$
$$r = 8\csc\theta$$



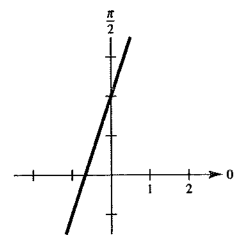
32. $x = 12$

$$r\cos\theta = 12$$
$$r = 12\sec\theta$$



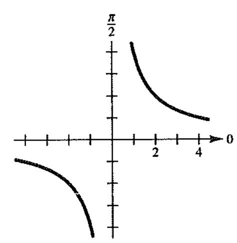
33. $3x - y + 2 = 0$

$$3r\cos\theta - r\sin\theta + 2 = 0$$
$$r(3\cos\theta - \sin\theta) = -2$$
$$r = \frac{-2}{3\cos\theta - \sin\theta}$$



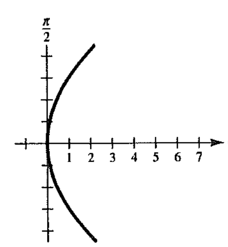
34. $xy = 4$

$$(r\cos\theta)(r\sin\theta) = 4$$
$$r^2 = 4\sec\theta\csc\theta$$
$$= 8\csc 2\theta$$

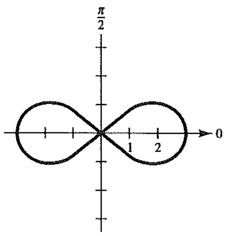


35. $y^2 = 9x$

$$r^2\sin^2\theta = 9r\cos\theta$$
$$r = \frac{9\cos\theta}{\sin^2\theta}$$
$$r = 9\csc^2\theta\cos\theta$$



36.
$$(x^2 + y^2)^2 - 9(x^2 - y^2) = 0$$
$$(r^2)^2 - 9(r^2 \cos^2 \theta - r^2 \sin^2 \theta) = 0$$
$$r^2[r^2 - 9(\cos 2\theta)] = 0$$
$$r^2 = 9 \cos 2\theta$$

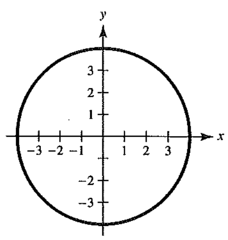


37. $r = 4$

$r^2 = 16$

$x^2 + y^2 = 16$

Circle

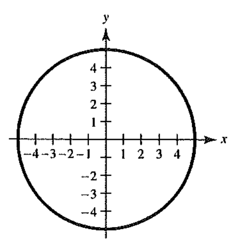


38. $r = -5$

$r^2 = 25$

$x^2 + y^2 = 25$

Circle



39. $r = 3 \sin \theta$

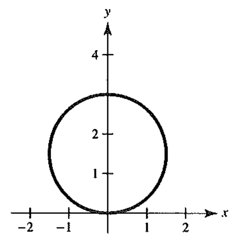
$r^2 = 3r \sin \theta$

$x^2 + y^2 = 3y$

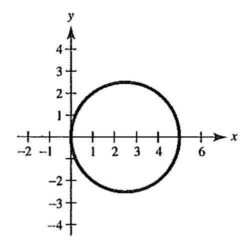
$x^2 + \left(y^2 - 3y + \frac{9}{4}\right) = \frac{9}{4}$

$x^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{4}$

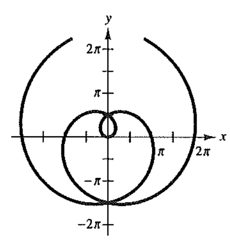
Circle



40.
$$r = 5 \cos \theta$$
$$r^2 = 5r \cos \theta$$
$$x^2 + y^2 = 5x$$
$$x^2 - 5x + \frac{25}{4} + y^2 = \frac{25}{4}$$
$$\left(x - \frac{5}{2}\right)^2 + y^2 = \left(\frac{5}{2}\right)^2$$



41.
$$r = \theta$$
$$\tan r = \tan \theta$$
$$\tan \sqrt{x^2 + y^2} = \frac{y}{x}$$
$$\sqrt{x^2 + y^2} = \arctan \frac{y}{x}$$

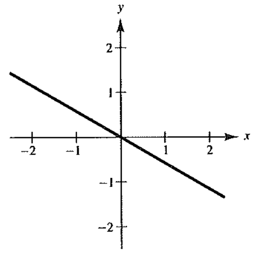


42. $\theta = \dfrac{5\pi}{6}$

$\tan\theta = \tan\dfrac{5\pi}{6}$

$\dfrac{y}{x} = -\dfrac{\sqrt{3}}{3}$

$y = -\dfrac{\sqrt{3}}{3}x$

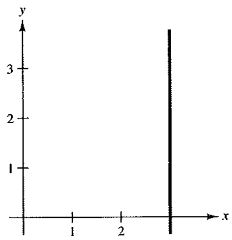


43. $r = 3\sec\theta$

$r\cos\theta = 3$

$x = 3$

$x - 3 = 0$

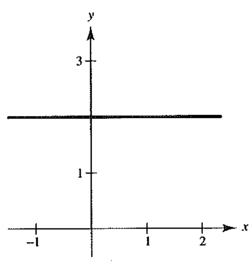


44. $r = 2\csc\theta$

$r\sin\theta = 2$

$y = 2$

$y - 2 = 0$



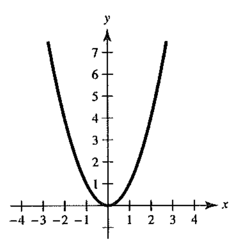
45. $r = \sec\theta\tan\theta$

$r\cos\theta = \tan\theta$

$x = \dfrac{y}{x}$

$y = x^2$

Parabola



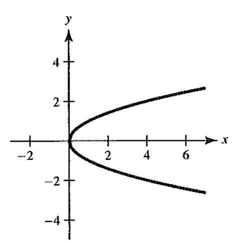
46. $r = \cot\theta\csc\theta$

$r\sin\theta = \cot\theta$

$y = \dfrac{x}{y}$

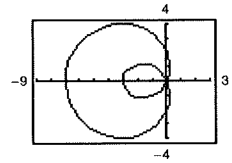
$x = y^2$

Parabola



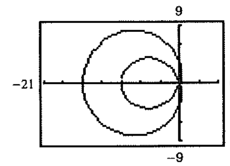
47. $r = 2 - 5\cos\theta$

$0 \le \theta < 2\pi$



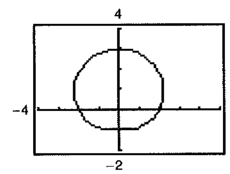
48. $r = 3(1 - 4\cos\theta)$

$0 \le \theta < 2\pi$



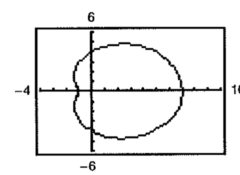
49. $r = 2 + \sin\theta$

$0 \le \theta < 2\pi$



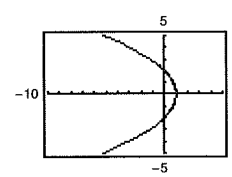
50. $r = 4 + 3\cos\theta$

$0 \le \theta < 2\pi$



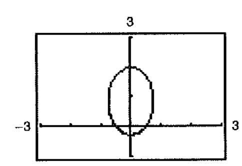
51. $r = \dfrac{2}{1 + \cos\theta}$

Traced out once on $-\pi < \theta < \pi$



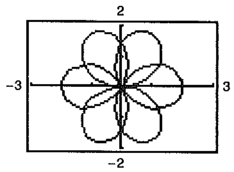
52. $r = \dfrac{2}{4 - 3\sin\theta}$

Traced out once on $0 \le \theta \le 2\pi$



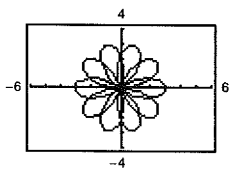
53. $r = 2\cos\left(\frac{3\theta}{2}\right)$

$0 \le \theta < 4\pi$



54. $r = 3\sin\left(\frac{5\theta}{2}\right)$

$0 \le \theta < 4\pi$

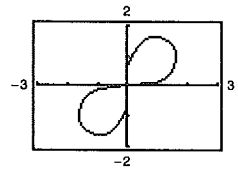


55. $r^2 = 4\sin 2\theta$

$r_1 = 2\sqrt{\sin 2\theta}$

$r_2 = -2\sqrt{\sin 2\theta}$

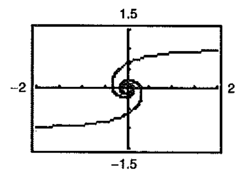
$0 \le \theta < \frac{\pi}{2}$



56. $r^2 = \frac{1}{\theta}$.

Graph as $r_1 = \frac{1}{\sqrt{\theta}}, r_2 = -\frac{1}{\sqrt{\theta}}$.

It is traced out once on $[0, \infty)$.



57.

$$\begin{aligned} r &= 2(h\cos\theta + k\sin\theta) \\ r^2 &= 2r(h\cos\theta + k\sin\theta) \\ r^2 &= 2\left[h(r\cos\theta) + k(r\sin\theta)\right] \\ x^2 + y^2 &= 2(hx + ky) \\ x^2 + y^2 - 2hx - 2ky &= 0 \\ \left(x^2 - 2hx + h^2\right) + \left(y^2 - 2ky + k^2\right) &= 0 + h^2 + k^2 \\ (x - h)^2 + (y - k)^2 &= h^2 + k^2 \end{aligned}$$

Radius: $\sqrt{h^2 + k^2}$

Center: (h, k)

58. (a) The rectangular coordinates of (r_1, θ_1) are $(r_1\cos\theta_1, r_1\sin\theta_1)$. The rectangular coordinates of (r_2, θ_2) are $(r_2\cos\theta_2, r_2\sin\theta_2)$.

$$\begin{aligned} d^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (r_2\cos\theta_2 - r_1\cos\theta_1)^2 + (r_2\sin\theta_2 - r_1\sin\theta_1)^2 \\ &= r_2^2\cos^2\theta_2 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\cos^2\theta_1 + r_2^2\sin^2\theta_2^2 - 2r_1r_2\sin\theta_1\sin\theta_2 + r_1^2\sin^2\theta_1 \\ &= r_2^2\left(\cos^2\theta_2 + \sin^2\theta_2\right) + r_1^2\left(\cos^2\theta_1 + \sin^2\theta_1\right) - 2r_1r_2\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right) \\ &= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2) \\ d &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)} \end{aligned}$$

(b) If $\theta_1 = \theta_2$, the points lie on the same line passing through the origin. In this case,

$$\begin{aligned} d &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(0)} \\ &= \sqrt{(r_1 - r_2)^2} = |r_1 - r_2|. \end{aligned}$$

(c) If $\theta_1 - \theta_2 = 90°$, then $\cos(\theta_1 - \theta_2) = 0$ and $d = \sqrt{r_1^2 + r_2^2}$, the Pythagorean Theorem!

(d) Many answers are possible. For example, consider the two points $(r_1, \theta_1) = (1, 0)$ and $(r_2, \theta_2) = \left(2, \frac{\pi}{2}\right)$.

$$d = \sqrt{1 + 2^2 - 2(1)(2)\cos\left(0 - \frac{\pi}{2}\right)} = \sqrt{5}$$

Using $(r_1, \theta_1) = (-1, \pi)$ and $(r_2, \theta_2) = \left[2, \left(\frac{5\pi}{2}\right)\right]$, $d = \sqrt{(-1)^2 + (2)^2 - 2(-1)(2)\cos\left(\pi - \frac{5\pi}{2}\right)} = \sqrt{5}.$

You always obtain the same distance.

59. $\left(1, \frac{5\pi}{6}\right), \left(4, \frac{\pi}{3}\right)$

$$d = \sqrt{1^2 + 4^2 - 2(1)(4)\cos\left(\frac{5\pi}{6} - \frac{\pi}{3}\right)}$$
$$= \sqrt{17 - 8\cos\frac{\pi}{2}} = \sqrt{17}$$

60. $\left(8, \frac{7\pi}{4}\right), (5, \pi)$

$$d = \sqrt{8^2 + 5^2 - 2(8)(5)\cos\left(\frac{7\pi}{4} - \pi\right)}$$
$$= \sqrt{89 - 80\cos\frac{3\pi}{4}}$$
$$= \sqrt{89 - 80\left(-\frac{\sqrt{2}}{2}\right)}$$
$$= \sqrt{89 + 40\sqrt{2}} \approx 12.0652$$

61. $(2, 0.5), (7, 1.2)$

$$d = \sqrt{2^2 + 7^2 - 2(2)(7)\cos(0.5 - 1.2)}$$
$$= \sqrt{53 - 28\cos(-0.7)} \approx 5.6$$

62. $(4, 2.5), (12, 1)$

$$d = \sqrt{4^2 + 12^2 - 2(4)(12)\cos(2.5 - 1)}$$
$$= \sqrt{160 - 96\cos 1.5} \approx 12.3$$

63. $r = 2 + 3\sin\theta$

$$\frac{dy}{dx} = \frac{3\cos\theta\sin\theta + \cos\theta(2 + 3\sin\theta)}{3\cos\theta\cos\theta - \sin\theta(2 + 3\sin\theta)}$$
$$= \frac{2\cos\theta(3\sin\theta + 1)}{3\cos 2\theta - 2\sin\theta} = \frac{2\cos\theta(3\sin\theta + 1)}{6\cos^2\theta - 2\sin\theta - 3}$$

At $\left(5, \frac{\pi}{2}\right)$, $\frac{dy}{dx} = 0$.

At $(2, \pi)$, $\frac{dy}{dx} = -\frac{2}{3}$.

At $\left(-1, \frac{3\pi}{2}\right)$, $\frac{dy}{dx} = 0$.

64. $r = 2(1 - \sin\theta)$

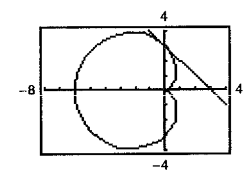
$$\frac{dy}{dx} = \frac{-2\cos\theta\sin\theta + 2\cos\theta(1 - \sin\theta)}{-2\cos\theta\cos\theta - 2\sin\theta(1 - \sin\theta)}$$

At $(2, 0)$, $\frac{dy}{dx} = -1$.

At $\left(3, \frac{7\pi}{6}\right)$, $\frac{dy}{dx}$ is undefined.

At $\left(4, \frac{3\pi}{2}\right)$, $\frac{dy}{dx} = 0$.

65. (a), (b) $r = 3(1 - \cos\theta)$



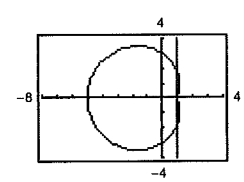
$(r, \theta) = \left(3, \frac{\pi}{2}\right) \Rightarrow (x, y) = (0, 3)$

Tangent line: $y - 3 = -1(x - 0)$

$y = -x + 3$

(c) At $\theta = \frac{\pi}{2}$, $\frac{dy}{dx} = -1.0$.

66. (a), (b) $r = 3 - 2\cos\theta$

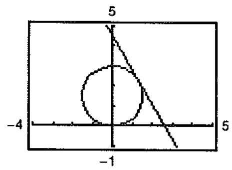


$(r, \theta) = (1, 0) \Rightarrow (x, y) = (1, 0)$

Tangent line: $x = 1$

(c) At $\theta = 0$, $\frac{dy}{dx}$ does not exist (vertical tangent).

67. (a), (b) $r = 3\sin\theta$



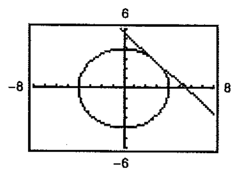
$$(r, \theta) = \left(\frac{3\sqrt{3}}{2}, \frac{\pi}{3}\right) \Rightarrow (x, y) = \left(\frac{3\sqrt{3}}{4}, \frac{9}{4}\right)$$

Tangent line: $y - \frac{9}{4} = -\sqrt{3}\left(x - \frac{3\sqrt{3}}{4}\right)$

$$y = -\sqrt{3}x + \frac{9}{2}$$

(c) At $\theta = \frac{\pi}{3}, \frac{dy}{dx} = -\sqrt{3} \approx -1.732$.

68. (a), (b) $r = 4$



$$(r, \theta) = \left(4, \frac{\pi}{4}\right) \Rightarrow (x, y) = \left(2\sqrt{2}, 2\sqrt{2}\right)$$

Tangent line: $y - 2\sqrt{2} = -1\left(x - 2\sqrt{2}\right)$

$$y = -x + 4\sqrt{2}$$

(c) At $\theta = \frac{\pi}{4}, \frac{dy}{dx} = -1$.

69. $r = 1 - \sin\theta$

$$\frac{dy}{d\theta} = (1 - \sin\theta)\cos\theta - \cos\theta\sin\theta$$
$$= \cos\theta(1 - 2\sin\theta) = 0$$

$\cos\theta = 0$ or $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$

Horizontal tangents: $\left(2, \frac{3\pi}{2}\right), \left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right)$

$$\frac{dx}{d\theta} = (-1 + \sin\theta)\sin\theta - \cos\theta\cos\theta$$
$$= -\sin\theta + \sin^2\theta + \sin^2\theta - 1$$
$$= 2\sin^2\theta - \sin\theta - 1$$
$$= (2\sin\theta + 1)(\sin\theta - 1) = 0$$

$\sin\theta = 1$ or $\sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

Vertical tangents: $\left(\frac{3}{2}, \frac{7\pi}{6}\right), \left(\frac{3}{2}, \frac{11\pi}{6}\right)$

70. $r = a\sin\theta$

$$\frac{dy}{d\theta} = a\sin\theta\cos\theta + a\cos\theta\sin\theta$$
$$= 2a\sin\theta\cos\theta = 0$$

$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

$$\frac{dx}{d\theta} = -a\sin^2\theta + a\cos^2\theta = a\left(1 - 2\sin^2\theta\right) = 0$$

$\sin\theta = \pm\frac{1}{\sqrt{2}}, \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Horizontal: $(0, 0), \left(a, \frac{\pi}{2}\right)$

Vertical: $\left(\frac{a\sqrt{2}}{2}, \frac{\pi}{4}\right), \left(\frac{a\sqrt{2}}{2}, \frac{3\pi}{4}\right)$

71. $r = 2\csc\theta + 3$

$$\frac{dy}{d\theta} = (2\csc\theta + 3)\cos\theta + (-2\csc\theta\cot\theta)\sin\theta$$
$$= 3\cos\theta = 0$$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

Horizontal: $\left(5, \frac{\pi}{2}\right), \left(1, \frac{3\pi}{2}\right)$

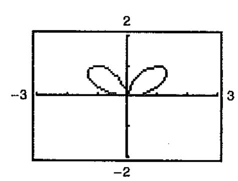
72. $r = a\sin\theta\cos^2\theta$

$$\frac{dy}{d\theta} = a\sin\theta\cos^3\theta + \left[-2a\sin^2\theta\cos\theta + a\cos^3\theta\right]\sin\theta$$
$$= 2a\left[\sin\theta\cos^3\theta - \sin^3\theta\cos\theta\right]$$
$$= 2a\sin\theta\cos\theta\left(\cos^2\theta - \sin^2\theta\right) = 0$$

$\theta = 0, \tan^2\theta = 1, \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

Horizontal: $\left(\frac{\sqrt{2}a}{4}, \frac{\pi}{4}\right), \left(\frac{\sqrt{2}a}{4}, \frac{3\pi}{4}\right), (0, 0)$

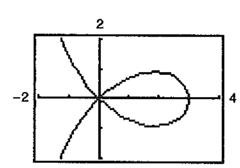
73. $r = 4\sin\theta\cos^2\theta$



Horizontal tangents:

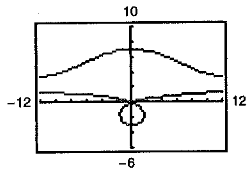
$(r, \theta) = (0, 0), (1.4142, 0.7854), (1.4142, 2.3562)$

74. $r = 3\cos 2\theta\sec\theta$



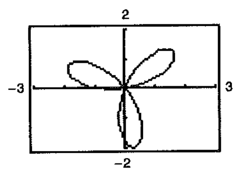
Horizontal tangents: $(r, \theta) = (2.061, \pm 0.452)$

75. $r = 2 \csc \theta + 5$



Horizontal tangents: $(r, \theta) = \left(7, \frac{\pi}{2}\right), \left(3, \frac{3\pi}{2}\right)$

76. $r = 2\cos(3\theta - 2)$



Horizontal tangents:

$(r, \theta) = (1.894, 0.776), (1.755, 2.594),$
$(1.998, -1.442), (-0.423, 0.072)$

77. $r = 5 \sin \theta$

$r^2 = 5r \sin \theta$

$x^2 + y^2 = 5y$

$x^2 + \left(y^2 - 5y + \frac{25}{4}\right) = \frac{25}{4}$

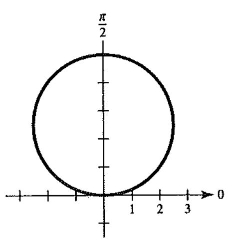
$x^2 + \left(y - \frac{5}{2}\right)^2 = \frac{25}{4}$

Circle: center: $\left(0, \frac{5}{2}\right)$, radius: $\frac{5}{2}$

Tangent at pole: $\theta = 0$

Note: $f(\theta) = r = 5 \sin \theta$

$f(0) = 0, f'(0) \neq 0$



78. $r = 5 \cos \theta$

$r^2 = 5r \cos \theta$

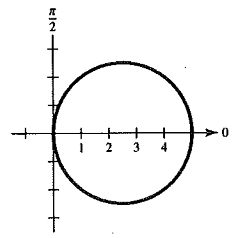
$x^2 + y^2 = 5x$

$\left(x^2 - 5x + \frac{25}{4}\right) + y^2 = \frac{25}{4}$

$\left(x - \frac{5}{2}\right)^2 + y^2 = \frac{25}{4}$

Circle: center: $\left(\frac{5}{2}, 0\right)$, radius: $\frac{5}{2}$

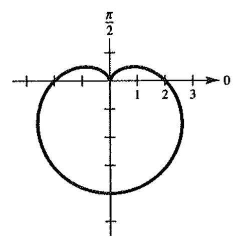
Tangent at pole: $\theta = \frac{\pi}{2}$.



79. $r = 2(1 - \sin \theta)$

Cardioid

Symmetric to y-axis, $\theta = \frac{\pi}{2}$

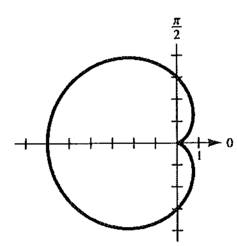


80. $r = 3(1 - \cos \theta)$

Cardioid

Symmetric to polar axis since r is a function of $\cos \theta$.

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	0	$\frac{3}{2}$	3	$\frac{9}{2}$	6

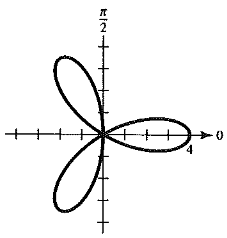


81. $r = 4\cos 3\theta$

Rose curve with three petals.

Tangents at pole: $(r = 0, r' \neq 0)$:

$$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$



82. $r = -\sin(5\theta)$

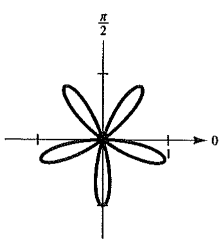
Rose curve with five petals

Symmetric to $\theta = \dfrac{\pi}{2}$

Relative extrema occur when

$$\frac{dr}{d\theta} = -5\cos(5\theta) = 0 \text{ at } \theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}.$$

Tangents at the pole: $\theta = 0, \dfrac{\pi}{5}, \dfrac{2\pi}{5}, \dfrac{3\pi}{5}, \dfrac{4\pi}{5}$



83. $r = 3\sin 2\theta$

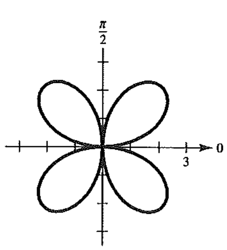
Rose curve with four petals

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $\left(\pm 3, \dfrac{\pi}{4}\right), \left(\pm 3, \dfrac{5\pi}{4}\right)$

Tangents at the pole: $\theta = 0, \dfrac{\pi}{2}$

$\left(\theta = \pi, \dfrac{3\pi}{2} \text{ give the same tangents.}\right)$



84. $r = 3\cos 2\theta$

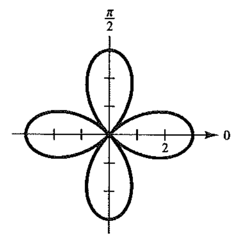
Rose curve with four petals

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $(3, 0), \left(-3, \dfrac{\pi}{2}\right), (3, \pi), \left(-3, \dfrac{3\pi}{2}\right)$

Tangents at the pole: $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

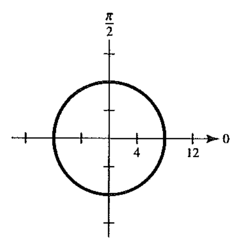
$\theta = \dfrac{5\pi}{4}$ and $\dfrac{7\pi}{4}$ given the same tangents.



85. $r = 8$

Circle radius 8

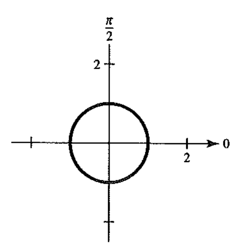
$x^2 + y^2 = 64$



86. $r = 1$

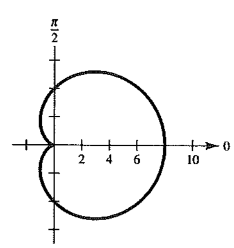
Circle radius 1

$x^2 + y^2 = 1$



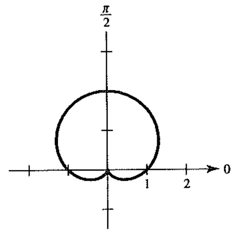
87. $r = 4(1 + \cos\theta)$

Cardioid



88. $r = 1 + \sin\theta$

Cardioid

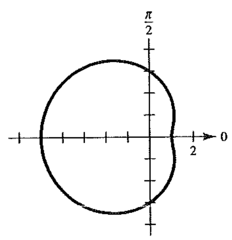


89. $r = 3 - 2\cos\theta$

Limaçon

Symmetric to polar axis

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	1	2	3	4	5

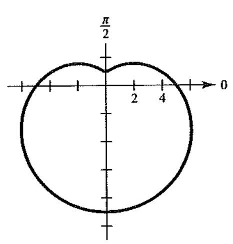


90. $r = 5 - 4\sin\theta$

Limaçon

Symmetric to $\theta = \frac{\pi}{2}$

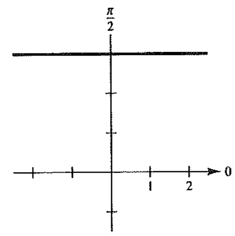
θ	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$
r	9	7	5	3	1



91.

$$r = 3\csc\theta$$
$$r\sin\theta = 3$$
$$y = 3$$

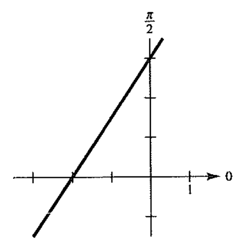
Horizontal line



92.

$$r = \frac{6}{2\sin\theta - 3\cos\theta}$$
$$2r\sin\theta - 3r\cos\theta = 6$$
$$2y - 3x = 6$$

Line



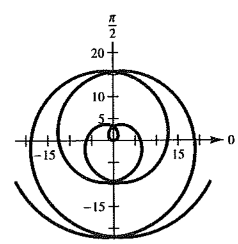
93. $r = 2\theta$

Spiral of Archimedes

Symmetric to $\theta = \frac{\pi}{2}$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π

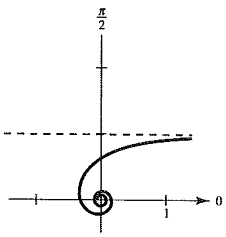
Tangent at the pole: $\theta = 0$



94. $r = \dfrac{1}{\theta}$

Hyperbolic spiral

θ	$\dfrac{\pi}{4}$	$\dfrac{\pi}{2}$	$\dfrac{3\pi}{4}$	π	$\dfrac{5\pi}{4}$	$\dfrac{3\pi}{2}$
r	$\dfrac{4}{\pi}$	$\dfrac{2}{\pi}$	$\dfrac{4}{3\pi}$	$\dfrac{1}{\pi}$	$\dfrac{4}{5\pi}$	$\dfrac{2}{3\pi}$



95. $r^2 = 4\cos(2\theta)$

$r = 2\sqrt{\cos 2\theta}, \quad 0 \le \theta \le 2\pi$

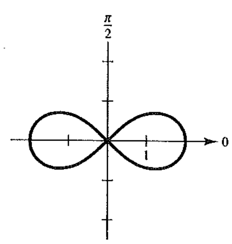
Lemniscate

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $(\pm 2, 0)$

θ	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$
r	± 2	$\pm\sqrt{2}$	0

Tangents at the pole: $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$



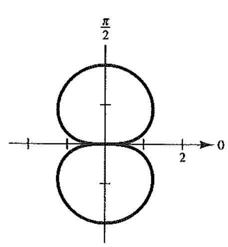
96. $r^2 = 4\sin\theta$

Lemniscate

Symmetric to the polar axis,

$\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $\left(\pm 2, \dfrac{\pi}{2}\right)$



θ	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{2}$	$\dfrac{5\pi}{6}$	π
r	0	$\pm\sqrt{2}$	± 2	$\pm\sqrt{2}$	0

Tangent at the pole: $\theta = 0$

97. Because

$$r = 2 - \sec\theta = 2 - \frac{1}{\cos\theta},$$

the graph has polar axis symmetry and the tangents at the pole are

$$\theta = \frac{\pi}{3}, -\frac{\pi}{3}.$$

Furthermore,

$$r \Rightarrow -\infty \text{ as } \theta \Rightarrow \frac{\pi}{2^-}$$

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow -\frac{\pi}{2^+}.$$

Also,

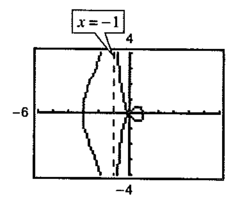
$$r = 2 - \frac{1}{\cos\theta}$$

$$= 2 - \frac{r}{r\cos\theta} = 2 - \frac{r}{x}$$

$$rx = 2x - r$$

$$r = \frac{2x}{1+x}.$$

So, $r \Rightarrow \pm\infty$ as $x \Rightarrow -1$.



98. Because

$$r = 2 + \csc\theta = 2 + \frac{1}{\sin\theta},$$

the graphs has symmetry with respect to $\theta = \pi/2$. Furthermore,

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow 0^+$$

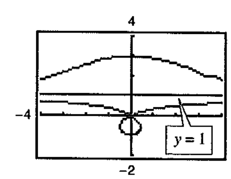
$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \pi^-.$$

Also, $r = 2 + \dfrac{1}{\sin\theta} = 2 + \dfrac{r}{r\sin\theta} = 2 + \dfrac{r}{y}$

$$ry = 2y + r$$

$$r = \frac{2y}{y-1}.$$

So, $r \Rightarrow \pm\infty$ as $y \Rightarrow 1$.



99. $r = \dfrac{2}{\theta}$

Hyperbolic spiral

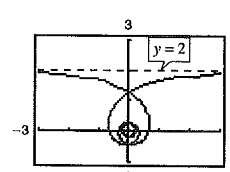
$r \Rightarrow \infty$ as $\theta \Rightarrow 0$

$$r = \frac{2}{\theta} \Rightarrow \theta = \frac{2}{r} = \frac{2\sin\theta}{r\sin\theta} = \frac{2\sin\theta}{y}$$

$$y = \frac{2\sin\theta}{\theta}$$

$$\lim_{\theta\to 0} \frac{2\sin\theta}{\theta} = \lim_{\theta\to 0} \frac{2\cos\theta}{1}$$

$$= 2$$



100. $r = 2\cos 2\theta \sec\theta$

Strophoid

$r \Rightarrow -\infty$ as $\theta \Rightarrow \dfrac{\pi^-}{2}$

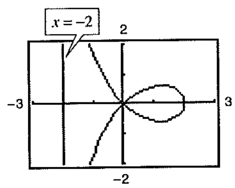
$r \Rightarrow \infty$ as $\theta \Rightarrow \dfrac{-\pi^+}{2}$

$$r = 2\cos 2\theta \sec\theta = 2\left(2\cos^2\theta - 1\right)\sec\theta$$

$$r\cos\theta = 4\cos^2\theta - 2$$

$$x = 4\cos^2\theta - 2$$

$$\lim_{\theta\to\pm\pi/2}\left(4\cos^2\theta - 2\right) = -2$$



101. The rectangular coordinate system consists of all points of the form (x, y) where x is the directed distance from the y-axis to the point, and y is the directed distance from the x-axis to the point.

Every point has a unique representation.

The polar coordinate system uses (r, θ) to designate the location of a point.

r is the directed distance to the origin and θ is the angle the point makes with the positive x-axis, measured counterclockwise.

Points do not have a unique polar representation.

102. $x = r\cos\theta,\ y = r\sin\theta$

$x^2 + y^2 = r^2,\ \tan\theta = \dfrac{y}{x}$

103. Slope of tangent line to graph of $r = f(\theta)$ at (r, θ) is

$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}.$$

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then $\theta = \alpha$ is tangent at the pole.

104. (a) $r = 7$: Circle radius 7 centered at origin

(b) $r^2 = 7$: Circle radius $\sqrt{7}$ centered at origin

(c) $r = \dfrac{7}{\cos\theta} \Rightarrow r\cos\theta = x = 7$: Vertical line through the point $(7, 0)$

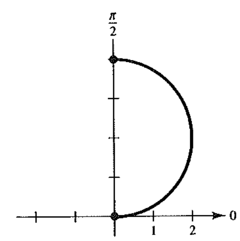
(d) $r = \dfrac{7}{\sin\theta} \Rightarrow r\sin\theta = y = 7$: Horizontal line through the point $(0, 7)$

(e) $r = 7\cos\theta$: Circle radius $\dfrac{7}{2}$ centered at $\left(\dfrac{7}{2}, 0\right)$

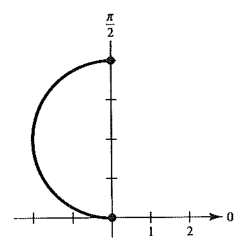
(f) $r = 7\sin\theta$: Circle radius $\dfrac{7}{2}$ centered at $\left(0, \dfrac{7}{2}\right)$

105. $r = 4\sin\theta$

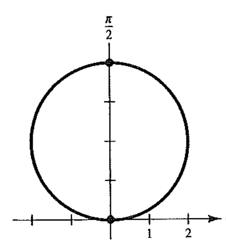
(a) $0 \le \theta \le \dfrac{\pi}{2}$



(b) $\dfrac{\pi}{2} \le \theta \le \pi$

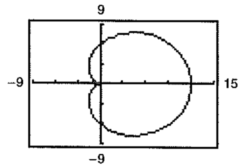


(c) $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

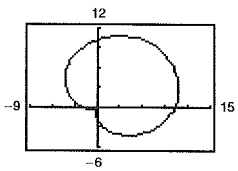


106. $r = 6[1 + \cos(\theta - \phi)]$

(a) $\phi = 0, r = 6[1 + \cos\theta]$



(b) $\theta = \dfrac{\pi}{4}, r = 6\left[1 + \cos\left(\theta - \dfrac{\pi}{4}\right)\right]$

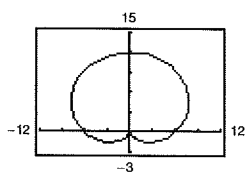


The graph of $r = 6[1 + \cos\theta]$ is rotated through the angle $\pi/4$.

(c) $\theta = \dfrac{\pi}{2}$

$$r = 6\left[1 + \cos\left(\theta - \frac{\pi}{2}\right)\right]$$

$$= 6\left[1 + \cos\theta\cos\frac{\pi}{2} + \sin\theta\sin\frac{\pi}{2}\right] = 6[1 + \sin\theta]$$



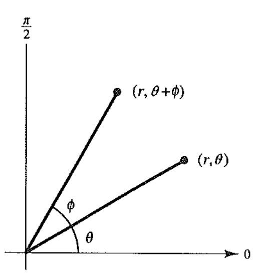
The graph of $r = 6[1 + \cos\theta]$ is rotated through the angle $\pi/2$.

107. Let the curve $r = f(\theta)$ be rotated by ϕ to form the curve $r = g(\theta)$. If (r_1, θ_1) is a point on $r = f(\theta)$, then $(r_1, \theta_1 + \phi)$ is on $r = g(\theta)$. That is,

$g(\theta_1 + \phi) = r_1 = f(\theta_1)$.

Letting $\theta = \theta_1 + \phi$, or $\theta_1 = \theta - \phi$, you see that

$g(\theta) = g(\theta_1 + \phi) = f(\theta_1) = f(\theta - \phi)$.



108. (a) $\sin\left(\theta - \dfrac{\pi}{2}\right) = \sin\theta\cos\left(\dfrac{\pi}{2}\right) - \cos\theta\sin\left(\dfrac{\pi}{2}\right)$

$$= -\cos\theta$$

$$r = f\left[\sin\left(\theta - \frac{\pi}{2}\right)\right]$$

$$= f(-\cos\theta)$$

(b) $\sin(\theta - \pi) = \sin\theta\cos\pi - \cos\theta\sin\pi$

$$= -\sin\theta$$

$$r = f[\sin(\theta - \pi)]$$

$$= f(-\sin\theta)$$

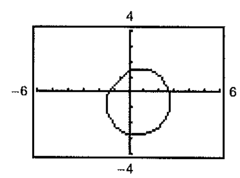
(c) $\sin\left(\theta - \dfrac{3\pi}{2}\right) = \sin\theta\cos\left(\dfrac{3\pi}{2}\right) - \cos\theta\sin\left(\dfrac{3\pi}{2}\right)$

$$= \cos\theta$$

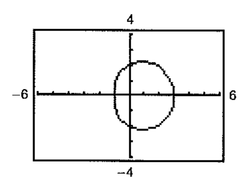
$$r = f\left[\sin\left(\theta - \frac{3\pi}{2}\right)\right] = f(\cos\theta)$$

109. $r = 2 - \sin\theta$

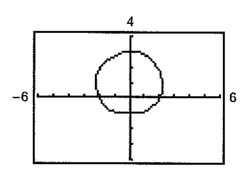
(a) $r = 2 - \sin\left(\theta - \dfrac{\pi}{4}\right) = 2 - \dfrac{\sqrt{2}}{2}(\sin\theta - \cos\theta)$



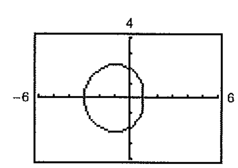
(b) $r = 2 - \sin\left(\theta - \dfrac{\pi}{2}\right) = 2 - (-\cos\theta) = 2 + \cos\theta$



(c) $r = 2 - \sin(\theta - \pi) = 2 - (-\sin\theta) = 2 + \sin\theta$

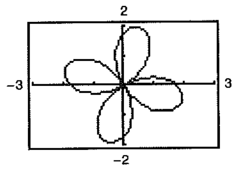


(d) $r = 2 - \sin\left(\theta - \dfrac{3\pi}{2}\right) = 2 - \cos\theta$

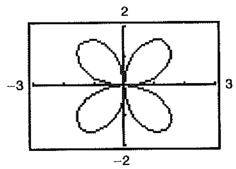


110. $r = 2\sin 2\theta = 4\sin\theta\cos\theta$

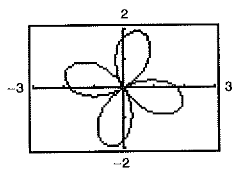
(a) $r = 4\sin\left(\theta - \dfrac{\pi}{6}\right)\cos\left(\theta - \dfrac{\pi}{6}\right)$



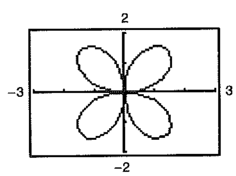
(b) $r = 4\sin\left(\theta - \dfrac{\pi}{2}\right)\cos\left(\theta - \dfrac{\pi}{2}\right) = -4\sin\theta\cos\theta$



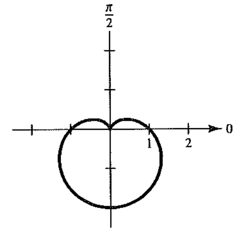
(c) $r = 4\sin\left(\theta - \dfrac{2\pi}{3}\right)\cos\left(\theta - \dfrac{2\pi}{3}\right)$



(d) $r = 4\sin(\theta - \pi)\cos(\theta - \pi) = 4\sin\theta\cos\theta$



111. (a) $r = 1 - \sin\theta$

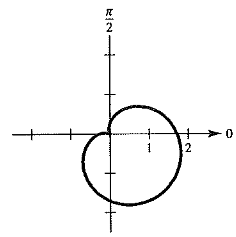


(b) $r = 1 - \sin\left(\theta - \dfrac{\pi}{4}\right)$

Rotate the graph of

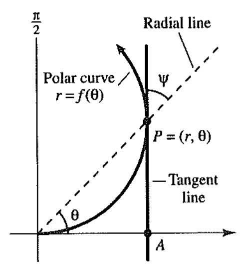
$r = 1 - \sin\theta$

through the angle $\pi/4$.



112. By Theorem 9.11, the slope of the tangent line through A and P is

$$\frac{f\cos\theta + f'\sin\theta}{-f\sin\theta + f'\cos\theta}.$$



This is equal to

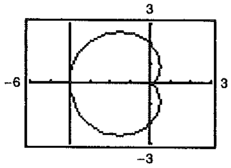
$$\tan(\theta + \psi) = \frac{\tan\theta + \tan\psi}{1 - \tan\theta\tan\psi} = \frac{\sin\theta + \cos\theta\tan\psi}{\cos\theta - \sin\theta\tan\psi}.$$

Equating the expressions and cross-multiplying, you obtain

$$\begin{aligned}
(f\cos\theta + f'\sin\theta)(\cos\theta - \sin\theta\tan\psi) &= (\sin\theta + \cos\theta\tan\psi)(-f\sin\theta + f'\cos\theta)\\
f\cos^2\theta - f\cos\theta\sin\theta\tan\psi + f'\sin\theta\cos\theta - f'\sin^2\theta\tan\psi &= -f\sin^2\theta - f\sin\theta\cos\theta\tan\psi + f'\sin\theta\cos\theta\\
&\quad + f'\cos^2\theta\tan\psi\\
f(\cos^2\theta + \sin^2\theta) &= f'\tan\psi(\cos^2\theta + \sin^2\theta)\\
\tan\psi &= \frac{f}{f'} = \frac{r}{dr/d\theta}.
\end{aligned}$$

113. $\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{2(1-\cos\theta)}{2\sin\theta}$

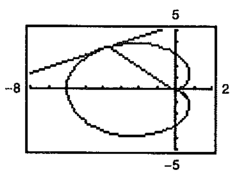
At $\theta = \pi$, $\tan\psi$ is undefined $\Rightarrow \psi = \dfrac{\pi}{2}$.



114. $\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{3(1-\cos\theta)}{3\sin\theta}$

At $\theta = \dfrac{3\pi}{4}$, $\tan\psi = \dfrac{1+\left(\sqrt{2}/2\right)}{\sqrt{2}/2} = \dfrac{2+\sqrt{2}}{\sqrt{2}}$.

$\psi = \arctan\left(\dfrac{2+\sqrt{2}}{\sqrt{2}}\right) \approx 1.178(\approx 67.5°)$

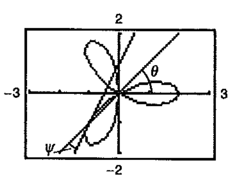


115. $r = 2\cos 3\theta$

$\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{2\cos 3\theta}{-6\sin 3\theta} = -\dfrac{1}{3}\cot 3\theta$

At $\theta = \dfrac{\pi}{4}$, $\tan\psi = -\dfrac{1}{3}\cot\left(\dfrac{3\pi}{4}\right) = \dfrac{1}{3}$.

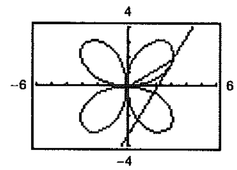
$\psi = \arctan\left(\dfrac{1}{3}\right) \approx 18.4°$



116. $\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{4\sin 2\theta}{8\cos 2\theta}$

At $\theta = \dfrac{\pi}{6}$, $\tan\psi = \dfrac{\sin(\pi/3)}{2\cos(\pi/3)} = \dfrac{\sqrt{3}}{2}$.

$\psi = \arctan\left(\dfrac{\sqrt{3}}{2}\right) \approx 0.7137(\approx 40.89°)$

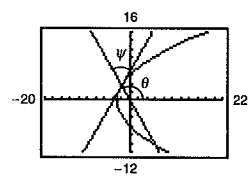


117. $r = \dfrac{6}{1-\cos\theta} = 6(1-\cos\theta)^{-1} \Rightarrow \dfrac{dr}{d\theta} = \dfrac{6\sin\theta}{(1-\cos\theta)^2}$

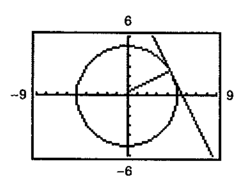
$\tan\psi = \dfrac{r}{\dfrac{dr}{d\theta}} = \dfrac{\dfrac{6}{(1-\cos\theta)}}{\dfrac{-6\sin\theta}{(1-\cos\theta)^2}} = \dfrac{1-\cos\theta}{-\sin\theta}$

At $\theta = \dfrac{2\pi}{3}$, $\tan\psi = \dfrac{1-\left(-\dfrac{1}{2}\right)}{-\dfrac{\sqrt{3}}{2}} = -\sqrt{3}$.

$\psi = \dfrac{\pi}{3}, (60°)$



118. $\tan\psi = \dfrac{r}{dr/d\theta} = \dfrac{5}{0}$ undefined $\Rightarrow \psi = \dfrac{\pi}{2}$



119. True

120. True

121. True

122. True

Section 10.5 Area and Arc Length in Polar Coordinates

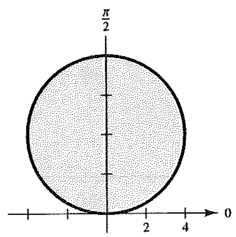
1. $A = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2\,d\theta$

$= \frac{1}{2}\int_0^{\pi/2}[4\sin\theta]^2\,d\theta = 8\int_0^{\pi/2}\sin^2\theta\,d\theta$

2. $A = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2\,d\theta = \frac{1}{2}\int_{3\pi/4}^{5\pi/4}(\cos 2\theta)^2\,d\theta$

3. $A = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2\,d\theta = \frac{1}{2}\int_{\pi/2}^{3\pi/2}[3 - 2\sin\theta]^2\,d\theta$

4. (a) $r = 8\sin\theta$



$A = \pi(4)^2 = 16\pi$

(b) $A = 2\left(\frac{1}{2}\right)\int_0^{\pi/2}[8\sin\theta]^2\,d\theta$

$= 64\int_0^{\pi/2}\sin^2\theta\,d\theta$

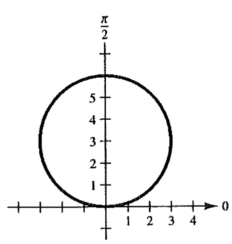
$= 32\int_0^{\pi/2}(1 - \cos 2\theta)\,d\theta$

$= 32\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = 16\pi$

5. $A = \frac{1}{2}\int_0^{\pi}[6\sin\theta]^2\,d\theta$

$= 18\int_0^{\pi}\frac{1 - \cos 2\theta}{2}\,d\theta = 9\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = 9\pi$

Note: $r = 6\sin\theta$ is circle of radius 3, $0 \le \theta \le \pi$.

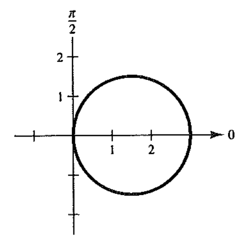


6. $A = \frac{1}{2}\int_0^{\pi}[3\cos\theta]^2\,d\theta$

$= \frac{9}{2}\int_0^{\pi}\frac{1 + \cos 2\theta}{2}\,d\theta$

$= \frac{9}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi} = \frac{9}{4}\pi$

Note: $r = 3\cos\theta$ is circle of radius $\frac{3}{2}$, $0 \le \theta \le \pi$.



7. $A = 2\left[\frac{1}{2}\int_0^{\pi/6}(2\cos 3\theta)^2\,d\theta\right] = 2\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = \frac{\pi}{3}$

8. $A = \frac{1}{2}\int_0^{\pi/3}[4\sin 3\theta]^2\,d\theta$

$= 8\int_0^{\pi/3}\sin^2 3\theta\,d\theta$

$= 8\int_0^{\pi/3}\frac{1 - \cos 6\theta}{2}\,d\theta$

$= 4\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/3}$

$= 4\left[\frac{\pi}{3}\right] = \frac{4\pi}{3}$

9. $A = \frac{1}{2}\int_0^{\pi/2}[\sin 2\theta]^2\,d\theta$

$= \frac{1}{2}\int_0^{\pi/2}\frac{1 - \cos 4\theta}{2}\,d\theta$

$= \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2}$

$= \frac{1}{4}\left[\frac{\pi}{2}\right] = \frac{\pi}{8}$

10. $A = 2\left[\frac{1}{2}\int_0^{\pi/10}(\cos 5\theta)^2\,d\theta\right]$

$= \frac{1}{2}\left[\theta + \frac{1}{10}\sin(10\theta)\right]_0^{\pi/10} = \frac{\pi}{20}$

11. $A = 2\left[\frac{1}{2}\int_{-\pi/2}^{\pi/2}(1 - \sin\theta)^2\,d\theta\right]$

$= \left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}$

12. $A = 2\left[\frac{1}{2}\int_0^{\pi/2} (1 - \sin\theta)^2\, d\theta\right]$

$= \left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{3\pi - 8}{4}$

13. $A = \frac{1}{2}\int_0^{2\pi} [5 + 2\sin\theta]^2\, d\theta$

$= \frac{1}{2}\int_0^{2\pi} \left[25 + 20\sin\theta + 4\sin^2\theta\right] d\theta$

$= \frac{1}{2}\int_0^{2\pi} \left[25 + 20\sin\theta + 2(1 - \cos 2\theta)\right] d\theta$

$= \frac{1}{2}[27\theta - 20\cos\theta - \sin 2\theta]_0^{2\pi}$

$= \frac{1}{2}\left[27(2\pi)\right] = 27\pi$

14. $A = \frac{1}{2}\int_0^{2\pi} [4 - 4\cos\theta]^2\, d\theta$

$= 8\int_0^{2\pi} (1 - \cos\theta)^2\, d\theta$

$= 8\int_0^{2\pi} \left(1 - 2\cos\theta + \cos^2\theta\right) d\theta$

$= 8\int_0^{2\pi} \left(1 - 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta$

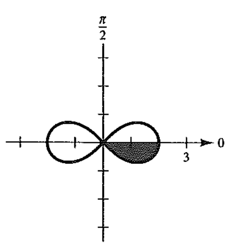
$= 8\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$

$= 8\left[\frac{3}{2}(2\pi)\right] = 24\pi$

15. On the interval $-\frac{\pi}{4} \le \theta \le 0$, $r = 2\sqrt{\cos 2\theta}$ traces out one-half of one leaf of the lemniscate. So,

$A = 4\frac{1}{2}\int_{-\pi/4}^{0} 4\cos 2\theta\, d\theta$

$= 8\left[\frac{\sin 2\theta}{2}\right]_{-\pi/4}^{0} = 8\left[\frac{1}{2}\right] = 4.$



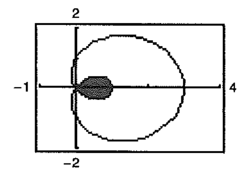
16. On the interval $0 \le \theta \le \pi/2$, $r = \sqrt{6\sin 2\theta}$ traces out half of the lemniscate. So

$A = 2 \cdot \frac{1}{2}\int_0^{\pi/2} 6\sin 2\theta\, d\theta$

$= 6\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2} = 6\left[\frac{1}{2} + \frac{1}{2}\right] = 6.$

17. $A = \left[2\frac{1}{2}\int_{2\pi/3}^{\pi} (1 + 2\cos\theta)^2\, d\theta\right]$

$= [3\theta + 4\sin\theta + \sin 2\theta]_{2\pi/3}^{\pi} = \frac{2\pi - 3\sqrt{3}}{2}$



18. Half of the inner loop of $r = 2 - 4\cos\theta$ is traced out on the interval $0 \le \theta \le \frac{\pi}{3}$. So

$A = 2 \cdot \frac{1}{2}\int_0^{\pi/3} (2 - 4\cos\theta)^2\, d\theta$

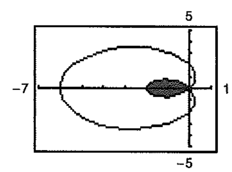
$= \int_0^{\pi/3} \left[4 - 16\cos\theta + 16\cos^2\theta\right] d\theta$

$= \int_0^{\pi/3} \left[4 - 16\cos\theta + 8[1 + \cos 2\theta]\right] d\theta$

$= [12\theta - 16\sin\theta + 4\sin 2\theta]_0^{\pi/3}$

$= 12(\pi/3) - 16\left(\sqrt{3}/2\right) + 4\left(\sqrt{3}/2\right)$

$= 4\pi - 6\sqrt{3}.$



19. The inner loop of $r = 1 + 2\sin\theta$ is traced out on the interval $\frac{7\pi}{6} \le \theta \le \frac{11\pi}{6}$. So,

$A = \frac{1}{2}\int_{7\pi/6}^{11\pi/6} [1 + 2\sin\theta]^2\, d\theta$

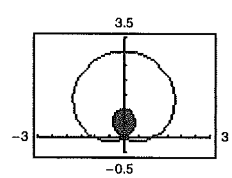
$= \frac{1}{2}\int_{7\pi/6}^{11\pi/6} \left[1 + 4\sin\theta + 4\sin^2\theta\right] d\theta$

$= \frac{1}{2}\int_{7\pi/6}^{11\pi/6} \left[1 + 4\sin\theta + 2(1 - \cos 2\theta)\right] d\theta$

$= \frac{1}{2}[3\theta - 4\cos\theta - \sin 2\theta]_{7\pi/6}^{11\pi/6}$

$= \frac{1}{2}\left[\left(\frac{11\pi}{2} - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right) - \left(\frac{7\pi}{2} + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right)\right]$

$= \frac{1}{2}\left[2\pi - 3\sqrt{3}\right].$

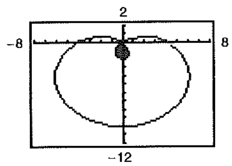


20. $A = 2\left[\frac{1}{2}\int_{\arcsin(2/3)}^{\pi/2} (4 - 6\sin\theta)^2\, d\theta\right]$

$= \int_{\arcsin(2/3)}^{\pi/2} \left[16 - 48\sin\theta + 36\sin^2\theta\right] d\theta$

$= \int_{\arcsin(2/3)}^{\pi/2} \left[16 - 48\sin\theta + 36\left(\frac{1 - \cos 2\theta}{2}\right)\right] d\theta$

$= \left[34\theta + 48\cos\theta - 9\sin 2\theta\right]_{\arcsin(2/3)}^{\pi/2} \approx 1.7635$



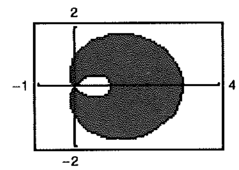
21. The area inside the outer loop is

$2\left[\frac{1}{2}\int_0^{2\pi/3} (1 + 2\cos\theta)^2\, d\theta\right] = \left[3\theta + 4\sin\theta + \sin 2\theta\right]_0^{2\pi/3}$

$= \frac{4\pi + 3\sqrt{3}}{2}.$

From the result of Exercise 17, the area between the loops is

$A = \left(\frac{4\pi + 3\sqrt{3}}{2}\right) - \left(\frac{2\pi - 3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3}.$



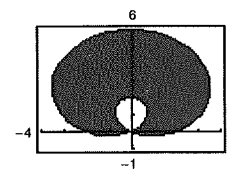
22. Four times the area in Exercise 21, $A = 4\left(\pi + 3\sqrt{3}\right)$. More specifically, you see that the area inside the outer loop is

$2\left[\frac{1}{2}\int_{-\pi/6}^{\pi/2} \left(2(1 + 2\sin\theta)\right)^2\, d\theta\right] = \int_{-\pi/6}^{\pi/2} \left(4 + 16\sin\theta + 16\sin^2\theta\right) d\theta = 8\pi + 6\sqrt{3}.$

The area inside the inner loop is

$2\frac{1}{2}\left[\int_{7\pi/6}^{3\pi/2} \left(2(1 + 2\sin\theta)\right)^2\, d\theta\right] = 4\pi - 6\sqrt{3}.$

So, the area between the loops is $\left(8\pi + 6\sqrt{3}\right) - \left(4\pi - 6\sqrt{3}\right) = 4\pi + 12\sqrt{3}.$



23. The area inside the outer loop is

$A = 2\cdot\frac{1}{2}\int_{5\pi/6}^{3\pi/2} \left[3 - 6\sin\theta\right]^2 d\theta$

$= \int_{5\pi/6}^{3\pi/2} \left[9 - 36\sin\theta + 36\sin^2\theta\right] d\theta$

$= \int_{5\pi/6}^{3\pi/2} \left[9 - 36\sin\theta + 18(1 - \cos 2\theta)\right] d\theta$

$= \left[27\theta + 36\cos\theta - 9\sin 2\theta\right]_{5\pi/6}^{3\pi/2} = \left[\frac{81\pi}{2} - \left(\frac{45\pi}{2} - 18\sqrt{3} + \frac{9\sqrt{3}}{2}\right)\right] = 18\pi + \frac{27\sqrt{3}}{2}.$

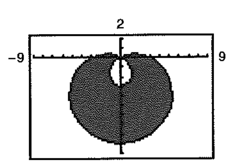
The area inside the inner loop is

$A = 2\cdot\frac{1}{2}\int_{\pi/6}^{\pi/2} \left[3 - 6\sin\theta\right]^2 d\theta$

$= \left[27\theta + 36\cos\theta - 9\sin 2\theta\right]_{\pi/6}^{\pi/2} = \left[\frac{27\pi}{2} - \left(\frac{9\pi}{2} + 18\sqrt{3} - \frac{9\sqrt{3}}{2}\right)\right] = 9\pi - \frac{27\sqrt{3}}{2}.$

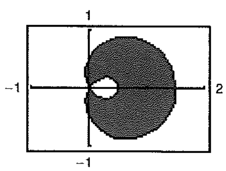
Finally, the area between the loops is

$\left[18\pi + \frac{27\sqrt{3}}{2}\right] - \left[9\pi - \frac{27\sqrt{3}}{2}\right] = 9\pi + 27\sqrt{3}.$



24. The area inside the outer loop is

$$A = 2 \cdot \frac{1}{2}\int_0^{2\pi/3}\left[\frac{1}{2} + \cos\theta\right]^2 d\theta$$
$$= \int_0^{2\pi/3}\left[\frac{1}{4} + \cos\theta + \frac{1 + \cos 2\theta}{2}\right] d\theta$$
$$= \left[\frac{3}{4}\theta + \sin\theta + \frac{\sin 2\theta}{4}\right]_0^{2\pi/3}$$
$$= \frac{3}{4}\left(\frac{2\pi}{3}\right) + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8}$$
$$= \frac{\pi}{2} + \frac{3\sqrt{3}}{8}.$$



The area inside the inner loop is

$$A = 2 \cdot \frac{1}{2}\int_{2\pi/3}^{\pi}\left[\frac{1}{2} + \cos\theta\right]^2 d\theta$$
$$= \left[\frac{3}{4}\theta + \sin\theta + \frac{\sin 2\theta}{4}\right]_{2\pi/3}^{\pi}$$
$$= \frac{3}{4}\pi - \left(\frac{\pi}{2} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8}\right) = \frac{\pi}{4} - \frac{3\sqrt{3}}{8}$$

Finally, the area between the loops is

$$\left[\frac{\pi}{2} + \frac{3\sqrt{3}}{8}\right] - \left[\frac{\pi}{4} - \frac{3\sqrt{3}}{8}\right] = \frac{\pi}{4} + \frac{3\sqrt{3}}{4}.$$

25. $r = 1 + \cos\theta$

$r = 1 - \cos\theta$

Solving simultaneously,

$$1 + \cos\theta = 1 - \cos\theta$$
$$2\cos\theta = 0$$
$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}.$$

Replacing r by $-r$ and θ by $\theta + \pi$ in the first equation and solving, $-1 + \cos\theta = 1 - \cos\theta$, $\cos\theta = 1$, $\theta = 0$. Both curves pass through the pole, $(0, \pi)$, and $(0, 0)$, respectively.

Points of intersection: $\left(1, \frac{\pi}{2}\right), \left(1, \frac{3\pi}{2}\right), (0, 0)$

26. $r = 3(1 + \sin\theta)$

$r = 3(1 - \sin\theta)$

Solving simultaneously,

$$3(1 + \sin\theta) = 3(1 - \sin\theta)$$
$$2\sin\theta = 0$$
$$\theta = 0, \pi.$$

Replacing r by $-r$ and θ by $\theta + \pi$ in the first equation and solving, $-3(1 - \sin\theta) = 3(1 - \sin\theta)$, $\sin\theta = 1$, $\theta = \pi/2$. Both curves pass through the pole, $(0, 3\pi/2)$, and $(0, \pi/2)$, respectively.

Points of intersection: $(3, 0), (3, \pi), (0, 0)$

27. $r = 1 + \cos\theta$

$r = 1 - \sin\theta$

Solving simultaneously,

$$1 + \cos\theta = 1 - \sin\theta$$
$$\cos\theta = -\sin\theta$$
$$\tan\theta = -1$$
$$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}.$$

Replacing r by $-r$ and θ by $\theta + \pi$ in the first equation and solving, $-1 + \cos\theta = 1 - \sin\theta$, $\sin\theta + \cos\theta = 2$, which has no solution. Both curves pass through the pole, $(0, \pi)$, and $(0, \pi/2)$, respectively.

Points of intersection:

$\left(\frac{2 - \sqrt{2}}{2}, \frac{3\pi}{4}\right), \left(\frac{2 + \sqrt{2}}{2}, \frac{7\pi}{4}\right), (0, 0)$

28. $r = 2 - 3\cos\theta$

$r = \cos\theta$

Solving simultaneously,

$$2 - 3\cos\theta = \cos\theta$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}.$$

Both curves pass through the pole, $(0, \arccos 2/3)$, and $(0, \pi/2)$, respectively.

Points of intersection: $\left(\frac{1}{2}, \frac{\pi}{3}\right), \left(\frac{1}{2}, \frac{5\pi}{3}\right), (0, 0)$

29. $r = 4 - 5\sin\theta$

$r = 3\sin\theta$

Solving simultaneously,

$4 - 5\sin\theta = 3\sin\theta$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

Both curves pass through the pole, $(0, \arcsin 4/5)$, and $(0, 0)$, respectively.

Points of intersection: $\left(\frac{3}{2}, \frac{\pi}{6}\right), \left(\frac{3}{2}, \frac{5\pi}{6}\right), (0, 0)$

30. $r = 1 + \cos\theta$

$r = 3\cos\theta$

Solving simultaneously,

$1 + \cos\theta = 3\cos\theta$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}.$$

Both curves pass through the pole, $(0, \pi)$, and $(0, \pi/2)$, respectively.

Points of intersection: $\left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, \frac{5\pi}{3}\right), (0, 0)$

31. $r = \frac{\theta}{2}$

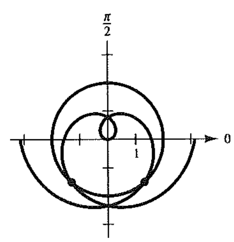
$r = 2$

Solving simultaneously, you have

$\theta/2 = 2, \theta = 4.$

Points of intersection:

$(2, 4), (-2, -4)$

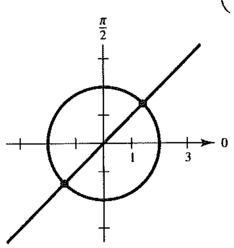


32. $\theta = \frac{\pi}{4}$

$r = 2$

Line of slope 1 passing through the pole and a circle of radius 2 centered at the pole.

Points of intersection: $\left(2, \frac{\pi}{4}\right), \left(-2, \frac{\pi}{4}\right)$



33. $r = 2\sin 2\theta$

$r = 1$

$r = 2\sin 2\theta$ is a 4-leaved rose curve. The circle $r = 1$ intersects at 8 points. For the petal in the first quadrant,

$2\sin 2\theta = 1$

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

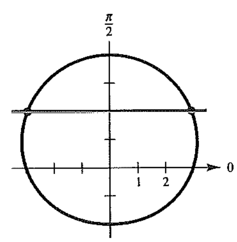
The points of intersection for one petal are $\left(1, \frac{\pi}{12}\right)$, $\left(1, \frac{5\pi}{12}\right)$. By symmetry, the other points are

$\left(1, \frac{7\pi}{12}\right), \left(1, \frac{11\pi}{12}\right), \left(1, \frac{13\pi}{12}\right), \left(1, \frac{17\pi}{12}\right),$

$\left(1, \frac{19\pi}{12}\right), \left(1, \frac{23\pi}{12}\right).$

34. $r = 3 + \sin\theta$

$r = 2\csc\theta$



The graph of $r = 3 + \sin\theta$ is a limaçon symmetric to $\theta = \pi/2$, and the graph of $r = 2\csc\theta$ is the horizontal line $y = 2$. So, there are two points of intersection.

Solving simultaneously,

$$3 + \sin\theta = 2\csc\theta$$

$$\sin^2\theta + 3\sin\theta - 2 = 0$$

$$\sin\theta = \frac{-3 \pm \sqrt{17}}{2}$$

$$\theta = \arcsin\left(\frac{\sqrt{17} - 3}{2}\right) \approx 0.596.$$

Points of intersection:

$\left(\frac{\sqrt{17} + 3}{2}, \arcsin\left(\frac{\sqrt{17} - 3}{2}\right)\right),$

$\left(\frac{\sqrt{17} + 3}{2}, \pi - \arcsin\left(\frac{\sqrt{17} - 3}{2}\right)\right),$

$(3.56, 0.596), (3.56, 2.545)$

35. $r = 2 + 3\cos\theta$

$r = \dfrac{\sec\theta}{2}$

The graph of $r = 2 + 3\cos\theta$ is a limaçon with an inner loop $(b > a)$ and is symmetric to the polar axis. The graph of $r = (\sec\theta)/2$ is the vertical line $x = 1/2$. So, there are four points of intersection. Solving simultaneously,

$$2 + 3\cos\theta = \frac{\sec\theta}{2}$$

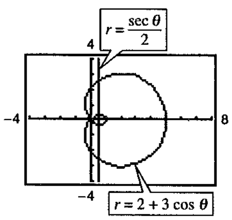
$$6\cos^2\theta + 4\cos\theta - 1 = 0$$

$$\cos\theta = \frac{-2 \pm \sqrt{10}}{6}$$

$$\theta = \arccos\left(\frac{-2 + \sqrt{10}}{6}\right) \approx 1.376$$

$$\theta = \arccos\left(\frac{-2 - \sqrt{10}}{6}\right) \approx 2.6068.$$

Points of intersection:
$(-0.581, \pm 2.607), (2.581, \pm 1.376)$



36. $r = 3(1 - \cos\theta)$

$r = \dfrac{6}{1 - \cos\theta}$

The graph of $r = 3(1 - \cos\theta)$ is a cardioid with polar axis symmetry. The graph of

$r = 6/(1 - \cos\theta)$

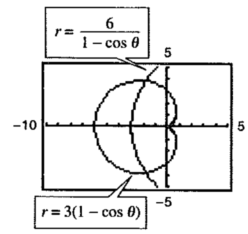
is a parabola with focus at the pole, vertex$(3, \pi)$, and polar axis symmetry. So, there are two points of intersection. Solving simultaneously,

$$3(1 - \cos\theta) = \frac{6}{1 - \cos\theta}$$

$$(1 - \cos\theta)^2 = 2$$

$$\cos\theta = 1 \pm \sqrt{2}$$

$$\theta = \arccos\left(1 - \sqrt{2}\right).$$



Points of intersection:

$\left(3\sqrt{2}, \arccos\left(1 - \sqrt{2}\right)\right) \approx (4.243, 1.998)$,

$\left(3\sqrt{2}, 2\pi - \arccos\left(1 - \sqrt{2}\right)\right) \approx (4.243, 4.285)$

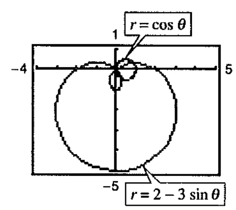
37. $r = \cos\theta$

$r = 2 - 3\sin\theta$

Points of intersection:

$(0, 0), (0.935, 0.363), (0.535, -1.006)$

The graphs reach the pole at different times (θ values).

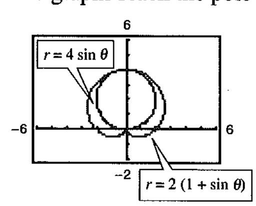


38. $r = 4\sin\theta$

$r = 2(1 + \sin\theta)$

Points of intersection: $(0, 0), \left(4, \dfrac{\pi}{2}\right)$

The graphs reach the pole at different times (θ values).



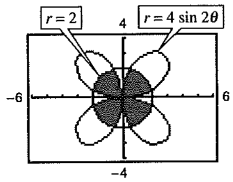
39. From Exercise 25, the points of intersection for one petal are $(2, \pi/12)$ and $(2, 5\pi/12)$. The area within one petal is

$$A = \frac{1}{2}\int_0^{\pi/12} (4 \sin 2\theta)^2 \, d\theta + \frac{1}{2}\int_{\pi/12}^{5\pi/12} (2)^2 \, d\theta + \frac{1}{2}\int_{5\pi/12}^{\pi/2} (4 \sin 2\theta)^2 \, d\theta$$

$$= 16\int_0^{\pi/12} \sin^2(2\theta) \, d\theta + 2\int_{\pi/12}^{5\pi/12} d\theta \text{ (by symmetry of the petal)}$$

$$= 8\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/12} + \left[2\theta\right]_{\pi/12}^{5\pi/12} = \frac{4\pi}{3} - \sqrt{3}.$$

$$\text{Total area} = 4\left(\frac{4\pi}{3} - \sqrt{3}\right) = \frac{16\pi}{3} - 4\sqrt{3} = \frac{4}{3}\left(4\pi - 3\sqrt{3}\right)$$



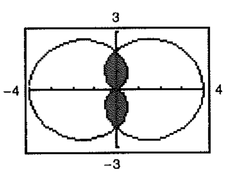
40. The common interior is 4 times the area in the first quadrant.

$$A = 4\frac{1}{2}\int_0^{\pi/2} \left[2(1 - \cos\theta)\right]^2 \, d\theta$$

$$= 8\int_0^{\pi/2} \left(1 - 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta$$

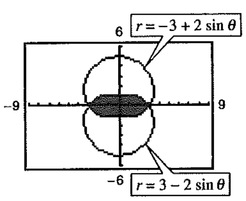
$$= 8\left[\frac{3\theta}{2} - 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= 8\left[\frac{3}{2}\left(\frac{\pi}{2}\right) - 2\right] = 6\pi - 16$$



41. $A = 4\left[\frac{1}{2}\int_0^{\pi/2} (3 - 2\sin\theta)^2 \, d\theta\right]$

$$= 2\left[11\theta + 12\cos\theta - \sin(2\theta)\right]_0^{\pi/2} = 11\pi - 24$$



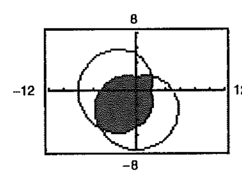
42. $r = 5 - 3\sin\theta$ and $r = 5 - 3\cos\theta$ intersect at $\theta = \pi/4$ and $\pi = 5\pi/4$.

$$A = 2\left[\frac{1}{2}\int_{\pi/4}^{5\pi/4} (5 - 3\sin\theta)^2 \, d\theta\right]$$

$$= \left[\frac{59}{2}\theta + 30\cos\theta - \frac{9}{4}\sin 2\theta\right]_{\pi/4}^{5\pi/4}$$

$$= \left(\frac{59}{2}\left(\frac{5\pi}{4}\right) - 30\frac{\sqrt{2}}{2} - \frac{9}{4}\right) - \left(\frac{59}{2}\left(\frac{\pi}{4}\right) + 30\frac{\sqrt{2}}{2} - \frac{9}{4}\right)$$

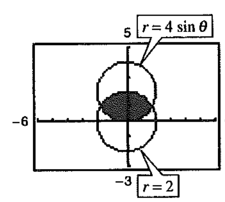
$$= \frac{59\pi}{2} - 30\sqrt{2} \approx 50.251$$



43. $A = 2\left[\frac{1}{2}\int_0^{\pi/6} (4\sin\theta)^2 \, d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} (2)^2 \, d\theta\right]$

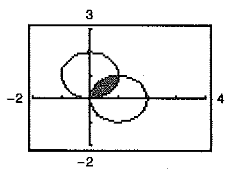
$$= 16\left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right]_0^{\pi/6} + \left[4\theta\right]_{\pi/6}^{\pi/2}$$

$$= \frac{8\pi}{3} - 2\sqrt{3} = \frac{2}{3}\left(4\pi - 3\sqrt{3}\right)$$



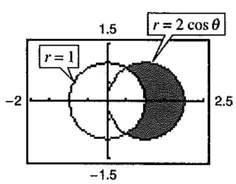
44. The common interior is given by

$$A = 2\frac{1}{2}\int_{\pi/4}^{\pi/2} [2\cos\theta]^2\,d\theta$$
$$= 4\int_{\pi/4}^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta$$
$$= 2\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/2}$$
$$= 2\left[\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right]$$
$$= \frac{\pi}{2} - 1$$



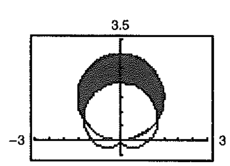
45. $r = 2\cos\theta = 1 \Rightarrow \theta = \pi/3$

$$A = 2\cdot\frac{1}{2}\int_0^{\pi/3}\left([2\cos\theta]^2 - 1\right)d\theta$$
$$= \int_0^{\pi/3}\left[2(1+\cos 2\theta) - 1\right]d\theta$$
$$= \left[\theta + \sin 2\theta\right]_0^{\pi/3}$$
$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$



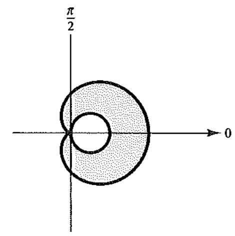
46. $3\sin\theta = 1 + \sin\theta \Rightarrow \sin\theta = 1/2 \Rightarrow \theta = \pi/6$

$$A = 2\cdot\frac{1}{2}\int_{\pi/6}^{\pi/2}\left([3\sin\theta]^2 - [1+\sin\theta]^2\right)d\theta$$
$$= \int_{\pi/6}^{\pi/2}\left[9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta\right]d\theta$$
$$= \int_{\pi/6}^{\pi/2}\left[4(1-\cos 2\theta) - 1 - 2\sin\theta\right]d\theta$$
$$= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\pi/6}^{\pi/2}$$
$$= 3\frac{\pi}{2} - 3\frac{\pi}{6} + 2\frac{\sqrt{3}}{2} - 2\frac{\sqrt{3}}{2}$$
$$= \pi$$



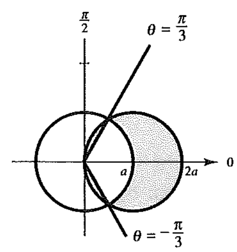
47. $A = 2\left[\frac{1}{2}\int_0^{\pi}\left[a(1+\cos\theta)\right]^2 d\theta\right] - \frac{a^2\pi}{4}$

$$= a^2\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi} - \frac{a^2\pi}{4}$$
$$= \frac{3a^2\pi}{2} - \frac{a^2\pi}{4} = \frac{5a^2\pi}{4}$$



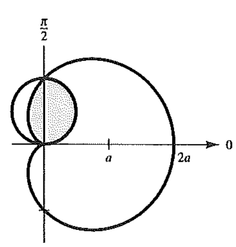
48. Area = Area of $r = 2a\cos\theta$ − Area of sector − twice area between $r = 2a\cos\theta$ and the lines

$$\theta = \frac{\pi}{3}, \theta = \frac{\pi}{2}.$$
$$A = \pi a^2 - \left(\frac{\pi}{3}\right)a^2 - 2\left[\frac{1}{2}\int_{\pi/3}^{\pi/2}(2a\cos\theta)^2\,d\theta\right]$$
$$= \frac{2\pi a^2}{3} - 2a^2\int_{\pi/3}^{\pi/2}(1+\cos 2\theta)\,d\theta$$
$$= \frac{2\pi a^2}{3} - 2a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/3}^{\pi/2}$$
$$= \frac{2\pi a^2}{3} - 2a^2\left[\frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4}\right] = \frac{2\pi a^2 + 3\sqrt{3}a^2}{6}$$



49. $A = \frac{\pi a^2}{8} + \frac{1}{2}\int_{\pi/2}^{\pi}\left[a(1+\cos\theta)\right]^2 d\theta$

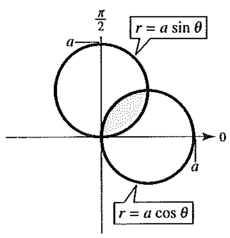
$$= \frac{\pi a^2}{8} + \frac{a^2}{2}\int_{\pi/2}^{\pi}\left(\frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2}\right)d\theta$$
$$= \frac{\pi a^2}{8} + \frac{a^2}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_{\pi/2}^{\pi}$$
$$= \frac{\pi a^2}{8} + \frac{a^2}{2}\left[\frac{3\pi}{2} - \frac{3\pi}{4} - 2\right] = \frac{a^2}{2}[\pi - 2]$$



50. $r = a\cos\theta,\ r = a\sin\theta$

$\tan\theta = 1,\ \theta = \pi/4$

$$\begin{aligned}A &= 2\left[\frac{1}{2}\int_0^{\pi/4}(a\sin\theta)^2\,d\theta\right]\\ &= a^2\int_0^{\pi/4}\frac{1-\cos 2\theta}{2}\,d\theta\\ &= \frac{1}{2}a^2\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/4}\\ &= \frac{1}{2}a^2\left[\frac{\pi}{4} - \frac{1}{2}\right]\\ &= \frac{1}{8}a^2\pi - \frac{1}{4}a^2\end{aligned}$$

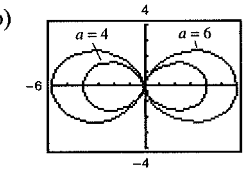


51. (a) $r = a\cos^2\theta$

$r^3 = ar^2\cos^2\theta$

$\left(x^2 + y^2\right)^{3/2} = ax^2$

(b)



(c) $$\begin{aligned}A &= 4\left(\frac{1}{2}\right)\int_0^{\pi/2}\left[\left(6\cos^2\theta\right)^2 - \left(4\cos^2\theta\right)^2\right]d\theta\\ &= 40\int_0^{\pi/2}\cos^4\theta\,d\theta\\ &= 10\int_0^{\pi/2}(1+\cos 2\theta)^2\,d\theta\\ &= 10\int_0^{\pi/2}\left(1 + 2\cos 2\theta + \frac{1-\cos 4\theta}{2}\right)d\theta\\ &= 10\left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{\pi/2} = \frac{15\pi}{2}\end{aligned}$$

52. By symmetry, $A_1 = A_2$ and $A_3 = A_4$.

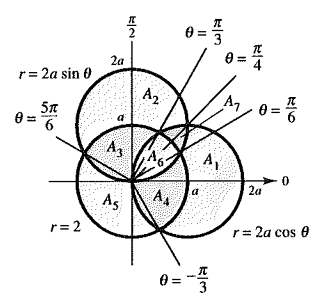
$$\begin{aligned}A_1 = A_2 &= \frac{1}{2}\int_{-\pi/3}^{\pi/6}\left[(2a\cos\theta)^2 - (a)^2\right]d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/4}\left[(2a\cos\theta)^2 - (2a\sin\theta)^2\right]d\theta\\ &= \frac{a^2}{2}\int_{-\pi/3}^{\pi/6}\left(4\cos^2\theta - 1\right)d\theta + 2a^2\int_{\pi/6}^{\pi/4}\cos 2\theta\,d\theta\\ &= \frac{a^2}{2}[\theta + \sin 2\theta]_{-\pi/3}^{\pi/6} + a^2[\sin 2\theta]_{\pi/6}^{\pi/4} = \frac{a^2}{2}\left(\frac{\pi}{2} + \sqrt{3}\right) + a^2\left(1 - \frac{\sqrt{3}}{2}\right) = a^2\left(\frac{\pi}{4} + 1\right)\end{aligned}$$

$$A_3 = A_4 = \frac{1}{2}\left(\frac{\pi}{2}\right)a^2 = \frac{\pi a^2}{4}$$

$$\begin{aligned}A_5 &= \frac{1}{2}\left(\frac{5\pi}{6}\right)a^2 - 2\left(\frac{1}{2}\right)\int_{5\pi/6}^{\pi}(2a\sin\theta)^2\,d\theta\\ &= \frac{5\pi a^2}{12} - 2a^2\int_{5\pi/6}^{\pi}(1-\cos 2\theta)\,d\theta\\ &= \frac{5\pi a^2}{12} - a^2[2\theta - \sin 2\theta]_{5\pi/6}^{\pi} = \frac{5\pi a^2}{12} - a^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = a^2\left(\frac{\pi}{12} + \frac{\sqrt{3}}{2}\right)\end{aligned}$$

$$\begin{aligned}A_6 &= 2\left(\frac{1}{2}\right)\int_0^{\pi/6}(2a\sin\theta)^2\,d\theta + 2\left(\frac{1}{2}\right)\int_{\pi/6}^{\pi/4}a^2\,d\theta\\ &= 2a^2\int_0^{\pi/6}(1-\cos 2\theta)\,d\theta + \left[a^2\theta\right]_{\pi/6}^{\pi/4}\\ &= a^2[2\theta - \sin 2\theta]_0^{\pi/6} + \frac{\pi a^2}{12} = a^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) + \frac{\pi a^2}{12} = a^2\left(\frac{5\pi}{12} - \frac{\sqrt{3}}{2}\right)\end{aligned}$$

$$\begin{aligned}A_7 &= 2\left(\frac{1}{2}\right)\int_{\pi/6}^{\pi/4}\left[(2a\sin\theta)^2 - (a)^2\right]d\theta\\ &= a^2\int_{\pi/6}^{\pi/4}\left(4\sin^2\theta - 1\right)d\theta = a^2[\theta - \sin 2\theta]_{\pi/6}^{\pi/4} = a^2\left(\frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2}\right)\end{aligned}$$



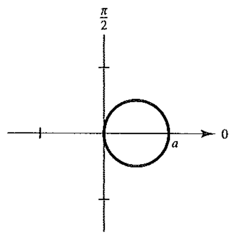
[**Note:** $A_1 + A_6 + A_7 + A_4 = \pi a^2$ = area of circle of radius a]

53. $r = a\cos(n\theta)$

For $n = 1$:

$r = a\cos\theta$

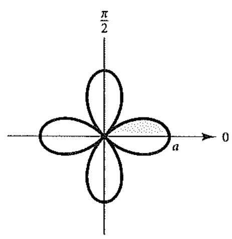
$$A = \pi\left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{4}$$



For $n = 2$:

$r = a\cos 2\theta$

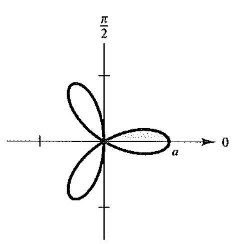
$$A = 8\left(\frac{1}{2}\right)\int_0^{\pi/4}(a\cos 2\theta)^2\,d\theta = \frac{\pi a^2}{2}$$



For $n = 3$:

$r = a\cos 3\theta$

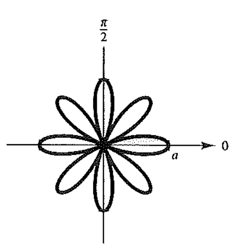
$$A = 6\left(\frac{1}{2}\right)\int_0^{\pi/6}(a\cos 3\theta)^2\,d\theta = \frac{\pi a^2}{4}$$



For $n = 4$:

$r = a\cos 4\theta$

$$A = 16\left(\frac{1}{2}\right)\int_0^{\pi/8}(a\cos 4\theta)^2\,d\theta = \frac{\pi a^2}{2}$$



In general, the area of the region enclosed by $r = a\cos(n\theta)$ for $n = 1, 2, 3, \ldots$ is $(\pi a^2)/4$ if n is odd and is $(\pi a^2)/2$ if n is even.

54. $r = \sec\theta - 2\cos\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$r\cos\theta = 1 - 2\cos^2\theta$$

$$x = 1 - 2\frac{r^2\cos^2\theta}{r^2} = 1 - 2\left(\frac{x^2}{x^2 + y^2}\right)$$

$$(x^2 + y^2)x = x^2 + y^2 - 2x^2$$

$$y^2(x - 1) = -x^2 - x^3$$

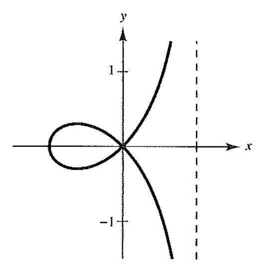
$$y^2 = \frac{x^2(1 + x)}{1 - x}$$

$$A = 2\left(\frac{1}{2}\right)\int_0^{\pi/4}(\sec\theta - 2\cos\theta)^2\,d\theta$$

$$= \int_0^{\pi/4}\left(\sec^2\theta - 4 + 4\cos^2\theta\right)d\theta$$

$$= \int_0^{\pi/4}\left(\sec^2\theta - 4 + 2(1 + \cos 2\theta)\right)d\theta$$

$$= \left[\tan\theta - 2\theta + \sin 2\theta\right]_0^{\pi/4} = 2 - \frac{\pi}{2}$$



55. $r = 8, r^1 = 0$

$$s = \int_0^{2\pi}\sqrt{8^2 + 0^2}\,d\theta = 8\theta\Big]_0^{2\pi} = 16\pi$$

(circumference of circle of radius 8)

56. $r = a$

$r' = 0$

$$s = \int_0^{2\pi}\sqrt{a^2 + 0^2}\,d\theta = \left[a\theta\right]_0^{2\pi} = 2\pi a$$

(circumference of circle of radius a)

57. $r = 4\sin\theta$

$r' = 4\cos\theta$

$$s = \int_0^{\pi}\sqrt{(4\sin\theta)^2 + (4\cos\theta)^2}\,d\theta$$

$$= \int_0^{\pi}4\,d\theta = \left[4\theta\right]_0^{\pi} = 4\pi$$

(circumference of circle of radius 2)

58. $r = 2a\cos\theta$

$r' = -2a\sin\theta$

$$s = \int_{-\pi/2}^{\pi/2}\sqrt{(2a\cos\theta)^2 + (-2a\sin\theta)^2}\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}2a\,d\theta = \left[2a\theta\right]_{-\pi/2}^{\pi/2} = 2\pi a$$

59. $r = 1 + \sin\theta$

$r' = \cos\theta$

$$s = 2\int_{\pi/2}^{3\pi/2} \sqrt{(1 + \sin\theta)^2 + (\cos\theta)^2}\, d\theta$$

$$= 2\sqrt{2}\int_{\pi/2}^{3\pi/2} \sqrt{1 + \sin\theta}\, d\theta$$

$$= 2\sqrt{2}\int_{\pi/2}^{3\pi/2} \frac{-\cos\theta}{\sqrt{1 - \sin\theta}}\, d\theta$$

$$= \left[4\sqrt{2}\sqrt{1 - \sin\theta}\right]_{\pi/2}^{3\pi/2}$$

$$= 4\sqrt{2}\left(\sqrt{2} - 0\right) = 8$$

60. $r = 8(1 + \cos\theta), 0 \le \theta \le 2\pi$

$r' = -8\sin\theta$

$$s = 2\int_0^{\pi} \sqrt{\left[8(1 + \cos\theta)\right]^2 + \left(-8\sin\theta\right]^2}\, d\theta$$

$$= 16\int_0^{\pi} \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$$

$$= 16\sqrt{2}\int_0^{\pi} \sqrt{1 + \cos\theta}\, d\theta$$

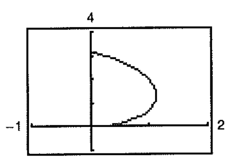
$$= 16\sqrt{2}\int_0^{\pi} \sqrt{1 + \cos\theta} \cdot \left(\frac{\sqrt{1 - \cos\theta}}{\sqrt{1 - \cos\theta}}\right) d\theta$$

$$= 16\sqrt{2}\int_0^{\pi} \frac{\sin\theta}{\sqrt{1 - \cos\theta}}\, d\theta$$

$$= \left[32\sqrt{2}\sqrt{1 - \cos\theta}\right]_0^{\pi}$$

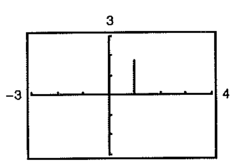
$$= 64$$

61. $r = 2\theta, 0 \le \theta \le \dfrac{\pi}{2}$



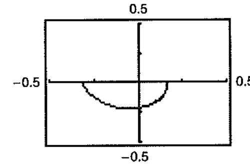
Length ≈ 4.16

62. $r = \sec\theta, 0 \le \theta \le \dfrac{\pi}{3}$



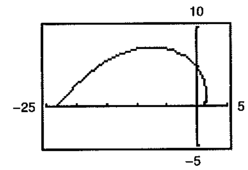
Length ≈ 1.73 (exact $\sqrt{3}$)

63. $r = \dfrac{1}{\theta}, \pi \le \theta \le 2\pi$



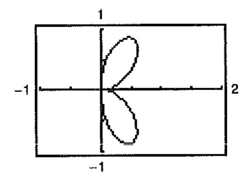
Length ≈ 0.71

64. $r = e^{\theta}, 0 \le \theta \le \pi$



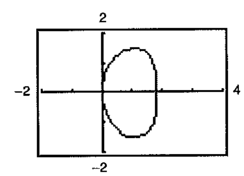
Length ≈ 31.31

65. $r = \sin(3\cos\theta), 0 \le \theta \le \pi$



Length ≈ 4.39

66. $r = 2\sin(2\cos\theta), 0 \le \theta \le \pi$



Length ≈ 7.78

67. $r = 6\cos\theta$

$r' = -6\sin\theta$

$$S = 2\pi\int_0^{\pi/2} 6\cos\theta\sin\theta\sqrt{36\cos^2\theta + 36\sin^2\theta}\, d\theta$$

$$= 72\pi\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta$$

$$= \left[36\pi\sin^2\theta\right]_0^{\pi/2}$$

$$= 36\pi$$

68. $r = a\cos\theta$

$r' = -a\sin\theta$

$$S = 2\pi\int_0^{\pi/2} a\cos\theta(\cos\theta)\sqrt{a^2\cos\theta + a^2\sin^2\theta}\,d\theta$$

$$= 2\pi a^2\int_0^{\pi/2}\cos^2\theta\,d\theta = \pi a^2\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$$

$$= \left[\pi a^2\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{\pi/2} = \frac{\pi^2a^2}{2}$$

69. $r = e^{a\theta}$

$r' = ae^{a\theta}$

$$S = 2\pi\int_0^{\pi/2} e^{a\theta}\cos\theta\sqrt{\left(e^{a\theta}\right)^2 + \left(ae^{a\theta}\right)^2}\,d\theta$$

$$= 2\pi\sqrt{1+a^2}\int_0^{\pi/2} e^{2a\theta}\cos\theta\,d\theta$$

$$= 2\pi\sqrt{1+a^2}\left[\frac{e^{2a\theta}}{4a^2+1}(2a\cos\theta + \sin\theta)\right]_0^{\pi/2}$$

$$= \frac{2\pi\sqrt{1+a^2}}{4a^2+1}\left(e^{\pi a} - 2a\right)$$

70. $r = a(1+\cos\theta)$

$r' = -a\sin\theta$

$$S = 2\pi\int_0^{\pi} a(1+\cos\theta)\sin\theta\sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\,d\theta = 2\pi a^2\int_0^{\pi}\sin\theta(1+\cos\theta)\sqrt{2+2\cos\theta}\,d\theta$$

$$= -2\sqrt{2}\pi a^2\int_0^{\pi}(1+\cos\theta)^{3/2}(-\sin\theta)\,d\theta = -\frac{4\sqrt{2}\pi a^2}{5}\left[(1+\cos\theta)^{5/2}\right]_0^{\pi} = \frac{32\pi a^2}{5}$$

71. $r = 4\cos 2\theta$

$r' = -8\sin 2\theta$

$$S = 2\pi\int_0^{\pi/4} 4\cos 2\theta\sin\theta\sqrt{16\cos^2 2\theta + 64\sin^2\theta\,2\theta}\,d\theta = 32\pi\int_0^{\pi/4}\cos 2\theta\sin\theta\sqrt{\cos^2 2\theta + 4\sin^2 2\theta}\,d\theta \approx 21.87$$

72. $r = \theta$

$r' = 1$

$$S = 2\pi\int_0^{\pi}\theta\sin\theta\sqrt{\theta^2+1}\,d\theta \approx 42.32$$

73. You will only find simultaneous points of intersection. There may be intersection points that do not occur with the same coordinates in the two graphs.

74. (a) is correct: $s \approx 33.124$.

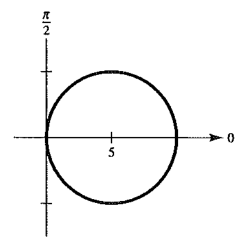
75. (a) $S = 2\pi\int_\alpha^\beta f(\theta)\sin\theta\sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta$

(b) $S = 2\pi\int_\alpha^\beta f(\theta)\cos\theta\sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta$

76. (a) $r = 10\cos\theta,\ 0 \le \theta < \pi$

Circle of radius 5

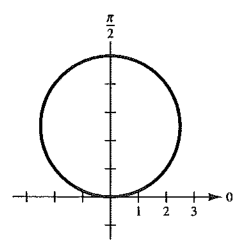
Area $= 25\pi$



(b) $r = 5\sin\theta,\ 0 \le \theta < \pi$

Circle radius $5/2$

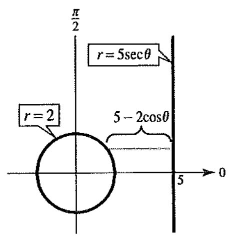
Area $= \frac{25}{4}\pi$



77. Revolve $r = 2$ about the line $r = 5 \sec \theta$.

$f(\theta) = 2, f'(\theta) = 0$

$$S = 2\pi \int_0^{2\pi} (5 - 2\cos\theta)\sqrt{2^2 + 0^2}\, d\theta$$
$$= 4\pi \int_0^{2\pi} (5 - 2\cos\theta)\, d\theta$$
$$= 4\pi[5\theta - 2\sin\theta]_0^{2\pi}$$
$$= 40\pi^2$$

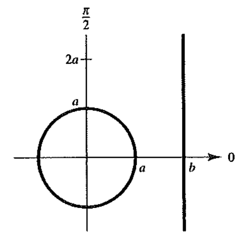


78. Revolve $r = a$ about the line $r = b \sec \theta$ where $b > a > 0$.

$f(\theta) = a$

$f'(\theta) = 0$

$$S = 2\pi \int_0^{2\pi} [b - a\cos\theta]\sqrt{a^2 + 0^2}\, d\theta$$
$$= 2\pi a[b\theta - a\sin\theta]_0^{2\pi}$$
$$= 2\pi a(2\pi b) = 4\pi^2 ab$$



79. $r = 8\cos\theta, 0 \le \theta \le \pi$

(a) $A = \frac{1}{2}\int_0^{\pi} r^2\, d\theta = \frac{1}{2}\int_0^{\pi} 64\cos^2\theta\, d\theta = 32\int_0^{\pi} \frac{1 + \cos 2\theta}{2}\, d\theta = 16\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi} = 16\pi$

(Area circle $= \pi r^2 = \pi 4^2 = 16\pi$)

(b)

θ	0.2	0.4	0.6	0.8	1.0	1.2	1.4
A	6.32	12.14	17.06	20.80	23.27	24.60	25.08

(c), (d) For $\frac{1}{4}$ of area $(4\pi \approx 12.57)$: 0.42

For $\frac{1}{2}$ of area $(8\pi \approx 25.13)$: $1.57\left(\frac{\pi}{2}\right)$

For $\frac{3}{4}$ of area $(12\pi \approx 37.70)$: 2.73

(e) No, it does not depend on the radius.

80. $r = 3\sin\theta, 0 \le \theta \le \pi$

(a) $A = \frac{1}{2}\int_0^{\pi} r^2\, d\theta = \frac{9}{2}\int_0^{\pi} \sin^2\theta\, d\theta = \frac{9}{4}\int_0^{\pi} (1 - \cos 2\theta)\, d\theta = \frac{9}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi} = \frac{9}{4}\pi$

$\left[\textbf{Note:} \text{ radius of circle is } \frac{3}{2} \Rightarrow A = \pi\left(\frac{3}{2}\right)^2 = \frac{9}{4}\pi\right]$

(b)

θ	0.2	0.4	0.6	0.8	1.0	1.2	1.4
A	0.0119	0.0930	0.3015	0.6755	1.2270	1.9401	2.7731

(c), (d) For $\frac{1}{8}$ of area $\left(\frac{1}{8}\frac{9}{4}\pi \approx 0.8836\right)$: $\theta \approx 0.88$

For $\frac{1}{4}$ of area $\left(\frac{1}{4}\frac{9}{4}\pi \approx 1.7671\right)$: $\theta \approx 1.15$

For $\frac{1}{2}$ of area $\left(\frac{1}{2}\frac{9}{4}\pi \approx 3.5343\right)$: $\theta = \frac{\pi}{2} \approx 1.57$

81. $$r = a\sin\theta + b\cos\theta$$
$$r^2 = ar\sin\theta + br\cos\theta$$
$$x^2 + y^2 = ay + bx$$
$x^2 + y^2 - bx - ay = 0$ represents a circle.

82. $r = \sin\theta + \cos\theta$, Circle

$$A = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta$$
$$= \frac{1}{2}\int_0^\pi (\sin\theta + \cos\theta)^2\,d\theta$$
$$= \frac{1}{2}\int_0^\pi (1 + 2\sin\theta\cos\theta)\,d\theta = \frac{1}{2}\Big[\theta + \sin^2\theta\Big]_0^\pi = \frac{\pi}{2}$$

Converting to rectangular form:

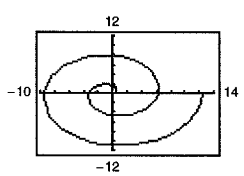
$$r^2 = r\sin\theta + r\cos\theta$$
$$x^2 + y^2 = y + x$$
$$\left(x^2 - x + \frac{1}{4}\right) + \left(y^2 - y + \frac{1}{4}\right) = \frac{1}{2}$$
$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$$

Circle of radius $\frac{1}{\sqrt{2}}$ and center $\left(\frac{1}{2}, \frac{1}{2}\right)$

$$\text{Area} = \pi\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{\pi}{2}$$

83. (a) $r = \theta, \theta \ge 0$

As a increases, the spiral opens more rapidly. If $\theta < 0$, the spiral is reflected about the y-axis.



(b) $r = a\theta, \theta \ge 0$, crosses the polar axis for $\theta = n\pi$, n and integer. To see this

$r = a\theta \Rightarrow r\sin\theta = y = a\theta\sin\theta = 0$

for $\theta = n\pi$. The points are

$(r, \theta) = (an\pi, n\pi), n = 1, 2, 3, \ldots$

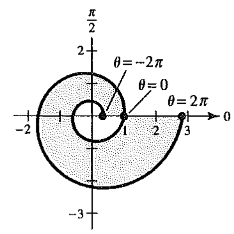
(c) $f(\theta) = \theta, f'(\theta) = 1$

$$s = \int_0^{2\pi} \sqrt{\theta^2 + 1}\,d\theta$$
$$= \frac{1}{2}\left[\ln\left(\sqrt{x^2+1} + x\right) + x\sqrt{x^2+1}\right]_0^{2\pi}$$
$$= \frac{1}{2}\ln\left(\sqrt{4\pi^2+1} + 2\pi\right) + \pi\sqrt{4\pi^2+1} \approx 21.2563$$

(d) $A = \frac{1}{2}\int_\alpha^\beta r^2\,dr = \frac{1}{2}\int_0^{2\pi}\theta^2\,d\theta = \left[\frac{\theta^3}{6}\right]_0^{2\pi} = \frac{4}{3}\pi^3$

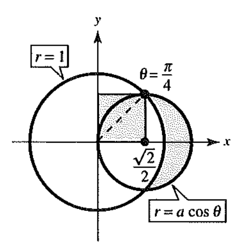
84. $r = e^{\theta/6}$

$$A = \frac{1}{2}\int_0^{2\pi}\left(e^{\theta/6}\right)^2 d\theta - \frac{1}{2}\int_{-2\pi}^0\left(e^{\theta/6}\right)^2 d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} e^{\theta/3}\,d\theta - \frac{1}{2}\int_{-2\pi}^0 e^{\theta/3}\,d\theta$$
$$= \left[\frac{3}{2}e^{\theta/3}\right]_0^{2\pi} - \left[\frac{3}{2}e^{\theta/3}\right]_{-2\pi}^0$$
$$= \frac{3}{2}e^{2\pi/3} - \frac{3}{2} - \frac{3}{2} + \frac{3}{2}e^{-2\pi/3} = \frac{3}{2}\left[e^{2\pi/3} + e^{-2\pi/3} - 2\right]$$
$$\approx 9.3655$$



85. The smaller circle has equation $r = a\cos\theta$. The area of the shaded lune is:

$$A = 2\left(\frac{1}{2}\right)\int_0^{\pi/4}\left[(a\cos\theta)^2 - 1\right]d\theta$$
$$= \int_0^{\pi/4}\left[\frac{a^2}{2}(1 + \cos 2\theta) - 1\right]d\theta$$
$$= \left[\frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) - \theta\right]_0^{\pi/4}$$
$$= \frac{a^2}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) - \frac{\pi}{4}$$



This equals the area of the square, $\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$.

$$\frac{a^2}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) - \frac{\pi}{4} = \frac{1}{2}$$
$$\pi a^2 + 2a^2 - 2\pi - 4 = 0$$
$$a^2 = \frac{4 + 2\pi}{2 + \pi} = 2$$
$$a = \sqrt{2}$$

Smaller circle: $r = \sqrt{2}\cos\theta$

86. $x = \frac{3t}{1 + t^3},\ y = \frac{3t^2}{1 + t^3}$

(a) $x^3 + y^3 = \frac{27(t^3 + t^6)}{(1 + t^3)^3} = \frac{27t^3}{(1 + t^3)^2}$

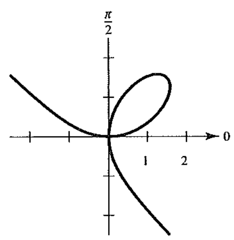
$3xy = \frac{27t^3}{(1 + t^3)^2}$

So, $x^3 + y^3 = 3xy$.

$(r \cos \theta)^3 + (r \sin \theta)^3 = 3(r \cos \theta)(r \sin \theta)$

$r = \frac{3 \cos \theta \sin \theta}{\cos^3 \theta + \sin^3 \theta}$

(b)



(c) $A = \frac{1}{2}\int_0^{\pi/2} r^2\, d\theta = \frac{3}{2}$

87. False. $f(\theta) = 1$ and $g(\theta) = -1$ have the same graphs.

88. False. $f(\theta) = 0$ and $g(\theta) = \sin 2\theta$ have only one point of intersection.

89. In parametric form,

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Using θ instead of t, you have
$x = r \cos \theta = f(\theta) \cos \theta$ and
$y = r \sin \theta = f(\theta) \sin \theta$. So,

$\frac{dx}{d\theta} = f'(\theta) \cos \theta - f(\theta) \sin \theta$ and

$\frac{dy}{d\theta} = f'(\theta) \sin \theta + f(\theta) \cos \theta$.

It follows that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f(\theta)]^2 + [f'(\theta)]^2.$$

So, $s = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$.

Section 10.6 Polar Equations of Conics and Kepler's Laws

1. $r = \frac{2e}{1 + e \cos \theta}$

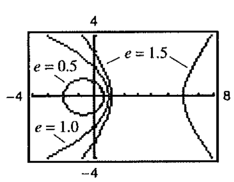
(a) $e = 1,\ r = \frac{2}{1 + \cos \theta}$, parabola

(b) $e = 0.5$,

$r = \frac{1}{1 + 0.5 \cos \theta} = \frac{2}{2 + \cos \theta}$, ellipse

(c) $e = 1.5$,

$r = \frac{3}{1 + 1.5 \cos \theta} = \frac{6}{2 + 3 \cos \theta}$, hyperbola



2. $r = \frac{2e}{1 - e \cos \theta}$

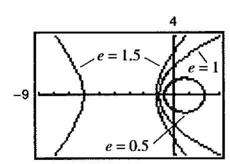
(a) $e = 1,\ r = \frac{2}{1 - \cos \theta}$, parabola

(b) $e = 0.5$,

$r = \frac{1}{1 - 0.5 \cos \theta} = \frac{2}{2 - \cos \theta}$, ellipse

(c) $e = 1.5$,

$r = \frac{3}{1 - 1.5 \cos \theta} = \frac{6}{2 - 3 \cos \theta}$, hyperbola



3. $r = \dfrac{2e}{1 - e \sin \theta}$

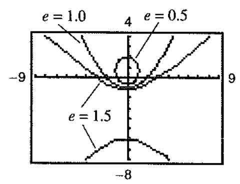
(a) $e = 1, r = \dfrac{2}{1 - \sin \theta}$, parabola

(b) $e = 0.5$,

$$r = \frac{1}{1 - 0.5 \sin \theta} = \frac{2}{2 - \sin \theta}, \text{ ellipse}$$

(c) $e = 1.5$,

$$r = \frac{3}{1 - 1.5 \sin \theta} = \frac{6}{2 - 3 \sin \theta}, \text{ hyperbola}$$



4. $r = \dfrac{2e}{1 + e \sin \theta}$

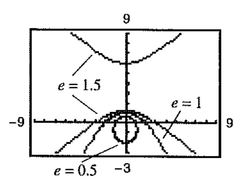
(a) $e = 1, r = \dfrac{2}{1 + \sin \theta}$, parabola

(b) $e = 0.5$,

$$r = \frac{1}{1 + 0.5 \sin \theta} = \frac{2}{2 + \sin \theta}, \text{ ellipse}$$

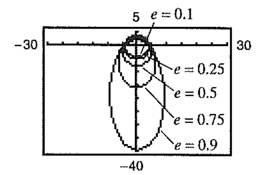
(c) $e = 1.5$,

$$r = \frac{3}{1 + 1.5 \sin \theta} = \frac{6}{2 + 3 \sin \theta}, \text{ hyperbola}$$

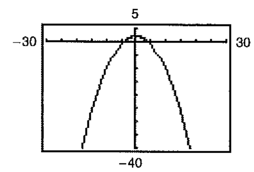


5. $r = \dfrac{4}{1 + e \sin \theta}$

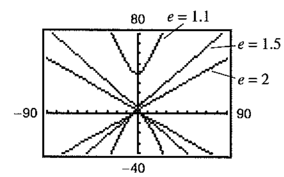
(a) The conic is an ellipse. As $e \to 1^-$, the ellipse becomes more elliptical, and as $e \to 0^+$, it becomes more circular.



(b) The conic is a parabola.



(c) The conic is a hyperbola. As $e \to 1^+$, the hyperbola opens more slowly, and as $e \to \infty$, it opens more rapidly.



6. $r = \dfrac{4}{1 - 0.4 \cos \theta}$

(a) Because $e = 0.4 < 1$, the conic is an ellipse with vertical directrix to the left of the pole.

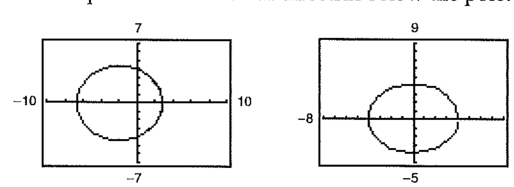
(b) $r = \dfrac{4}{1 + 0.4 \cos \theta}$

The ellipse is shifted to the left. The vertical directrix is to the right of the pole.

$$r = \frac{4}{1 - 0.4 \sin \theta}.$$

The ellipse has a horizontal directrix below the pole.

(c)



7. Parabola; Matches (c)

8. Ellipse; Matches (f)

9. Hyperbola; Matches (a)

10. Parabola; Matches (e)

11. Ellipse; Matches (b)

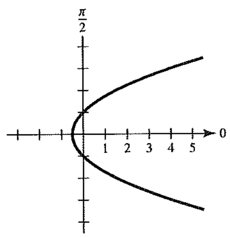
12. Hyperbola; Matches (d)

13. $r = \dfrac{1}{1 - \cos\theta}$

Parabola because $e = 1, d = 1$.

Distance from pole to directrix: $|d| = 1$

Directrix: $x = -d = -1$

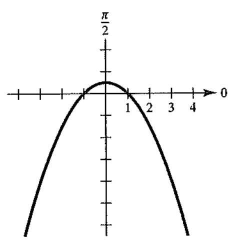


14. $r = \dfrac{1}{1 + \sin\theta}$

Parabola because $e = 1, d = 1$

Distance from pole to directrix: $|d| = 1$

Directrix: $y = 1$

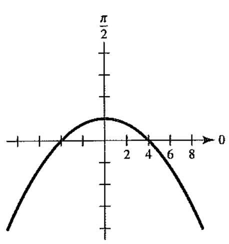


15. $r = \dfrac{-4}{1 - \sin\theta} = \dfrac{1(-4)}{1 - \sin\theta}$

$e = 1, d = -4$ Parabola

Distance from pole to directrix: $|d| = 4$

Directrix: $y = 4$

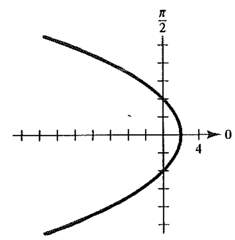


16. $r = \dfrac{4}{1 + \cos\theta}$

$e = 1, d = 4$ Parabola

Distance from pole to directrix: $|d| = 4$

Directrix: $x = 4$



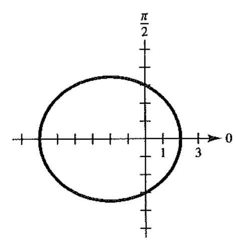
17. $r = \dfrac{6}{2 + \cos\theta} = \dfrac{3}{1 + (1/2)\cos\theta}$

Ellipse because $e = \dfrac{1}{2}; d = 6$

Directrix: $x = 6$

Distance from pole to directrix: $|d| = 6$

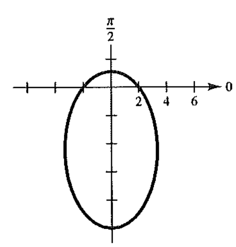
Vertices: $(r, \theta) = (2, 0), (6, \pi)$



18. $r = \dfrac{10}{5 + 4\sin\theta} = \dfrac{2}{1 + \left(\frac{4}{5}\right)\sin\theta} = \dfrac{\left(\frac{4}{5}\right)\left(\frac{5}{2}\right)}{1 + \left(\frac{4}{5}\right)\sin\theta}$

$e = \dfrac{4}{5} < 1, d = \dfrac{5}{2}$ Ellipse

Distance from pole to directrix: $|d| = \dfrac{5}{2}$



19. $r(2 + \sin\theta) = 4$

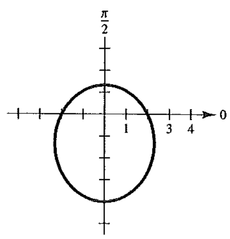
$$r = \frac{4}{2 + \sin\theta} = \frac{2}{1 + (1/2)\sin\theta}$$

Ellipse because $e = 1/2$; $d = 4$

Directrix: $y = 4$

Distance from pole to directrix: $|d| = 4$

Vertices: $(r, \theta) = (4/3, \pi/2), (4, 3\pi/2)$



20. $r(3 - 2\cos\theta) = 6$

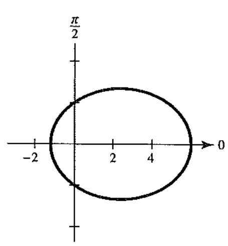
$$r = \frac{6}{3 - 2\cos\theta} = \frac{2}{1 - (2/3)\cos\theta}$$

Ellipse because $e = 2/3 < 1$; $d = 3$

Directrix: $x = -3$

Distance from pole to directrix: $|d| = 3$

Vertices: $(r, \theta) = (6, 0), (6/5, \pi)$



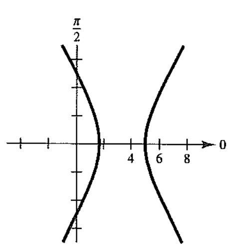
21. $r = \dfrac{5}{-1 + 2\cos\theta} = \dfrac{-5}{1 - 2\cos\theta}$

Hyperbola because $e = 2 > 1$; $d = -5/2$

Directrix: $x = 5/2$

Distance from pole to directrix: $|d| = 5/2$

Vertices: $(r, \theta) = (5, 0), (-5/3, \pi)$



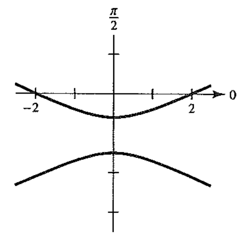
22. $r = \dfrac{-6}{3 + 7\sin\theta} = \dfrac{-2}{1 + (7/3)\sin\theta}$

Hyperbola because $e = 7/3 > 1$; $d = -6/7$

Directrix: $y = -6/7$

Distance from pole to directrix: $|d| = 6/7$

Vertices: $(r, \theta) = (-3/5, \pi/2), (3/2, 3\pi/2)$



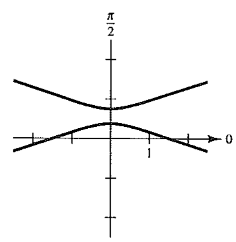
23. $r = \dfrac{3}{2 + 6\sin\theta} = \dfrac{3/2}{1 + 3\sin\theta}$

Hyperbola because $e = 3 > 0$; $d = 1/2$

Directrix: $y = 1/2$

Distance from pole to directrix: $|d| = 1/2$

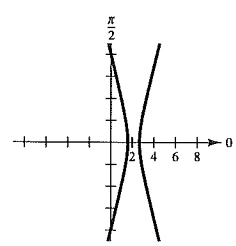
Vertices: $(r, \theta) = (3/8, \pi/2), (-3/4, 3\pi/2)$



24. $r = \dfrac{8}{1 + 4\cos\theta} = \dfrac{4(2)}{1 + 4\cos\theta}$

$e = 4 > 1$, $d = 2$ Hyperbola

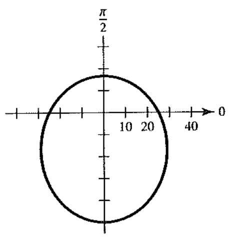
Distance from pole to directrix: $|d| = 2$



25. $r = \dfrac{300}{-12 + 6\sin\theta} = \dfrac{-25}{1 - \frac{1}{2}\sin\theta} = \dfrac{\frac{1}{2}(-50)}{1 - \frac{1}{2}\sin\theta}$

$e = \dfrac{1}{2}$, $d = -50$, Ellipse

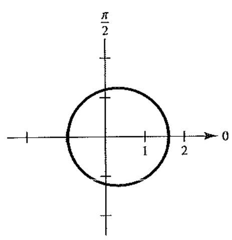
Distance from pole to directrix: $|d| = 50$



26. $r = \dfrac{180}{15 - 3.75\cos\theta} = \dfrac{12}{1 - \frac{1}{4}\cos\theta} = \dfrac{\frac{1}{4}(48)}{1 - \frac{1}{4}\cos\theta}$

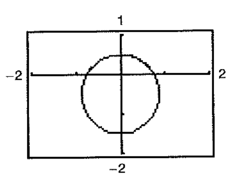
$e = \frac{1}{4}$, $d = 48$, Ellipse

Distance from pole to directrix: $|d| = 48$



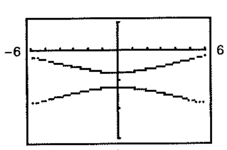
27. $r = \dfrac{3}{-4 + 2\sin\theta} = \dfrac{-\frac{3}{4}}{1 - \frac{1}{2}\sin\theta}$

$e = \dfrac{1}{2}$, Ellipse



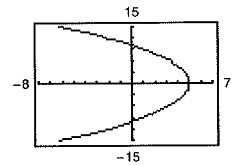
28. $r = \dfrac{-15}{2 + 8\sin\theta} = \dfrac{-\frac{15}{2}}{1 + 4\sin\theta}$

$e = 4$, Hyperbola



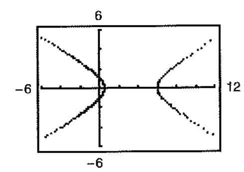
29. $r = \dfrac{-10}{1 - \cos\theta}$

$e = 1$, Parabola

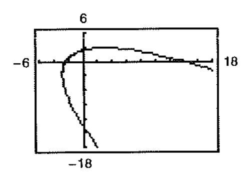


30. $r = \dfrac{6}{6 + 7\cos\theta} = \dfrac{1}{1 + \left(\frac{7}{6}\right)\cos\theta}$

$e = \dfrac{7}{6}$, Hyperbola



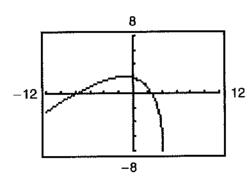
31. $r = \dfrac{-4}{1 - \sin\left(\theta - \dfrac{\pi}{4}\right)}$



Rotate the graph of $r = \dfrac{-4}{1 - \sin\theta}$

$\dfrac{\pi}{4}$ radian counterclockwise.

32. $r = \dfrac{4}{1 + \cos\left(\theta - \dfrac{\pi}{3}\right)}$



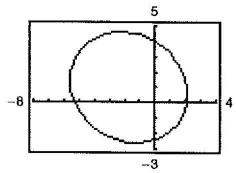
Rotate the graph of $r = \dfrac{4}{1 + \cos\theta}$

$\dfrac{\pi}{3}$ radian counterclockwise.

33. $r = \dfrac{6}{2 + \cos\left(\theta + \dfrac{\pi}{6}\right)}$

Rotate the graph of $r = \dfrac{6}{2 + \cos\theta}$

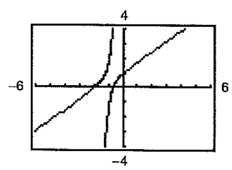
$\dfrac{\pi}{6}$ radian clockwise.



34. $r = \dfrac{-6}{3 + 7\sin(\theta + (2\pi/3))}$

Rotate graph of $r = \dfrac{-6}{3 + 7\sin\theta}$

$\dfrac{2\pi}{3}$ radians clockwise.



35. Change θ to $\theta + \dfrac{\pi}{6}$

$$r = \frac{8}{8 + 5\cos\left(\theta + \dfrac{\pi}{6}\right)}$$

36. Change θ to $\theta - \dfrac{\pi}{4}$

$$r = \frac{9}{1 + \sin\left(\theta - \dfrac{\pi}{4}\right)}$$

37. Parabola

$e = 1$

$x = -3 \Rightarrow d = 3$

$$r = \frac{ed}{1 - e\cos\theta} = \frac{3}{1 - \cos\theta}$$

38. Parabola

$e = 1, y = 4 \Rightarrow d = 4$

$$r = \frac{ed}{1 + e\sin\theta} = \frac{4}{1 + \sin\theta}$$

39. Ellipse

$e = \dfrac{1}{2}, y = 1, d = 1$

$$r = \frac{ed}{1 + e\sin\theta} = \frac{1/2}{1 + (1/2)\sin\theta} = \frac{1}{2 + \sin\theta}$$

40. Ellipse

$e = \dfrac{3}{4}, y = -2, d = 2$

$$r = \frac{ed}{1 - e\sin\theta} = \frac{2(3/4)}{1 - (3/4)\sin\theta} = \frac{6}{4 - 3\sin\theta}$$

41. Hyperbola

$e = 2, x = 1, d = 1$

$$r = \frac{ed}{1 + e\cos\theta} = \frac{2}{1 + 2\cos\theta}$$

42. Hyperbola

$e = \dfrac{3}{2}, x = -1, d = 1$

$$r = \frac{ed}{1 - e\cos\theta} = \frac{3/2}{1 - (3/2)\cos\theta} = \frac{3}{2 - 3\cos\theta}$$

43. Parabola

Vertex: $\left(1, -\dfrac{\pi}{2}\right)$

$e = 1, d = 2, r = \dfrac{2}{1 - \sin\theta}$

44. Parabola

Vertex: $(5, \pi)$

$e = 1, d = 10$

$$r = \frac{ed}{1 - e\cos\theta} = \frac{10}{1 - \cos\theta}$$

45. Ellipse

Vertices: $(2, 0), (8, \pi)$

$e = \dfrac{3}{5}, d = \dfrac{16}{3}$

$$r = \frac{ed}{1 + e\cos\theta} = \frac{16/5}{1 + (3/5)\cos\theta} = \frac{16}{5 + 3\cos\theta}$$

46. Ellipse

Vertices: $\left(2, \dfrac{\pi}{2}\right), \left(4, \dfrac{3\pi}{2}\right)$

$e = \dfrac{1}{3}, d = 8$

$$r = \frac{ed}{1 + e\sin\theta} = \frac{8/3}{1 + (1/3)\sin\theta} = \frac{8}{3 + \sin\theta}$$

47. Hyperbola

Vertices: $\left(1, \frac{3\pi}{2}\right), \left(9, \frac{3\pi}{2}\right)$

$e = \frac{5}{4}, d = \frac{9}{5}$

$$r = \frac{ed}{1 - e\sin\theta} = \frac{9/4}{1 - (5/4)\sin\theta} = \frac{9}{4 - 5\sin\theta}$$

48. Hyperbola

Vertices: $(2, 0), (10, 0)$

$e = \frac{3}{2}, d = \frac{10}{3}$

$$r = \frac{ed}{1 + e\cos\theta} = \frac{5}{1 + (3/2)\cos\theta} = \frac{10}{2 + 3\cos\theta}$$

49. Ellipse, $e = \frac{1}{2}$,

Directrix: $r = 4\sec\theta \Rightarrow x = r\cos\theta = 4$

$$r = \frac{ed}{1 + e\cos\theta} = \frac{\left(\frac{1}{2}\right)4}{1 + \frac{1}{2}\cos\theta} = \frac{4}{2 + \cos\theta}$$

50. Hyperbola, $e = 2$

Directrix: $r = -8\csc\theta \Rightarrow y = r\sin\theta = -8$

$$r = \frac{ed}{1 - e\sin\theta} = \frac{2(-8)}{1 - 2\sin\theta} = \frac{-16}{1 - 2\sin\theta}$$

51. Ellipse if $0 < e < 1$, parabola if $e = 1$, hyperbola if $e > 1$.

52. (a) Hyperbola $(e = 2 > 1)$

(b) Ellipse $\left(e = \frac{1}{10} < 1\right)$

(c) Parabola $(e = 1)$

(d) Rotated hyperbola $(e = 3)$

53. If the foci are fixed and $e \to 0$, then $d \to \infty$. To see this, compare the ellipses

$$r = \frac{1/2}{1 + (1/2)\cos\theta}, e = 1/2, d = 1$$

$$r = \frac{5/16}{1 + (1/4)\cos\theta}, e = 1/4, d = 5/4.$$

54. $r = \frac{4}{1 + \sin\theta}$ is a parabola with horizontal directrix above the pole.

(a) Parabola with vertical directrix to left of pole.

(b) Parabola with horizontal directrix below pole.

(c) Parabola with vertical directrix to right of pole.

(d) Parabola (b) rotated counterclockwise $\pi/4$.

55.

$$\begin{aligned}
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1\\
x^2b^2 + y^2a^2 &= a^2b^2\\
b^2r^2\cos^2\theta + a^2r^2\sin^2\theta &= a^2b^2\\
r^2\left[b^2\cos^2\theta + a^2(1 - \cos^2\theta)\right] &= a^2b^2\\
r^2\left[a^2 + \cos^2\theta(b^2 - a^2)\right] &= a^2b^2
\end{aligned}$$

$$r^2 = \frac{a^2b^2}{a^2 + (b^2 - a^2)\cos^2\theta} = \frac{a^2b^2}{a^2 - c^2\cos^2\theta}$$

$$= \frac{b^2}{1 - (c/a)^2\cos^2\theta} = \frac{b^2}{1 - e^2\cos^2\theta}$$

56.

$$\begin{aligned}
\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1\\
x^2b^2 - y^2a^2 &= a^2b^2\\
b^2r^2\cos^2\theta - a^2r^2\sin^2\theta &= a^2b^2\\
r^2\left[b^2\cos^2\theta - a^2(1 - \cos^2\theta)\right] &= a^2b^2\\
r^2\left[-a^2 + \cos^2\theta(a^2 + b^2)\right] &= a^2b^2
\end{aligned}$$

$$r^2 = \frac{a^2b^2}{-a^2 + c^2\cos^2\theta} = \frac{b^2}{-1 + (c^2/a^2)\cos^2\theta}$$

$$= \frac{-b^2}{1 - e^2\cos^2\theta}$$

57. $a = 5, c = 4, e = \frac{4}{5}, b = 3$

$$r^2 = \frac{9}{1 - (16/25)\cos^2\theta}$$

58. $a = 4, c = 5, b = 3, e = \frac{5}{4}$

$$r^2 = \frac{-9}{1 - (25/16)\cos^2\theta}$$

59. $a = 3, b = 4, c = 5, e = \frac{5}{3}$

$$r^2 = \frac{-16}{1 - (25/9)\cos^2\theta}$$

60. $a = 2, b = 1, c = \sqrt{3}, e = \frac{\sqrt{3}}{2}$

$$r^2 = \frac{1}{1 - (3/4)\cos^2\theta}$$

61. $A = 2\left[\frac{1}{2}\int_0^{\pi}\left(\frac{3}{2-\cos\theta}\right)^2 d\theta\right]$

$= 9\int_0^{\pi}\frac{1}{(2-\cos\theta)^2}\,d\theta \approx 10.88$

62. $A = \frac{1}{2}\int_0^{2\pi}\left(\frac{9}{4+\cos\theta}\right)^2 d\theta \approx 17.52$

63. $A = 2\left[\frac{1}{2}\int_{-\pi/2}^{\pi/2}\left(\frac{2}{3-2\sin\theta}\right)^2 d\theta\right]$

$= 4\int_{-\pi/2}^{\pi/2}\frac{1}{(3-2\sin\theta)^2}\,d\theta \approx 3.37$

64. $A = \frac{1}{2}\int_0^{2\pi}\left[\frac{3}{6+5\sin\theta}\right]^2 d\theta \approx 4.65$

65. Vertices: $(123{,}000 + 4000, 0) = (127{,}000, 0)$

$(119 + 4000, \pi) = (4119, \pi)$

$a = \frac{127{,}000 + 4119}{2} = 65{,}559.5$

$c = 65{,}559.5 - 4119 = 61{,}440.5$

$e = \frac{c}{a} = \frac{122{,}881}{131{,}119} \approx 0.93717$

$r = \frac{ed}{1 - e\cos\theta}$

$\theta = 0: r = \frac{ed}{1-e}, \theta = \pi: r = \frac{ed}{1+e}$

$2a = 2(65{,}559.5) = \frac{ed}{1-e} + \frac{ed}{1+e}$

$131{,}119 = d\left(\frac{e}{1-e} + \frac{e}{1+e}\right) = d\left(\frac{2e}{1-e^2}\right)$

$d = \frac{131{,}119(1-e^2)}{2e} \approx 8514.1397$

$r = \frac{7979.21}{1 - 0.93717\cos\theta} = \frac{1{,}046{,}226{,}000}{131{,}119 - 122{,}881\cos\theta}$

When $\theta = 60° = \frac{\pi}{3}$, $r \approx 15{,}015$.

Distance between earth and the satellite is $r - 4000 \approx 11{,}015$ miles.

66. (a) $r = \frac{ed}{1 - e\cos\theta}$

When $\theta = 0, r = c + a = ea + a = a(1+e)$.

So,

$a(1+e) = \frac{ed}{1-e}$

$a(1+e)(1-e) = ed$

$a(1-e^2) = ed.$

So, $r = \frac{(1-e^2)a}{1-e\cos\theta}$.

(b) The perihelion distance is $a - c = a - ea = a(1-e)$.

When $\theta = \pi, r = \frac{(1-e^2)a}{1+e} = a(1-e)$.

The aphelion distance is $a + c = a + ea = a(1+e)$.

When $\theta = 0, r = \frac{(1-e^2)a}{1-e} = a(1+e)$.

67. $a = 1.496 \times 10^8, e = 0.0167$

$r = \frac{(1-e^2)a}{1-e\cos\theta} = \frac{149{,}558{,}278.1}{1 - 0.0167\cos\theta}$

Perihelion distance: $a(1-e) \approx 147{,}101{,}680$ km

Aphelion distance: $a(1+e) \approx 152{,}098{,}320$ km

68. $a = 1.427 \times 10^9, e = 0.0542$

$r = \frac{(1-e^2)a}{1-e\cos\theta} = \frac{1{,}422{,}807{,}988}{1 - 0.0542\cos\theta}$

Perihelion distance: $a(1-e) \approx 1{,}349{,}656{,}600$ km

Aphelion distance: $a(1+e) \approx 1{,}504{,}343{,}400$ km

69. $a = 4.498 \times 10^9, e = 0.0086$

$r = \frac{(1-e^2)a}{1-e\cos\theta} = \frac{4{,}497{,}667{,}328}{1 - 0.0086\cos\theta}$

Perihelion distance: $a(1-e) \approx 4{,}459{,}317{,}200$ km

Aphelion distance: $a(1+e) \approx 4{,}536{,}682{,}800$ km

70. $a = 5.791 \times 10^7, e = 0.2056$

$r = \frac{(1-e^2)a}{1-e\cos\theta} \approx \frac{55{,}462{,}065.54}{1 - 0.2056\cos\theta}$

Perihelion distance $\approx a(1-e) \approx 46{,}003{,}704$ km

Aphelion distance $\approx a(1+e) \approx 69{,}816{,}296$ km

71. $r = \dfrac{4.498 \times 10^9}{1 - 0.0086 \cos \theta}$

(a) $A = \dfrac{1}{2}\int_0^{\pi/9} r^2 \, d\theta \approx 3.591 \times 10^{18} \text{ km}^2$

$$165\left[\frac{\frac{1}{2}\int_0^{\pi/2} r^2 \, d\theta}{\frac{1}{2}\int_0^{2\pi} r^2 \, d\theta}\right] \approx 9.322 \text{ yrs}$$

(b) $\dfrac{1}{2}\int_\pi^\alpha r^2 \, d\theta = 3.591 \times 10^{18}$

By trial and error, $\alpha \approx \pi + 0.361$

$0.361 > \pi/9 \approx 0.349$ because the rays in part (a) are longer than those in part (b)

(c) For part (a),

$$s = \int_0^{\pi/9} \sqrt{r^2 + (dr/d\theta)^2} \approx 1.583 \times 10^9 \text{ km}$$

$$\text{Average per year} = \frac{1.583 \times 10^9}{9.322} \approx 1.698 \times 10^8 \text{ km/yr}$$

For part (b),

$$s = \int_\pi^{\pi + 0.361} \sqrt{r^2 + (dr/d\theta)^2} \, d\theta \approx 1.610 \times 10^9 \text{ km}$$

$$\text{Average per year} = \frac{1.610 \times 10^9}{9.322} \approx 1.727 \times 10^8 \text{ km/yr}$$

72. $a = \dfrac{1}{2}(500) = 250 \text{ au}, e \approx 0.995$

(a) $e = \dfrac{c}{a} \Rightarrow c \approx 248.75$

$b^2 = a^2 - c^2 \Rightarrow b \approx 24.969 \Rightarrow \text{minor axis} = 2b \approx 49.9 \text{ au}$

(b) $r = \dfrac{(1 - e^2)a}{1 - e \cos \theta} = \dfrac{2.49375}{1 - 0.995 \cos \theta}$

(c) Perihelion distance: $a(1 - e) \approx 1.25$ au

Aphelion distance: $a(1 + e) \approx 498.75$ au

73. $r_1 = a + c, r_0 = a - c, r_1 - r_0 = 2c, r_1 + r_0 = 2a$

$$e = \frac{c}{a} = \frac{r_1 - r_0}{r_1 + r_0}$$

$$\frac{1 + e}{1 - e} = \frac{1 + \frac{c}{a}}{1 - \frac{c}{a}} = \frac{a + c}{a - c} = \frac{r_1}{r_0}$$

74. For a hyperbola,

$r_0 = c - a$ and $r_1 = c + a$.

So $r_1 + r_0 = 2c$ and $r_1 - r_0 = 2a$.

$$e = \frac{c}{a} = \frac{r_1 + r_0}{r_1 - r_0}$$

$$\frac{e + 1}{e - 1} = \frac{\frac{c}{a} + 1}{\frac{c}{a} - 1} = \frac{c + a}{c - a} = \frac{r_1}{r_0}$$

75. $r_1 = \dfrac{ed}{1 + \sin\theta}$ and $r_2 = \dfrac{ed}{1 - \sin\theta}$

Points of intersection: $(ed, 0), (ed, \pi)$

$$r_1: \frac{dy}{dx} = \frac{\left(\frac{ed}{1+\sin\theta}\right)(\cos\theta) + \left(\frac{-ed\cos\theta}{(1+\sin\theta)^2}\right)(\sin\theta)}{\left(\frac{-ed}{1+\sin\theta}\right)(\sin\theta) + \left(\frac{-ed\cos\theta}{(1+\sin\theta)^2}\right)(\cos\theta)}$$

At $(ed, 0)$, $\dfrac{dy}{dx} = -1$. At (ed, π), $\dfrac{dy}{dx} = 1$.

$$r_2: \frac{dy}{dx} = \frac{\left(\frac{ed}{1-\sin\theta}\right)(\cos\theta) + \left(\frac{ed\cos\theta}{(1-\sin\theta)^2}\right)(\sin\theta)}{\left(\frac{-ed}{1-\sin\theta}\right)(\sin\theta) + \left(\frac{ed\cos\theta}{(1-\sin\theta)^2}\right)(\cos\theta)}$$

At $(ed, 0)$, $\dfrac{dy}{dx} = 1$. At (ed, π), $\dfrac{dy}{dx} = -1$.

So, at $(ed, 0)$ you have $m_1 m_2 = (-1)(1) = -1$, and at (ed, π) you have $m_1 m_2 = 1(-1) = -1$. The curves intersect at right angles.

76. $r_1 = \dfrac{c}{1 + \cos\theta}$, $r_2 = \dfrac{d}{1 - \cos\theta}$ (Parabolas)

To find the intersection points:

$$\frac{c}{1+\cos\theta} = \frac{d}{1-\cos\theta}$$
$$c - c \cdot \cos\theta = d + d \cdot \cos\theta$$
$$\cos\theta = \frac{c-d}{c+d}$$
$$r_1 = \frac{c}{1 + \left(\frac{c-d}{c+d}\right)} = \frac{c(c+d)}{2c} = \frac{c+d}{2} = r_2$$

$$\frac{dr_1}{d\theta} = \frac{c \cdot \sin\theta}{(1+\cos\theta)^2}, \quad \frac{dr_2}{d\theta} = \frac{-d \cdot \sin\theta}{(1-\cos\theta)^2}$$

For the first parabola,

$$\frac{dy}{dx} = \frac{r_1\cos\theta + r_1'\sin\theta}{-r_1\sin\theta + r_1'\cos\theta}$$
$$= \frac{c \cdot \cos\theta(1+\cos\theta) + c \cdot \sin^2\theta}{-c \cdot \sin\theta(1+\cos\theta) + c \cdot \sin\theta\cos\theta}$$
$$= \frac{1+\cos\theta}{-\sin\theta}.$$

Similarly for the second parabola, $\dfrac{dy}{dx} = \dfrac{\sin\theta}{1+\cos\theta}$.

Because the product of the slopes is -1, they intersect at right angles.

Review Exercises for Chapter 10

1. $4x^2 + y^2 = 4$

Ellipse

Vertex: $(1, 0)$.

Matches (e)

2. $4x^2 - y^2 = 4$

Hyperbola

Vertex: $(1, 0)$

Matches (c)

3. $y^2 = -4x$

Parabola opening to left.

Matches (b)

4. $y^2 - 4x^2 = 4$

Hyperbola

Vertex: $(0, 2)$

Matches (d)

5. $x^2 + 4y^2 = 4$

Ellipse

Vertex: $(0, 1)$

Matches (a)

6. $x^2 = 4y$

Parabola opening upward.

Matches (f)

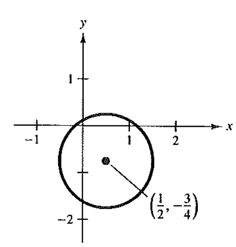
7. $16x^2 + 16y^2 - 16x + 24y - 3 = 0$

$$\left(x^2 - x + \tfrac{1}{4}\right) + \left(y^2 + \tfrac{3}{2}y + \tfrac{9}{16}\right) = \tfrac{3}{16} + \tfrac{1}{4} + \tfrac{9}{16}$$
$$\left(x - \tfrac{1}{2}\right)^2 + \left(y + \tfrac{3}{4}\right)^2 = 1$$

Circle

Center: $\left(\frac{1}{2}, -\frac{3}{4}\right)$

Radius: 1



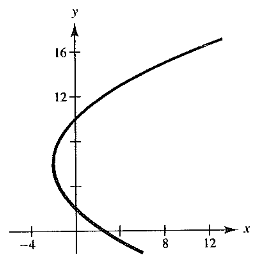
8. $y^2 - 12y - 8x + 20 = 0$

$$y^2 - 12y + 36 = 8x - 20 + 36$$

$$(y-6)^2 = 4(2)(x+2)$$

Parabola

Vertex: $(-2, 6)$



9. $3x^2 - 2y^2 + 24x + 12y + 24 = 0$

$$3(x^2 + 8x + 16) - 2(y^2 - 6y + 9) = -24 + 48 - 18$$

$$\frac{(x+4)^2}{2} - \frac{(y-3)^2}{3} = 1$$

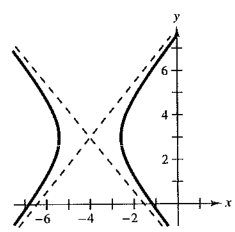
Hyperbola

Center: $(-4, 3)$

Vertices: $(-4 \pm \sqrt{2}, 3)$

Asymptotes:

$$y = 3 \pm \sqrt{\frac{3}{2}}(x+4)$$



10. $5x^2 + y^2 - 20x + 19 = 0$

$$5(x^2 - 4x + 4) + y^2 = -19 + 20$$

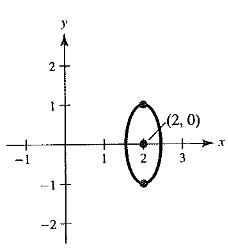
$$5(x-2)^2 + y^2 = 1$$

$$\frac{(x-2)^2}{(1/5)} + y^2 = 1$$

Ellipse

Center: $(2, 0)$

Vertices: $(2, \pm 1)$



11. $3x^2 + 2y^2 - 12x + 12y + 29 = 0$

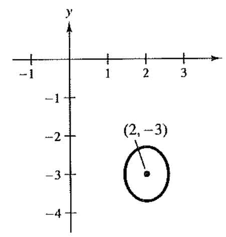
$$3(x^2 - 4x + 4) + 2(y^2 + 6y + 9) = -29 + 12 + 18$$

$$\frac{(x-2)^2}{1/3} + \frac{(y+3)^2}{1/2} = 1$$

Ellipse

Center: $(2, -3)$

Vertices: $\left(2, -3 \pm \frac{\sqrt{2}}{2}\right)$



12. $12x^2 - 12y^2 - 12x + 24y - 45 = 0$

$$12\left(x^2 - x + \frac{1}{4}\right) - 12(y^2 - 2y + 1) = 45 + 3 - 12$$

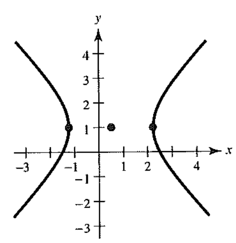
$$12\left(x - \frac{1}{2}\right)^2 - 12(y-1)^2 = 36$$

$$\frac{(x - 1/2)^2}{3} - \frac{(y-1)^2}{3} = 1$$

Hyperbola

Center: $\left(\frac{1}{2}, 1\right)$

Vertices: $\left(\frac{1}{2} \pm \sqrt{3}, 1\right)$



13. Vertex: $(0, 2)$

Directrix: $x = -3$

Parabola opens to the right.

$p = 3$

$(y-2)^2 = 4(3)(x-0)$

$y^2 - 4y - 12x + 4 = 0$

14. Vertex: $(2, 6)$

Focus: $(2, 4)$

Parabola opens downward, $p = -2$

$$(x - 2)^2 = 4(-2)(y - 6)$$
$$x^2 - 4x + 4 = -8y + 48$$
$$x^2 - 4x + 8y - 44 = 0$$

15. Vertices: $(-5, 0), (7, 0)$

Foci: $(-3, 0), (5, 0)$

Horizontal major axis

$a = 6, c = 4, b = \sqrt{36 - 16} = 2\sqrt{5}$

Center: $(1, 0)$

$$\frac{(x - 1)^2}{36} + \frac{y^2}{20} = 1$$

16. Center: $(0, 0)$

Solution points: $(1, 2), (2, 0)$

Substituting the values of the coordinates of the given points into

$$\left(\frac{x^2}{b^2}\right) + \left(\frac{y^2}{a^2}\right) = 1,$$

you obtain the system

$$\left(\frac{1}{b^2}\right) + \left(\frac{4}{a^2}\right) = 1, \frac{4}{b^2} = 1.$$

Solving the system, you have

$$a^2 = \frac{16}{3} \text{ and } b^2 = 4, \left(\frac{x^2}{4}\right) + \left(\frac{3y^2}{16}\right) = 1.$$

17. Vertices: $(\pm 7, 0)$

Foci: $(\pm 9, 0)$

Horizontal transverse axis

Center $(0, 0)$

$a = 7, c = 9, b = \sqrt{81 - 49} = \sqrt{32}$

$$\frac{x^2}{49} - \frac{y^2}{32} = 1$$

18. Foci: $(0, \pm 8)$

Asymptotes: $y = \pm 4x$

Center: $(0, 0)$

Vertical transverse axis

$c = 8$

$y = \frac{a}{b}x = 4x$ asymptote $\rightarrow a = 4b$

$$b^2 = c^2 - a^2 = 64 - (4b)^2 \Rightarrow 17b^2 = 64$$
$$\Rightarrow b^2 = \frac{64}{17} \Rightarrow a^2 = \frac{1024}{17}$$
$$\frac{y^2}{1024/17} - \frac{x^2}{64/17} = 1$$

19. $\frac{x^2}{9} + \frac{y^2}{4} = 1, a = 3, b = 2, c = \sqrt{5}, e = \frac{\sqrt{5}}{3}$

By Example 5 of Section 10.1,

$$C = 12\int_0^{\pi/2} \sqrt{1 - \left(\frac{5}{9}\right)\sin^2\theta}\, d\theta \approx 15.87.$$

20. $\frac{x^2}{4} + \frac{y^2}{25} = 1, a = 5, b = 2, c = \sqrt{21}, e = \frac{\sqrt{21}}{5}$

By Example 5 of Section 10.1,

$$C = 20\int_0^{\pi/2} \sqrt{1 - \frac{21}{25}\sin^2\theta}\, d\theta \approx 23.01.$$

21. $y = x - 2$ has a slope of 1. The perpendicular slope is -1.

$y = x^2 - 2x + 2$

$\frac{dy}{dx} = 2x - 2 = -1$ when $x = \frac{1}{2}$ and $y = \frac{5}{4}$.

Perpendicular line: $y - \frac{5}{4} = -1\left(x - \frac{1}{2}\right)$

$$4x + 4y - 7 = 0$$

22. $2x + y = 5$ has slope -2. The perpendicular slope is $\frac{1}{2}$.

$y = -3x^2 + x - 6$ Parabola

$$y' = -6x + 1 = \tfrac{1}{2}$$
$$6x = \tfrac{1}{2}$$
$$x = \tfrac{1}{12}, \; y = -\tfrac{95}{16}$$

Perpendicular line: $y + \frac{95}{16} = \frac{1}{2}\left(x - \frac{1}{12}\right)$

$$y = \tfrac{1}{2}x - \tfrac{287}{48}$$

23. $y = \frac{1}{200}x^2$

(a) $x^2 = 200y$

$x^2 = 4(50)y$

Focus: $(0, 50)$

(b) $y = \frac{1}{200}x^2$

$y' = \frac{1}{100}x$

$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{10{,}000}}$

$S = 2\pi \int_0^{100} x\sqrt{1 + \frac{x^2}{10{,}000}}\, dx \approx 38{,}294.49$

24. (a) $V = (\pi ab)(\text{Length}) = 12\pi(16) = 192\pi \text{ ft}^3$

(b) $F = 2(62.4)\int_{-3}^{3}(3 - y)\frac{4}{3}\sqrt{9 - y^2}\, dy = \frac{8}{3}(62.4)\left[3\int_{-3}^{3}\sqrt{9 - y^2}\, dy - \int_{-3}^{3} y\sqrt{9 - y^2}\, dy\right]$

$= \frac{8}{3}(62.4)\left[\frac{3}{2}\left(y\sqrt{9 - y^2} + 9 \arcsin \frac{y}{3}\right) + \frac{1}{3}(9 - y^2)^{3/2}\right]_{-3}^{3}$

$= \frac{8}{3}(62.4)\left[\frac{3}{2}\left(\frac{9\pi}{2}\right) - \frac{3}{2}\left(-\frac{9\pi}{2}\right)\right] = \frac{8}{3}(62.4)\left(\frac{27\pi}{2}\right) \approx 7057.274$

(c) You want $\frac{3}{4}$ of the total area of 12π covered. Find h so that

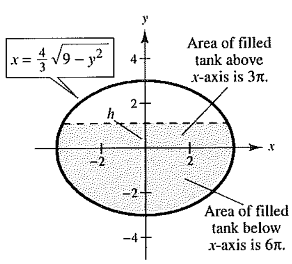
$$2\int_0^h \frac{4}{3}\sqrt{9 - y^2}\, dy = 3\pi$$

$$\int_0^h \sqrt{9 - y^2}\, dy = \frac{9\pi}{8}$$

$$\frac{1}{2}\left[y\sqrt{9 - y^2} + 9 \arcsin\left(\frac{y}{3}\right)\right]_0^h = \frac{9\pi}{8}$$

$$h\sqrt{9 - h^2} + 9 \arcsin\left(\frac{h}{3}\right) = \frac{9\pi}{4}.$$

By Newton's Method, $h \approx 1.212$. So, the total height of the water is $1.212 + 3 = 4.212$ ft.



(d) Area of ends $= 2(12\pi) = 24\pi$

Area of sides $= (\text{Perimeter})(\text{Length})$

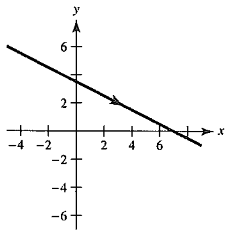
$= 16\int_0^{\pi/2}\left(\sqrt{1 - \left(\frac{7}{16}\right)\sin^2\theta}\right) d\theta(16)$ [from Example 5 of Section 9.1] ≈ 353.656

Total area $= 24\pi + 353.656 \approx 429.054$

25. $x = 1 + 8t,\ y = 3 - 4t$

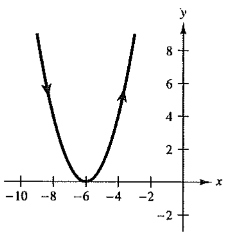
$$t = \frac{x-1}{8} \Rightarrow y = 3 - 4\left(\frac{x-1}{8}\right) = \frac{7}{2} - \frac{x}{2}$$

$x + 2y - 7 = 0$, Line



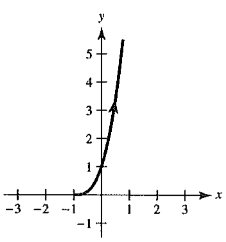
26. $x = t - 6,\ y = t^2$

$t = x + 6 \Rightarrow y = (x+6)^2$, Parabola



27. $x = e^t - 1,\ y = e^{3t}$

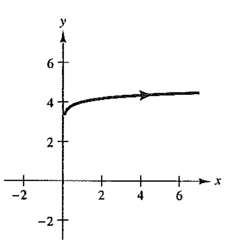
$e^t = x + 1 \Rightarrow y = (x+1)^3, x > -1$



28. $x = e^{4t},\ y = t + 4$

$t = y - 4 \Rightarrow x = e^{4y-16}$

or, $4t = \ln x \Rightarrow y = \dfrac{\ln x}{4} + 4, x > 0$

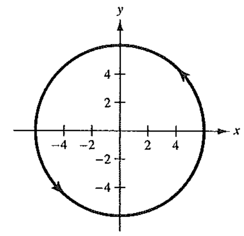


29. $x = 6\cos\theta,\ y = 6\sin\theta$

$$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$$

$x^2 + y^2 = 36$

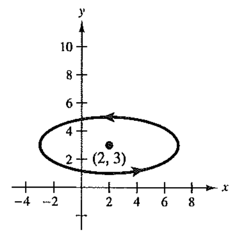
Circle



30. $x = 2 + 5\cos t,\ y = 3 + 2\sin t$

$$\left(\frac{x-2}{5}\right)^2 + \left(\frac{y-3}{2}\right)^2 = \cos^2 t + \sin^2 t = 1$$

$$\frac{(x-2)^2}{25} + \frac{(y-3)^2}{4} = 1 \quad \text{Ellipse}$$

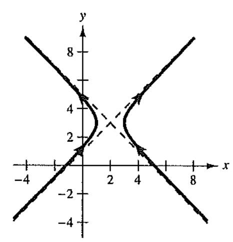


31. $x = 2 + \sec\theta,\ y = 3 + \tan\theta$

$(x-2)^2 = \sec^2\theta = 1 + \tan^2\theta = 1 + (y-3)^2$

$(x-2)^2 - (y-3)^2 = 1$

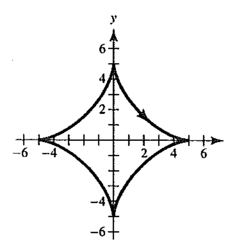
Hyperbola



32. $x = 5\sin^3\theta,\ y = 5\cos^3\theta$

$$\left(\frac{x}{5}\right)^{2/3} + \left(\frac{y}{5}\right)^{2/3} = 1$$

$$x^{2/3} + y^{2/3} = 5^{2/3}$$



33. $x = 3 + (3 - (-2))t = 3 + 5t$

$y = 2 + (2 - 6)t = 2 - 4t$

(other answers possible)

34. $(x-h)^2 + (y-k)^2 = r^2$

$(x+4)^2 + (y+5)^2 = 9$

$x = -4 + 3\cos t$

$y = -5 + 3\sin t$

35. $\dfrac{(x+3)^2}{16} + \dfrac{(y-4)^2}{9} = 1$

Let $\dfrac{(x+3)^2}{16} = \cos^2\theta$ and $\dfrac{(y-4)^2}{9} = \sin^2\theta$.

Then $x = -3 + 4\cos\theta$ and $y = 4 + 3\sin\theta$.

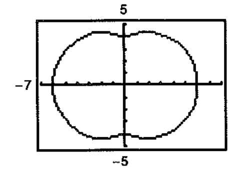
36. $a = 4, c = 5, b^2 = c^2 - a^2 = 9, \dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$

Let $\dfrac{y^2}{16} = \sec^2\theta$ and $\dfrac{x^2}{9} = \tan^2\theta$.

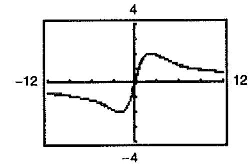
Then $x = 3\tan\theta$ and $y = 4\sec\theta$.

37. $x = \cos 3\theta + 5\cos\theta$

$y = \sin 3\theta + 5\sin\theta$



38. (a) $x = 2\cot\theta, y = 4\sin\theta\cos\theta, 0 < \theta < \pi$



(b) $(4 + x^2)y = (4 + 4\cot^2\theta)4\sin\theta\cos\theta$

$= 16\csc^2\theta \cdot \sin\theta \cdot \cos\theta$

$= 16\dfrac{\cos\theta}{\sin\theta} = 8(2\cot\theta) = 8x$

39. $x = 2 + 5t, y = 1 - 4t$

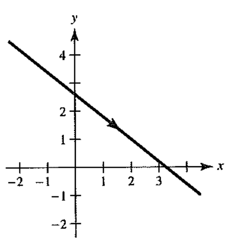
(a) $\dfrac{dy}{dx} = -\dfrac{4}{5}$

No horizontal tangent

(b) $t = \dfrac{x-2}{5} \Rightarrow y = 1 - 4\left(\dfrac{x-2}{5}\right) = 1 - \dfrac{4}{5}x + \dfrac{8}{5} = -\dfrac{4}{5}x + \dfrac{13}{5}$

$4x + 5y - 13 = 0$

(c)

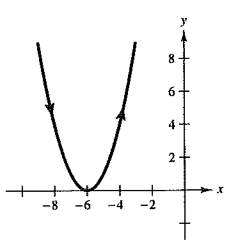


40. $x = t - 6, y = t^2$

(a) $\dfrac{dy}{dx} = \dfrac{2t}{1} = 2t$

(b) $t = x + 6 \Rightarrow y = (x+6)^2$, parabola

(c)



41. $x = \dfrac{1}{t}$

$y = 2t + 3$

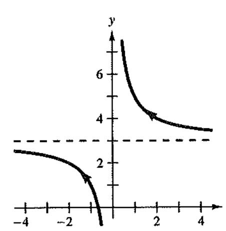
(a) $\dfrac{dy}{dx} = \dfrac{2}{-1/t^2} = -2t^2$

No horizontal tangents, $(t \neq 0)$

(b) $t = \dfrac{1}{x}$

$y = \dfrac{2}{x} + 3$

(c)



42. $x = \dfrac{1}{t}$

$y = t^2$

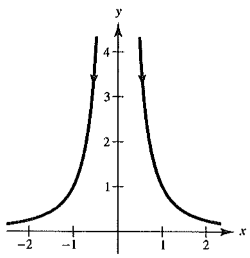
(a) $\dfrac{dy}{dx} = \dfrac{2t}{-1/t^2} = -2t^3$

No horizontal tangents, $(t \neq 0)$

(b) $t = \dfrac{1}{x}$

$y = \dfrac{1}{x^2}$

(c)



43. $x = \dfrac{1}{2t+1}$

$y = \dfrac{1}{t^2 - 2t}$

(a) $\dfrac{dy}{dx} = \dfrac{\dfrac{-(2t-2)}{(t^2-2t)^2}}{\dfrac{-2}{(2t+1)^2}} = \dfrac{(t-1)(2t+1)^2}{t^2(t-2)^2} = 0$

when $t = 1$.

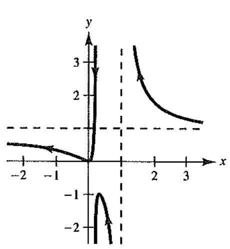
Point of horizontal tangency: $\left(\dfrac{1}{3}, -1\right)$

(b) $2t + 1 = \dfrac{1}{x} \Rightarrow t = \dfrac{1}{2}\left(\dfrac{1}{x} - 1\right)$

$y = \dfrac{1}{\dfrac{1}{2}\left(\dfrac{1-x}{x}\right)\left[\dfrac{1}{2}\left(\dfrac{1-x}{x}\right) - 2\right]}$

$= \dfrac{4x^2}{(1-x)^2 - 4x(1-x)} = \dfrac{4x^2}{(5x-1)(x-1)}, (x \neq 0)$

(c)



44. $x = 2t - 1$

$y = \dfrac{1}{t^2 - 2t}$

(a) $\dfrac{dy}{dx} = \dfrac{-(t^2-2t)^{-2}(2t-2)}{2}$

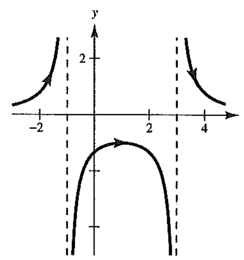
$= \dfrac{1-t}{t^2(t-2)^2} = 0$ when $t = 1$.

Point of horizontal tangency: $(1, -1)$

(b) $t = \dfrac{x+1}{2}$

$y = \dfrac{1}{[(x+1)/2]^2 - 2[(x+1)/2]} = \dfrac{4}{(x-3)(x+1)}$

(c)



45. $x = 5 + \cos\theta$, $y = 3 + 4\sin\theta$

(a) $\dfrac{dy}{dx} = \dfrac{4\cos\theta}{-\sin\theta} = -4\cot\theta$

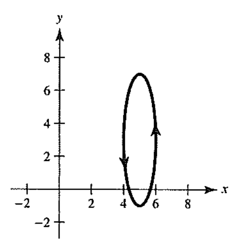
Horizontal tangents:

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2} \Rightarrow (5, 7), (5, -1)$

(b) $(x-5)^2 + \left(\dfrac{(y-3)}{4}\right)^2 = 1$

$(x-5)^2 + \dfrac{(y-3)^2}{16} = 1$, Ellipse

(c)



46. $x = 10\cos\theta,\ y = 10\sin\theta$

(a) $\dfrac{dy}{dx} = \dfrac{10\cos\theta}{-10\sin\theta} = -\cot\theta$

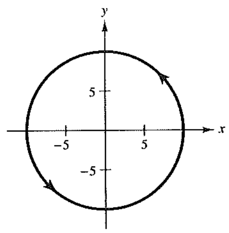
Horizontal tangents:

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2} \Rightarrow (0, 10), (0, -10)$

(b) $\left(\dfrac{x}{10}\right)^2 + \left(\dfrac{y}{10}\right)^2 = 1$

$x^2 + y^2 = 100$, Circle

(c)



47. $x = \cos^3\theta$

$y = 4\sin^3\theta$

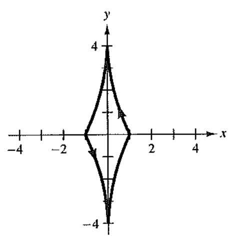
(a) $\dfrac{dy}{dx} = \dfrac{12\sin^2\theta\cos\theta}{3\cos^2\theta(-\sin\theta)} = \dfrac{-4\sin\theta}{\cos\theta} = -4\tan\theta = 0$

when $\theta = 0, \pi$.

But, $\dfrac{dy}{dt} = \dfrac{dx}{dt} = 0$ at $\theta = 0, \pi$. So no points of horizontal tangency.

(b) $x^{2/3} + \left(\dfrac{y}{4}\right)^{2/3} = 1$

(c)



48. $x = e^t$

$y = e^{-t}$

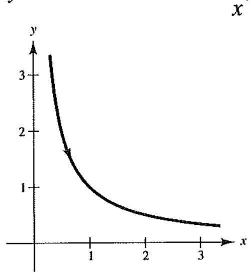
(a) $\dfrac{dy}{dx} = \dfrac{-e^{-t}}{e^t} = -\dfrac{1}{e^{2t}} = -\dfrac{1}{x^2}$

No horizontal tangents

(b) $t = \ln x$

$y = e^{-\ln x} = e^{\ln(1/x)} = \dfrac{1}{x},\ x > 0$

(c)



49. $x = 5 - t,\ y = 2t^2$

$\dfrac{dx}{dt} = -1, \dfrac{dy}{dt} = 4t$

Horizontal tangent at $t = 0$: $(5, 0)$

No vertical tangents

50. $x = t + 2,\ y = t^3 - 2t$

$\dfrac{dx}{dt} = 1, \dfrac{dy}{dt} = 3t^2 - 2$

$\dfrac{dy}{dt} = 0$ for $t = \pm\sqrt{\dfrac{2}{3}} = \dfrac{\pm\sqrt{6}}{3}$

Horizontal tangents:

$t = \dfrac{\sqrt{2}}{3}$: $(x, y) = \left(\dfrac{\sqrt{6}}{3} + 2, \dfrac{2\sqrt{6}}{9} - \dfrac{2}{3}\sqrt{6}\right)$

$\approx (2.8165, -1.0887)$

$t = -\dfrac{\sqrt{6}}{3}$: $(x, y) = \left(-\dfrac{\sqrt{6}}{3} + 2, \dfrac{2}{3}\sqrt{6} - \dfrac{2\sqrt{6}}{9}\right)$

$\approx (1.1835, 1.0887)$

No vertical tangents

51. $x = 2 + 2\sin\theta,\ y = 1 + \cos\theta$

$\dfrac{dx}{d\theta} = 2\cos\theta, \dfrac{dy}{d\theta} = -\sin\theta$

$\dfrac{dy}{d\theta} = 0$ for $\theta = 0, \pi, 2\pi, \ldots$

Horizontal tangents: $(x, y) = (2, 2), (2, 0)$

$\dfrac{dx}{d\theta} = 0$ for $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \ldots$

Vertical tangents: $(x, y) = (4, 1), (0, 1)$

52. $x = 2 - 2\cos\theta,\ y = 2\sin 2\theta$

$\dfrac{dx}{d\theta} = 2\sin\theta, \dfrac{dy}{d\theta} = 4\cos 2\theta$

$\dfrac{dy}{d\theta} = 0$ for $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}, \ldots$

Horizontal tangents: $(x, y) = \left(2 \pm \sqrt{2}, 2\right)$,

$\left(2 \pm \sqrt{2}, -2\right)$

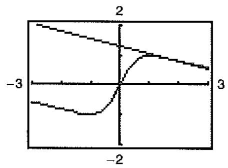
$\dfrac{dx}{d\theta} = 0$ for $\theta = 0, \pi, 2\pi, \ldots$

Vertical tangents: $(x, y) = (0, 0), (4, 0)$

53. $x = \cot\theta$

$y = \sin 2\theta = 2\sin\theta\cos\theta$

(a), (c)

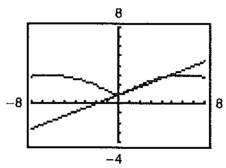


(b) At $\theta = \dfrac{\pi}{6}, \dfrac{dx}{d\theta} = -4, \dfrac{dy}{d\theta} = 1$, and $\dfrac{dy}{dx} = -\dfrac{1}{4}$.

54. $x = 2\theta - \sin\theta$

$y = 2 - \cos\theta$

(a), (c)



(b) At $\theta = \dfrac{\pi}{6}, \dfrac{dx}{d\theta} \approx 1.134, \left(2 - \dfrac{\sqrt{3}}{2}\right)$,

$\dfrac{dy}{dt} = 0.5$, and $\dfrac{dy}{dx} \approx 0.441$.

55. $x = r(\cos\theta + \theta\sin\theta)$

$y = r(\sin\theta - \theta\cos\theta)$

$$\frac{dx}{d\theta} = r\theta\cos\theta$$

$$\frac{dy}{d\theta} = r\theta\sin\theta$$

$$s = r\int_0^{\pi}\sqrt{\theta^2\cos^2\theta + \theta^2\sin^2\theta}\,d\theta$$

$$= r\int_0^{\pi}\theta\,d\theta = \frac{r}{2}\Big[\theta^2\Big]_0^{\pi} = \frac{1}{2}\pi^2 r$$

56. $x = 6\cos\theta$

$y = 6\sin\theta$

$$\frac{dx}{d\theta} = -6\sin\theta$$

$$\frac{dy}{d\theta} = 6\cos\theta$$

$$s = \int_0^{\pi}\sqrt{36\sin^2\theta + 36\cos^2\theta}\,d\theta = \Big[6\theta\Big]_0^{\pi} = 6\pi$$

(one-half circumference of circle)

57. $x = t, y = 3t, 0 \le t \le 2$

$$\frac{dx}{dt} = 1, \frac{dy}{dt} = -3, \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{1+9} = \sqrt{10}$$

(a) $S = 2\pi\int_0^2 3t\sqrt{10}\,dt = 6\sqrt{10}\,\pi\left[\dfrac{t^2}{2}\right]_0^2 = 12\sqrt{10}\,\pi \approx 119.215$

(b) $S = 2\pi\int_0^2\sqrt{10}\,dt = 2\pi\Big[\sqrt{10}t\Big]_0^2 = 4\pi\sqrt{10} \approx 39.738$

58. $x = 2\cos\theta, y = 2\sin\theta, 0 \le \theta \le \dfrac{\pi}{2}$

$$\frac{dx}{dt} = -2\sin\theta, \frac{dy}{dt} = 2\cos\theta, \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = 2$$

(a) $S = 2\pi\int_0^{\pi/2} 2\sin\theta(2)d\theta = 8\pi\Big[-\cos\theta\Big]_0^{\pi/2} = 8\pi$

(b) $S = 2\pi\int_0^{\pi/2} 2\cos\theta(2)d\theta = 8\pi\Big[\sin\theta\Big]_0^{\pi/2} = 8\pi$

[**Note:** The surface is a hemisphere: $\dfrac{1}{2}\left(4\pi\left(2^2\right)\right) = 8\pi$]

59. $x = 3\sin\theta, y = 2\cos\theta$

$$A = \int_a^b y\,dx = \int_{-\pi/2}^{\pi/2} 2\cos\theta(3\cos\theta)\,d\theta$$

$$= 6\int_{-\pi/2}^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta$$

$$= 3\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}$$

$$= 3\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 3\pi$$

60. $A = \int_a^b y\,dx = \int_{\pi}^{0}\sin\theta(-2\sin\theta)\,d\theta$

$$= -\int_{\pi}^{0}\frac{1-\cos 2\theta}{2}\,d\theta$$

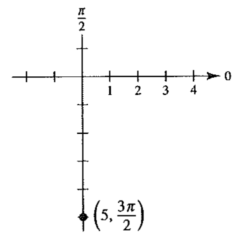
$$= -\left[\theta - \frac{\sin 2\theta}{2}\right]_{\pi}^{0} = \pi$$

61. $(r, \theta) = \left(5, \frac{3\pi}{2}\right)$

$x = r\cos\theta = 5\cos\frac{3\pi}{2} = 0$

$y = r\sin\theta = 5\sin\frac{3\pi}{2} = -5$

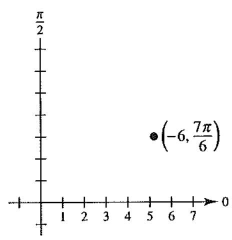
$(x, y) = (0, -5)$



62. $(r, \theta) = \left(-6, \frac{7\pi}{6}\right)$

$x = r\cos\theta = -6\cos\frac{7\pi}{6} = (-6)\left(-\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}$

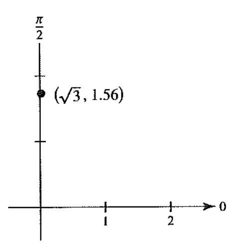
$y = r\sin\theta = -6\sin\frac{7\pi}{6} = 3$



63. $(r, \theta) = \left(\sqrt{3}, 1.56\right)$

$(x, y) = \left(\sqrt{3}\cos(1.56), \sqrt{3}\sin(1.56)\right)$

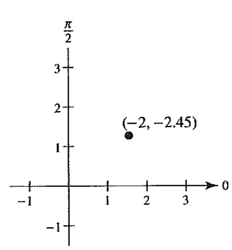
$\approx (0.0187, 1.7319)$



64. $(r, \theta) = (-2, -2.45)$

$(x, y) = \left(-2\cos(-2.45), -\sin(-2.45)\right)$

$\approx (1.5405, 1.2755)$

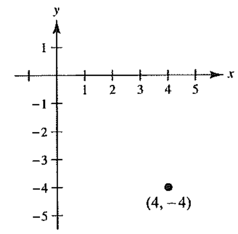


65. $(x, y) = (4, -4)$

$r = \sqrt{4^2 + (-4)^2} = 4\sqrt{2}$

$\theta = \frac{7\pi}{4}$

$(r, \theta) = \left(4\sqrt{2}, \frac{7\pi}{4}\right), \left(-4\sqrt{2}, \frac{3\pi}{4}\right)$

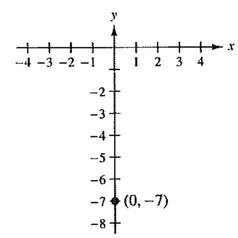


66. $(x, y) = (0, -7)$

$r = \sqrt{0^2 + (-7)^2} = 7$

$\tan\theta$ undefined $\Rightarrow \theta = \frac{3\pi}{2}$

$(r, \theta) = \left(7, \frac{3\pi}{2}\right), \left(-7, \frac{\pi}{2}\right)$

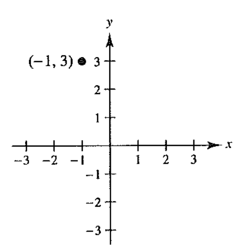


67. $(x, y) = (-1, 3)$

$r = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$

$\theta = \arctan(-3) \approx 1.89(108.43°)$

$(r, \theta) = \left(\sqrt{10}, 1.89\right), \left(-\sqrt{10}, 5.03\right)$

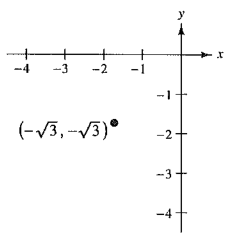


68. $(x, y) = \left(-\sqrt{3}, -\sqrt{3}\right)$

$r = \sqrt{3 + 3} = \sqrt{6}$

$\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$

$(r, \theta) = \left(\sqrt{6}, \frac{5\pi}{4}\right), \left(-\sqrt{6}, \frac{\pi}{4}\right)$



69.
$$r = 3\cos\theta$$
$$r^2 = 3r\cos\theta$$
$$x^2 + y^2 = 3x$$
$$x^2 + y^2 - 3x = 0$$

70.
$$r = 10$$
$$r^2 = 100$$
$$x^2 + y^2 = 100$$

71.
$$r = -2(1 + \cos\theta)$$
$$r^2 = -2r(1 + \cos\theta)$$
$$x^2 + y^2 = -2\left(\pm\sqrt{x^2 + y^2}\right) - 2x$$
$$\left(x^2 + y^2 + 2x\right)^2 = 4\left(x^2 + y^2\right)$$

72.
$$r = \frac{1}{2 - \cos\theta}$$
$$2r - r\cos\theta = 1$$
$$2\left(\pm\sqrt{x^2 + y^2}\right) - x = 1$$
$$4\left(x^2 + y^2\right) = (x + 1)^2$$
$$3x^2 + 4y^2 - 2x - 1 = 0$$

73.
$$r^2 = \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$r^4 = r^2\cos^2\theta - r^2\sin^2\theta$$
$$\left(x^2 + y^2\right)^2 = x^2 - y^2$$

74.
$$r = 4\sec\left(\theta - \frac{\pi}{3}\right) = \frac{4}{\cos\left[\theta - (\pi/3)\right]}$$
$$= \frac{4}{(1/2)\cos\theta + \left(\sqrt{3}/2\right)\sin\theta}$$
$$r\left(\cos\theta + \sqrt{3}\sin\theta\right) = 8$$
$$x + \sqrt{3}y = 8$$

75.
$$r = 4\cos 2\theta\sec\theta$$
$$= 4\left(2\cos^2\theta - 1\right)\left(\frac{1}{\cos\theta}\right)$$
$$r\cos\theta = 8\cos^2\theta - 4$$
$$x = 8\left(\frac{x^2}{x^2 + y^2}\right) - 4$$
$$x^3 + xy^2 = 4x^2 - 4y^2$$
$$y^2 = x^2\left(\frac{4 - x}{4 + x}\right)$$

76.
$$\theta = \frac{3\pi}{4}$$
$$\tan\theta = -1$$
$$\frac{y}{x} = -1$$
$$y = -x$$

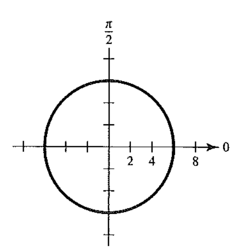
77.
$$\left(x^2 + y^2\right)^2 = ax^2y$$
$$r^4 = a\left(r^2\cos^2\theta\right)(r\sin\theta)$$
$$r = a\cos^2\theta\sin\theta$$

78.
$$x^2 + y^2 - 4x = 0$$
$$r^2 - 4r\cos\theta = 0$$
$$r = 4\cos\theta$$

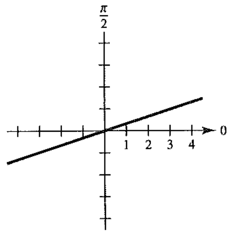
79.
$$x^2 + y^2 = a^2\left(\arctan\frac{y}{x}\right)^2$$
$$r^2 = a^2\theta^2$$

80.
$$\left(x^2 + y^2\right)\left(\arctan\frac{y}{x}\right)^2 = a^2$$
$$r^2\theta^2 = a^2$$

81. $r = 6$, Circle radius 6



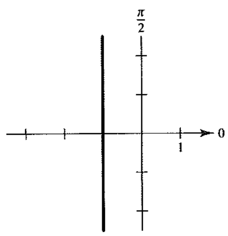
82. $\theta = \dfrac{\pi}{10}$, Line



83. $r = -\sec\theta = \dfrac{-1}{\cos\theta}$

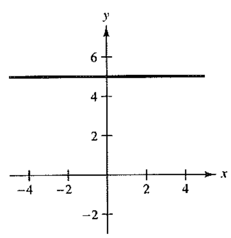
$r\cos\theta = -1, x = -1$

Vertical line



84. $r = 5\csc\theta \Rightarrow r\sin\theta = y = 5$

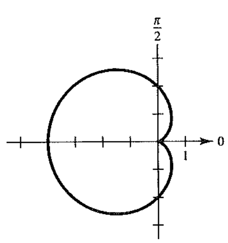
Horizontal line



85. $r = -2(1 + \cos\theta)$

Cardioid

Symmetric to polar axis

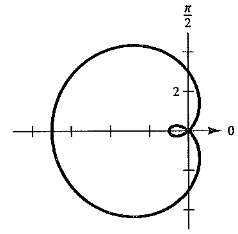


θ	0	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$	$\dfrac{2\pi}{3}$	π
r	-4	-3	-2	-1	0

86. $r = 3 - 4\cos\theta$

Limaçon

Symmetric to polar axis

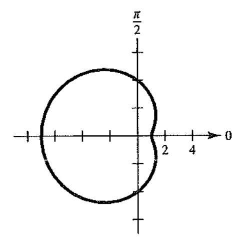


θ	0	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$	$\dfrac{2\pi}{3}$	π
r	-1	1	3	5	7

87. $r = 4 - 3\cos\theta$

Limaçon

Symmetric to polar axis

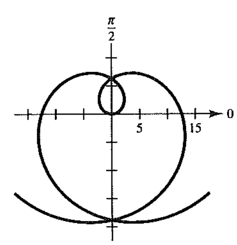


θ	0	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$	$\dfrac{2\pi}{3}$	π
r	1	$\dfrac{5}{2}$	4	$\dfrac{11}{2}$	7

88. $r = 4\theta$

Spiral

Symmetric to $\theta = \dfrac{\pi}{2}$



θ	0	$\dfrac{\pi}{4}$	$\dfrac{\pi}{2}$	$\dfrac{3\pi}{4}$	π	$\dfrac{3\pi}{2}$	2π
r	0	π	2π	3π	4π	6π	8π

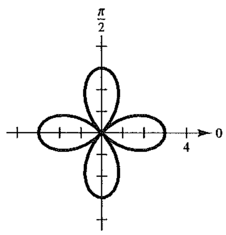
89. $r = -3\cos 2\theta$

Rose curve with four petals

Symmetric to polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $(-3, 0), \left(3, \dfrac{\pi}{2}\right), (-3, \pi), \left(3, \dfrac{3\pi}{2}\right)$

Tangents at the pole: $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$



90. $r = \cos 5\theta$

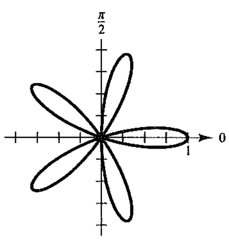
Rose curve with five petals

Symmetric to polar axis

Relative extrema:

$(1, 0), \left(-1, \dfrac{\pi}{5}\right), \left(1, \dfrac{2\pi}{5}\right), \left(-1, \dfrac{3\pi}{5}\right), \left(1, \dfrac{4\pi}{5}\right)$

Tangents at the pole: $\theta = \dfrac{\pi}{10}, \dfrac{3\pi}{10}, \dfrac{\pi}{2}, \dfrac{7\pi}{10}, \dfrac{9\pi}{10}$



91. $r^2 = 4\sin^2 2\theta$

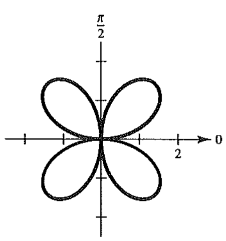
$r = \pm 2\sin(2\theta)$

Rose curve with four petals

Symmetric to the polar axis, $\theta = \dfrac{\pi}{2}$, and pole

Relative extrema: $\left(\pm 2, \dfrac{\pi}{4}\right), \left(\pm 2, \dfrac{3\pi}{4}\right)$

Tangents at the pole: $\theta = 0, \dfrac{\pi}{2}$



92. $r^2 = \cos 2\theta$

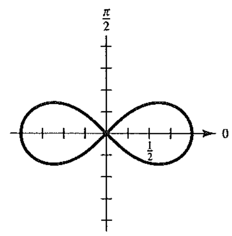
Lemniscate

Symmetric to the polar axis

Relative extrema: $(\pm 1, 0)$

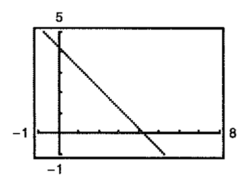
Tangents at the pole: $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

θ	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$
r	± 1	$\pm\dfrac{\sqrt{2}}{2}$	0



93. $r = \dfrac{3}{\cos\theta - (\pi/4)}$

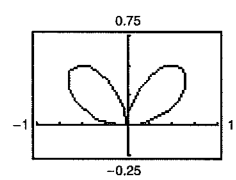
Graph of $r = 3\sec\theta$ rotated through an angle of $\pi/4$



94. $r = 2\sin\theta\cos^2\theta$

Bifolium

Symmetric to $\theta = \pi/2$



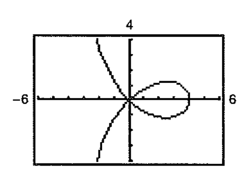
95. $r = 4\cos 2\theta\sec\theta$

Strophoid

Symmetric to the polar axis

$r \Rightarrow \infty$ as $\theta \Rightarrow \dfrac{\pi^-}{2}$

$r \Rightarrow \infty$ as $\theta \Rightarrow \dfrac{-\pi^+}{2}$



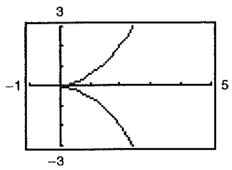
96. $r = 4(\sec\theta - \cos\theta)$

Semicubical parabola

Symmetric to the polar axis

$r \Rightarrow \infty$ as $\theta \Rightarrow \dfrac{\pi^-}{2}$

$r \Rightarrow \infty$ as $\theta \Rightarrow \dfrac{-\pi^+}{2}$



97. $r = 1 - 2\cos\theta$

(a) The graph has polar symmetry and the tangents at the pole are $\theta = \dfrac{\pi}{3}, -\dfrac{\pi}{3}$.

(b) $$\frac{dy}{dx} = \frac{2\sin^2\theta + (1 - 2\cos\theta)\cos\theta}{2\sin\theta\cos\theta - (1 - 2\cos\theta)\sin\theta}$$

Horizontal tangents: $-4\cos^2\theta + \cos\theta + 2 = 0$,

$$\cos\theta = \frac{-1 \pm \sqrt{1 + 32}}{-8} = \frac{1 \pm \sqrt{33}}{8}$$

When

$$\cos\theta = \frac{1 \pm \sqrt{33}}{8}, r = 1 - 2\left(\frac{1 + \sqrt{33}}{8}\right) = \frac{3 \mp \sqrt{33}}{4},$$

$$\left[\frac{3 - \sqrt{33}}{4}, \arccos\left(\frac{1 + \sqrt{33}}{8}\right)\right] \approx (-0.686, 0.568)$$

$$\left[\frac{3 - \sqrt{33}}{4}, -\arccos\left(\frac{1 + \sqrt{33}}{8}\right)\right] \approx (-0.686, -0.568)$$

$$\left[\frac{3 + \sqrt{33}}{4}, \arccos\left(\frac{1 - \sqrt{33}}{8}\right)\right] \approx (2.186, 2.206)$$

$$\left[\frac{3 + \sqrt{33}}{4}, -\arccos\left(\frac{1 - \sqrt{33}}{8}\right)\right] \approx (2.186, -2.206).$$

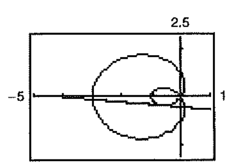
Vertical tangents:

$$\sin\theta(4\cos\theta - 1) = 0, \sin\theta = 0, \cos\theta = \frac{1}{4},$$

$$\theta = 0, \pi, \theta = \pm\arccos\left(\frac{1}{4}\right), (-1, 0), (3, \pi)$$

$$\left(\frac{1}{2}, \pm\arccos\frac{1}{4}\right) \approx (0.5, \pm 1.318)$$

(c)



98. $r^2 = 4\sin(2\theta)$

(a) $$2r\left(\frac{dr}{d\theta}\right) = 8\cos(2\theta)$$

$$\frac{dr}{d\theta} = \frac{4\cos(2\theta)}{r}$$

Tangents at the pole: $\theta = 0, \dfrac{\pi}{2}$

(b) $$\frac{dy}{dx} = \frac{r\cos\theta + [(4\cos 2\theta\sin\theta)/r]}{-r\sin\theta + [(4\cos 2\theta\cos\theta)/r]}$$

$$= \frac{\cos(2\theta)\sin\theta + \sin(2\theta)\cos\theta}{\cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta}$$

Horizontal tangents:

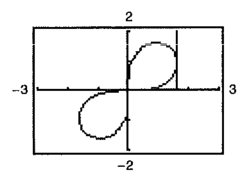
$\dfrac{dy}{dx} = 0$ when $\cos(2\theta)\sin\theta + \sin(2\theta)\cos\theta = 0$,

$$\tan\theta = -\tan(2\theta), \theta = 0, \frac{\pi}{3}, (0, 0), \left(\pm\sqrt{2\sqrt{3}}, \frac{\pi}{3}\right)$$

Vertical tangents when
$\cos 2\theta\cos\theta - \sin 2\theta\sin\theta = 0$:

$$\tan 2\theta\tan\theta = 1, \theta = 0, \frac{\pi}{6}, (0, 0), \left(\pm\sqrt{2\sqrt{3}}, \frac{\pi}{6}\right)$$

(c)



99. $r = 1 + \cos\theta, r = 1 - \cos\theta$

The points $(1, \pi/2)$ and $(1, 3\pi/2)$ are the two points of intersection (other than the pole). The slope of the graph of $r = 1 + \cos\theta$ is

$$m_1 = \frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{-\sin^2\theta + \cos\theta(1 + \cos\theta)}{-\sin\theta\cos\theta - \sin\theta(1 + \cos\theta)}.$$

At $(1, \pi/2)$, $m_1 = -1/-1 = 1$ and at $(1, 3\pi/2)$, $m_1 = -1/1 = -1$. The slope of the graph of $r = 1 - \cos\theta$ is

$$m_2 = \frac{dy}{dx} = \frac{\sin^2\theta + \cos\theta(1 - \cos\theta)}{\sin\theta\cos\theta - \sin\theta(1 - \cos\theta)}.$$

At $(1, \pi/2)$, $m_2 = 1/-1 = -1$ and at $(1, 3\pi/2)$, $m_2 = 1/1 = 1$. In both cases, $m_1 = -1/m_2$ and you conclude that the graphs are orthogonal at $(1, \pi/2)$ and $(1, 3\pi/2)$.

100. $r = a\sin\theta, r = a\cos\theta$

The points of intersection are $\left(a/\sqrt{2}, \pi/4\right)$ and $(0, 0)$. For $r = a\sin\theta$,

$$m_1 = \frac{dy}{dx} = \frac{a\cos\theta\sin\theta + a\sin\theta\cos\theta}{a\cos^2\theta - a\sin^2\theta} = \frac{2\sin\theta\cos\theta}{\cos 2\theta}.$$

At $\left(a/\sqrt{2}, \pi/4\right)$, m_1 is undefined and at $(0, 0)$, $m_1 = 0$.

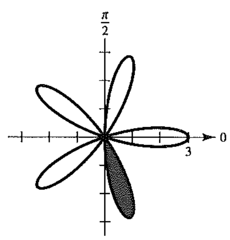
For $r = a\cos\theta$,

$$m_2 = \frac{dy}{dx} = \frac{-a\sin^2\theta + a\cos^2\theta}{-a\sin\theta\cos\theta - a\cos\theta\sin\theta} = \frac{\cos 2\theta}{-2\sin\theta\cos\theta}.$$

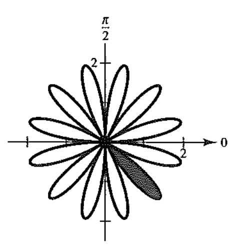
At $\left(a/\sqrt{2}, \pi/4\right)$, $m_2 = 0$ and at $(0, 0)$, m_2 is undefined.

So, the graphs are orthogonal at $\left(a/\sqrt{2}, \pi/4\right)$ and $(0, 0)$.

101. $$A = 2\cdot\frac{1}{2}\int_0^{\pi/10}[3\cos 5\theta]^2\,d\theta = \int_0^{\pi/10} 9\left(\frac{1+\cos(10\theta)}{2}\right)d\theta = \frac{9}{2}\left[\theta + \frac{\sin(10\theta)}{2}\right]_0^{\pi/10} = \frac{9}{2}\left[\frac{\pi}{10}\right] = \frac{9\pi}{20}$$

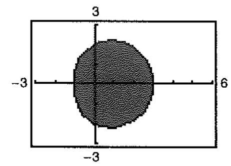


102. $$A = 2\cdot\frac{1}{2}\int_0^{\pi/12}[2\sin 6\theta]^2\,d\theta = \int_0^{\pi/12} 4\left(\frac{1-\cos 12\theta}{2}\right)d\theta = 2\left[\theta - \frac{\sin 12\theta}{12}\right]_0^{\pi/12} = 2\left[\frac{\pi}{12}\right] = \frac{\pi}{6}$$



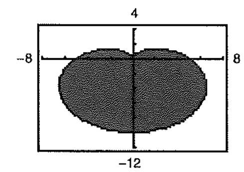
103. $r = 2 + \cos\theta$

$$A = 2\left[\frac{1}{2}\int_0^{\pi}(2+\cos\theta)^2\,d\theta\right] \approx 14.14, \left(\frac{9\pi}{2}\right)$$



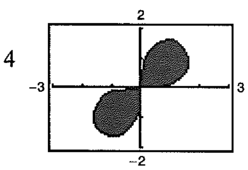
104. $r = 5(1 - \sin\theta)$

$$A = 2\left[\frac{1}{2}\int_{\pi/2}^{3\pi/2}\left[5(1-\sin\theta)\right]^2\right]d\theta \approx 117.81, \left(\frac{75\pi}{2}\right)$$



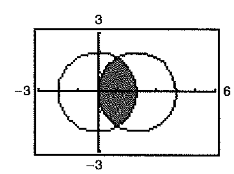
105. $r^2 = 4\sin 2\theta$

$$A = 2\left[\frac{1}{2}\int_0^{\pi/2} 4\sin 2\theta\,d\theta\right] = 4$$



106. $r = 4\cos\theta, r = 2$

$$A = 2\left[\frac{1}{2}\int_0^{\pi/3} 4\,d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2}(4\cos\theta)^2\,d\theta\right] \approx 4.91$$



107. $r = 1 - \cos\theta$

$r = 1 + \sin\theta$

The cardioids intersect at 3 points:

$1 - \cos\theta = 1 + \sin\theta$

$$\tan\theta = -1 \Rightarrow \theta = \frac{3\pi}{4}, \frac{7\pi}{4}.$$

$$\left(1 + \frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right), \left(1 - \frac{\sqrt{2}}{2}, \frac{7\pi}{4}\right), (0, 0)$$

108. The circle $r = 3\sin\theta$ and cardioid $r = 1 + \sin\theta$ intersect at 3 points:

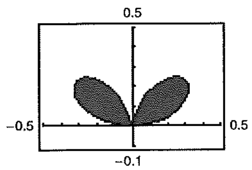
$3\sin\theta = 1 + \sin\theta$

$$\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

$$\left(\frac{3}{2}, \frac{\pi}{6}\right), \left(\frac{3}{2}, \frac{5\pi}{6}\right), (0, 0)$$

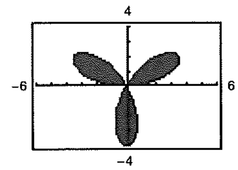
109. $r = \sin\theta\cos^2\theta$

$$A = 2\left[\frac{1}{2}\int_0^{\pi/2}\left(\sin\theta\cos^2\theta\right)^2 d\theta\right] \approx 0.10, \left(\frac{\pi}{32}\right)$$



110. $r = 4\sin 3\theta$

$$A = 3\left[\frac{1}{2}\int_0^{\pi/3}\left(4\sin 3\theta\right)^2 d\theta\right] \approx 12.57\,(4\pi)$$



111. $r = 3, r^2 = 18\sin 2\theta$

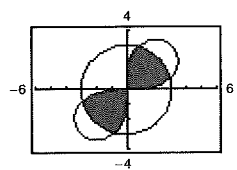
$9 = r^2 = 18\sin 2\theta$

$\sin 2\theta = \frac{1}{2}$

$\theta = \frac{\pi}{12}$

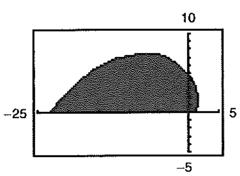
$$A = 2\left[\frac{1}{2}\int_0^{\pi/12} 18\sin 2\theta\,d\theta + \frac{1}{2}\int_{\pi/12}^{5\pi/12} 9\,d\theta + \frac{1}{2}\int_{5\pi/12}^{\pi/2} 18\sin 2\theta\,d\theta\right]$$

$\approx 1.2058 + 9.4248 + 1.2058 \approx 11.84$



112. $r = e^{\theta}, 0 \le \theta \le \pi$

$$A = \frac{1}{2}\int_0^{\pi}\left(e^{\theta}\right)^2 d\theta \approx 133.62$$



113. $r = a(1 - \cos\theta), 0 \le \theta \le \pi$

$$\frac{dr}{d\theta} = a\sin\theta$$

$$s = \int_0^{\pi}\sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\,d\theta$$

$$= \sqrt{2}a\int_0^{\pi}\sqrt{1-\cos\theta}\,d\theta$$

$$= \sqrt{2}a\int_0^{\pi}\frac{\sin\theta}{\sqrt{1+\cos\theta}}\,d\theta$$

$$= -2\sqrt{2}a\left[(1+\cos\theta)^{1/2}\right]_0^{\pi} = 4a$$

114. $r = a\cos 2\theta, -\pi/4 \le \theta \le \pi/2$

$$\frac{dr}{d\theta} = -2a\sin 2\theta$$

$$s = \int_{-\pi/2}^{\pi/2}\sqrt{a^2\cos^2 2\theta + 4a^2\sin^2 2\theta}\,d\theta$$

$$= a\int_{-\pi/2}^{\pi/2}\sqrt{1 + 3\sin^2 2\theta}\,d\theta$$

Using a graphing utility, $s \approx 4.8442a$.

115. $f(\theta) = 1 + 4\cos\theta$

$f'(\theta) = -4\sin\theta$

$$\sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{(1+4\cos\theta)^2 + (-4\sin\theta)^2}$$

$$= \sqrt{17 + 8\cos\theta}$$

$$S = 2\pi\int_0^{\pi/2}(1+4\cos\theta)\sin\theta\sqrt{17+8\cos\theta}\,d\theta$$

$$= \frac{34\pi\sqrt{17}}{5} \approx 88.08$$

116. $f(\theta) = 2\sin\theta$

$f'(\theta) = 2\cos\theta$

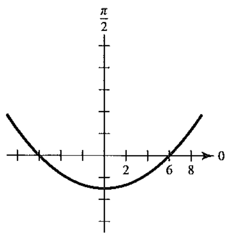
$$\sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{4\sin^2\theta + 4\cos^2\theta} = 2$$

$$S = 2\pi\int_0^{\pi/2} 2\sin\theta\cos\theta(2)\,d\theta = 4\pi$$

117. $r = \dfrac{6}{1 - \sin\theta}$

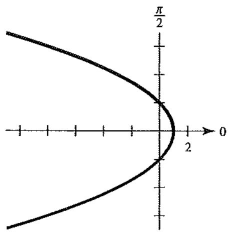
$e = 1,$

Parabola



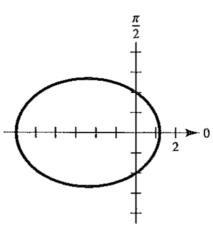
118. $r = \dfrac{2}{1 + \cos\theta}, e = 1$

Parabola



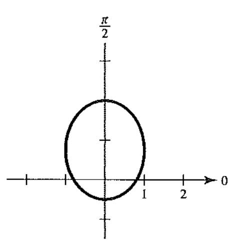
119. $r = \dfrac{6}{3 + 2\cos\theta} = \dfrac{2}{1 + (2/3)\cos\theta}, e = \dfrac{2}{3}$

Ellipse



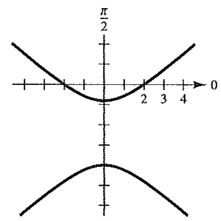
120. $r = \dfrac{4}{5 - 3\sin\theta} = \dfrac{4/5}{1 - (3/5)\sin\theta}, e = \dfrac{3}{5}$

Ellipse



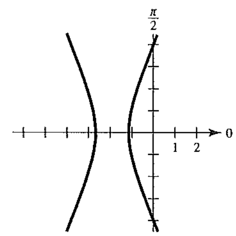
121. $r = \dfrac{4}{2 - 3\sin\theta} = \dfrac{2}{1 - (3/2)\sin\theta}, e = \dfrac{3}{2}$

Hyperbola



122. $r = \dfrac{8}{2 - 5\cos\theta} = \dfrac{4}{1 - (5/2)\cos\theta}, e = \dfrac{5}{2}$

Hyperbola



123. Circle

Center: $\left(5, \dfrac{\pi}{2}\right) = (0, 5)$ in rectangular coordinates

Solution point: $(0, 0)$

$$x^2 + (y - 5)^5 = 25$$
$$x^2 + y^2 - 10y = 0$$
$$r^2 - 10r\sin\theta = 0$$
$$r = 10\sin\theta$$

124. Line

Slope: $\sqrt{3}$

Solution point: $(0, 0)$

$y = \sqrt{3}x, r\sin\theta = \sqrt{3}\, r\cos\theta,$

$\tan\theta = \sqrt{3}, \theta = \dfrac{\pi}{3}$

125. Parabola

Vertex: $(2, \pi)$

Focus: $(0, 0)$

$e = 1, d = 4$

$r = \dfrac{4}{1 - \cos\theta}$

126. Parabola

Vertex: $\left(2, \dfrac{\pi}{2}\right)$

Focus: $(0, 0)$

$e = 1, d = 4$

$$r = \frac{4}{1 + \sin\theta}$$

127. Ellipse

Vertices: $(5, 0), (1, \pi)$

Focus: $(0, 0)$

$$a = 3, c = 2, e = \frac{2}{3}, d = \frac{5}{2}$$

$$r = \frac{\left(\frac{2}{3}\right)\left(\frac{5}{2}\right)}{1 - \left(\frac{2}{3}\right)\cos\theta} = \frac{5}{3 - 2\cos\theta}$$

128. Hyperbola

Vertices: $(1, 0), (7, 0)$

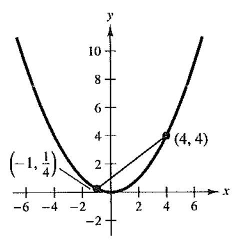
Focus: $(0, 0)$

$$a = 3, c = 4, e = \frac{4}{3}, d = \frac{7}{4}$$

$$r = \frac{\left(\frac{4}{3}\right)\left(\frac{7}{4}\right)}{1 + \left(\frac{4}{3}\right)\cos\theta} = \frac{7}{3 + 4\cos\theta}$$

Problem Solving for Chapter 10

1. (a)



(b) $x^2 = 4y$

$2x = 4y'$

$y' = \dfrac{1}{2}x$

$y - 4 = 2(x - 4) \Rightarrow y = 2x - 4$ Tangent line at $(4, 4)$

$y - \frac{1}{4} = -\frac{1}{2}(x + 1) \Rightarrow y = -\frac{1}{2}x - \frac{1}{4}$ Tangent line at $\left(-1, \frac{1}{4}\right)$

Tangent lines have slopes of 2 and $-\frac{1}{2} \Rightarrow$ perpendicular.

(c) Intersection:

$2x - 4 = -\frac{1}{2}x - \frac{1}{4}$

$8x - 16 = -2x - 1$

$10x = 15$

$x = \frac{3}{2} \Rightarrow \left(\frac{3}{2}, -1\right)$

Point of intersection, $\left(\frac{3}{2}, -1\right)$, is on directrix $y = -1$.

2. Assume $p > 0$.

Let $y = mx + p$ be the equation of the focal chord.

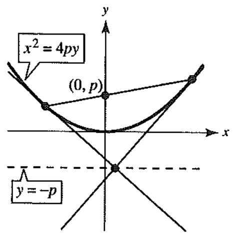
First find x-coordinates of focal chord endpoints:

$$x^2 = 4py = 4p(mx + p)$$

$$x^2 - 4pmx - 4p^2 = 0$$

$$x = \frac{4pm \pm \sqrt{16p^2m^2 + 16p^2}}{2} = 2pm \pm 2p\sqrt{m^2 + 1}$$

$$x^2 = 4py, 2x = 4py' \Rightarrow y' = \frac{x}{2p}.$$



(a) The slopes of the tangent lines at the endpoints are perpendicular because

$$\frac{1}{2p}\left[2pm + 2p\sqrt{m^2 + 1}\right]\frac{1}{2p}\left[2pm - 2p\sqrt{m^2 + 1}\right] = \frac{1}{4p^2}\left[4p^2m^2 - 4p^2(m^2 + 1)\right] = \frac{1}{4p^2}\left[-4p^2\right] = -1$$

(b) Finally, you show that the tangent lines intersect at a point on the directrix $y = -p$.

Let $b = 2pm + 2p\sqrt{m^2 + 1}$ and $c = 2pm - 2p\sqrt{m^2 + 1}$.

$$b^2 = 8p^2m^2 + 4p^2 + 8p^2m\sqrt{m^2 + 1}$$

$$c^2 = 8p^2m^2 + 4p^2 - 8p^2m\sqrt{m^2 + 1}$$

$$\frac{b^2}{4p} = 2pm^2 + p + 2pm\sqrt{m^2 + 1}$$

$$\frac{c^2}{4p} = 2pm^2 + p - 2pm\sqrt{m^2 + 1}$$

Tangent line at $x = b$: $y - \frac{b^2}{4p} = \frac{b}{2p}(x - b) \Rightarrow y = \frac{bx}{2p} - \frac{b^2}{4p}$

Tangent line at $x = c$: $y - \frac{c^2}{4p} = \frac{c}{2p}(x - c) \Rightarrow y = \frac{cx}{2p} - \frac{c^2}{4p}$

Intersection of tangent lines:

$$\frac{bx}{2p} - \frac{b^2}{4p} = \frac{cx}{2p} - \frac{c^2}{4p}$$

$$2bx - b^2 = 2cx - c^2$$

$$2x(b - c) = b^2 - c^2$$

$$2x\left(4p\sqrt{m^2 + 1}\right) = 16p^2m\sqrt{m^2 + 1}$$

$$x = 2pm$$

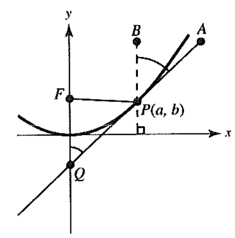
Finally, the corresponding y-value is $y - p$, which shows that the intersection point lies on the directrix.

3. Consider $x^2 = 4py$ with focus $F = (0, p)$.

Let $P(a, b)$ be point on parabola.

$$2x = 4py' \Rightarrow y' = \frac{x}{2p}$$

$$y - b = \frac{a}{2p}(x - a) \quad \text{Tangent line at } P$$



For $x = 0, y = b + \frac{a}{2p}(-a) = b - \frac{a^2}{2p} = b - \frac{4pb}{2p} = -b.$

So, $Q = (0, -b)$.

ΔFQP is isosceles because

$|FQ| = p + b$

$|FP| = \sqrt{(a-0)^2 + (b-p)^2} = \sqrt{a^2 + b^2 - 2bp + p^2} = \sqrt{4pb + b^2 - 2bp + p^2} = \sqrt{(b+p)^2} = b + p.$

So, $\measuredangle FQP = \measuredangle BPA = \measuredangle FPQ$.

4. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, a^2 + b^2 = c^2, MF_2 - MF_1 = 2a$

$$y' = \frac{b^2 x}{a^2 y}$$

Tangent line at $M(x_0, y_0)$: $\quad y - y_0 = \frac{b^2 x_0}{a^2 y_0}(x - x_0)$

$$\frac{yy_0 - y_0^2}{b^2} = \frac{x_0 x - x_0^2}{a^2}$$

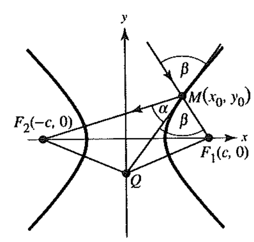
$$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}$$

$$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$$

At $x = 0, y = -\frac{b^2}{y_0} \Rightarrow Q = \left(0, -\frac{b^2}{y_0}\right)$.

$$QF_2 = QF_1 = \sqrt{c^2 + \frac{b^4}{y_0^2}} = d$$

$$MQ = \sqrt{x_0^2 + \left(y_0 + \frac{b^2}{y_0}\right)^2} = f$$



By the Law of Cosines,

$$(F_2Q)^2 = (MF_2)^2 + (MQ)^2 - 2(MF_2)(MQ)\cos\alpha$$

$$d^2 = (MF_2)^2 + f^2 - 2f(MF_2)\cos\alpha$$

$$(F_1Q)^2 = (MF_1)^2 + f^2 - 2f(MF_1)\cos\beta$$

$$d^2 = (MF_1)^2 + f^2 - 2f(MF_1)\cos\beta.$$

$$\cos\alpha = \frac{(MF_2)^2 f^2 - d^2}{2f(MF_2)}, \cos\beta = \frac{(MF_1)^2 + f^2 - d^2}{2f(MF_1)}$$

$MF_2 = MF_1 + 2a$. Let $z = MF_1$.

Slopes: $MF_1: \frac{y_0}{x_0 - c}$; $QF_1: \frac{-b^2}{y_0 c}$; $QF_2: \frac{b^2}{y_0 c}$

To show $\alpha = \beta$, consider

$$\left[(MF_2)^2 + f^2 - d^2\right]\left[2f(MF_1)\right] = \left[(MF_1)^2 + f^2 - d^2\right]\left[2f(MF_2)\right]$$

$$\Leftrightarrow \quad \left[(z + 2a)^2 + f^2 - d^2\right][z] = \left[z^2 + f^2 - d^2\right][z + 2a]$$

$$\Leftrightarrow \quad z^2 + 2az = f^2 - d^2$$

$$\Leftrightarrow \quad (x_0 - c)^2 + y_0^2 + 2az = \left(x_0^2 + \left(y_0 + \frac{b^2}{y_0}\right)^2\right) - \left(c^2 + \frac{b^4}{y_0^2}\right)$$

$$\Leftrightarrow \quad az - x_0c + a^2 = 0$$

$$\Leftrightarrow \quad a\sqrt{(x_0 - c)^2 + y_0^2} = x_0c - a^2$$

$$\Leftrightarrow \quad x_0^2b^2 - a^2y_0^2 = a^2b^2$$

$$\Leftrightarrow \quad \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1.$$

So, $\alpha = \beta$ and the reflective property is verified.

5. (a) In ΔOCB, $\cos\theta = \dfrac{2a}{OB} \Rightarrow OB = 2a \cdot \sec\theta.$

In ΔOAC, $\cos\theta = \dfrac{OA}{2a} \Rightarrow OA = 2a \cdot \cos\theta.$

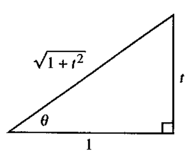
$$\begin{aligned} r = OP = AB = OB - OA &= 2a(\sec\theta - \cos\theta) \\ &= 2a\left(\frac{1}{\cos\theta} - \cos\theta\right) \\ &= 2a \cdot \frac{\sin^2\theta}{\cos\theta} \\ &= 2a \cdot \tan\theta \sin\theta \end{aligned}$$

(b) $x = r\cos\theta = (2a\tan\theta\sin\theta)\cos\theta = 2a\sin^2\theta$

$y = r\sin\theta = (2a\tan\theta\sin\theta)\sin\theta = 2a\tan\theta \cdot \sin^2\theta, -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$

Let $t = \tan\theta, -\infty < t < \infty.$

Then $\sin^2\theta = \dfrac{t^2}{1 + t^2}$ and $x = 2a\dfrac{t^2}{1 + t^2}$, $y = 2a\dfrac{t^3}{1 + t^2}$.



(c)

$$\begin{aligned} r &= 2a\tan\theta\sin\theta \\ r\cos\theta &= 2a\sin^2\theta \\ r^3\cos\theta &= 2a\,r^2\sin^2\theta \\ (x^2 + y^2)x &= 2ay^2 \\ y^2 &= \frac{x^3}{(2a - x)} \end{aligned}$$

6. (a) $A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx = \frac{4b}{a}\left(\frac{1}{2}\right)\left[x\sqrt{a^2 - x^2} + a^2 \arcsin\left(\frac{x}{a}\right)\right]_0^a = \pi ab$

(b) **Disk:** $V = 2\pi\int_0^b \frac{a^2}{b^2}\left(b^2 - y^2\right)dy = \frac{2\pi a^2}{b^2}\int_0^b \left(b^2 - y^2\right)dy = \frac{2\pi a^2}{b^2}\left[b^2 y - \frac{1}{3}y^3\right]_0^b = \frac{4}{3}\pi a^2 b$

$$S = 4\pi\int_0^b \frac{a}{b}\sqrt{b^2 - y^2}\left(\frac{\sqrt{b^4 + \left(a^2 - b^2\right)y^2}}{b\sqrt{b^2 - y^2}}\right)dy$$

$$= \frac{4\pi a}{b^2}\int_0^b \sqrt{b^4 + c^2y^2}\,dy = \frac{2\pi a}{b^2 c}\left[cy\sqrt{b^4 + c^2y^2} + b^4 \ln\left|cy + \sqrt{b^4 + c^2y^2}\right|\right]_0^b$$

$$= \frac{2\pi a}{b^2 c}\left[b^2 c\sqrt{b^2 + c^2} + b^4 \ln\left|cb + b\sqrt{b^2 + c^2}\right| - b^4 \ln\left(b^2\right)\right]$$

$$= 2\pi a^2 + \frac{\pi ab^2}{c}\ln\left(\frac{c + a}{e}\right)^2 = 2\pi a^2 + \left(\frac{\pi b^2}{e}\right)\ln\left(\frac{1 + e}{1 - e}\right)$$

(c) **Disk:** $V = 2\pi\int_0^a \frac{b^2}{a^2}\left(a^2 - x^2\right)dx = \frac{2\pi b^2}{a^2}\int_0^a \left(a^2 - x^2\right)dx = \frac{2\pi b^2}{a^2}\left[a^2 x - \frac{1}{3}x^3\right]_0^a = \frac{4}{3}\pi ab^2$

$$S = 2(2\pi)\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\left(\frac{\sqrt{a^4 - \left(a^2 - b^2\right)x^2}}{a\sqrt{a^2 - x^2}}\right)dx$$

$$= \frac{4\pi b}{a^2}\int_0^a \sqrt{a^4 - c^2x^2}\,dx = \frac{2\pi b}{a^2 c}\left[cx\sqrt{a^4 - c^2x^2} + a^4 \arcsin\left(\frac{cx}{a^2}\right)\right]_0^a$$

$$= \frac{a\pi b}{a^2 c}\left[a^2 c\sqrt{a^2 - c^2} + a^4 \arcsin\left(\frac{c}{a}\right)\right] = 2\pi b^2 + 2\pi\left(\frac{ab}{e}\right)\arcsin(e)$$

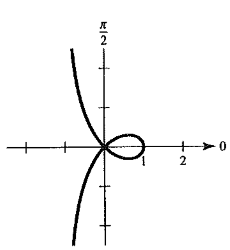
7. (a) $y^2 = \frac{t^2\left(1 - t^2\right)^2}{\left(1 + t^2\right)^2},\ x^2 = \frac{\left(1 - t^2\right)^2}{\left(1 + t^2\right)^2}$

$$\frac{1 - x}{1 + x} = \frac{1 - \left(\frac{1 - t^2}{1 + t^2}\right)}{1 + \left(\frac{1 - t^2}{1 + t^2}\right)} = \frac{2t^2}{2} = t^2$$

So, $y^2 = x^2\left(\frac{1 - x}{1 + x}\right)$.

(b)
$$\begin{aligned}
r^2 \sin^2\theta &= r^2 \cos^2\theta\left(\frac{1 - r\cos\theta}{1 + r\cos\theta}\right)\\
\sin^2\theta(1 + r\cos\theta) &= \cos^2\theta(1 - r\cos\theta)\\
r\cos\theta\sin^2\theta + \sin^2\theta &= \cos^2\theta - r\cos^3\theta\\
r\cos\theta\left(\sin^2\theta + \cos^2\theta\right) &= \cos^2\theta - \sin^2\theta\\
r\cos\theta &= \cos 2\theta\\
r &= \cos 2\theta \cdot \sec\theta
\end{aligned}$$

(c)



(d) $r(\theta) = 0$ for $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$.

So, $y = x$ and $y = -x$ are tangent lines to curve at the origin.

(e) $$y'(t) = \frac{(1+t^2)(1-3t^2) - (t-t^3)(2t)}{(1+t^2)^2} = \frac{1-4t^2-t^4}{(1+t^2)^2} = 0$$

$$t^4 + 4t^2 - 1 = 0 \Rightarrow t^2 = -2 \pm \sqrt{5} \Rightarrow x = \frac{1-\left(-2\pm\sqrt{5}\right)}{1+\left(-2\pm\sqrt{5}\right)} = \frac{3 \mp \sqrt{5}}{-1 \pm \sqrt{5}} = \frac{3-\sqrt{5}}{-1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$$

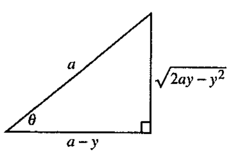
$$\left(\frac{\sqrt{5}-1}{2}, \pm\frac{\sqrt{5}-1}{2}\sqrt{-2+\sqrt{5}}\right)$$

8. $y = a(1-\cos\theta) \Rightarrow \cos\theta = \dfrac{a-y}{a}$

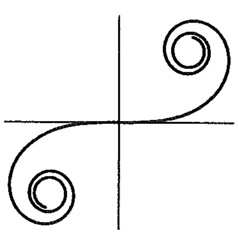
$$\theta = \arccos\left(\frac{a-y}{a}\right)$$

$$\begin{aligned} x &= a(\theta - \sin\theta) \\ &= a\left(\arccos\left(\frac{a-y}{a}\right) - \sin\left(\arccos\left(\frac{a-y}{a}\right)\right)\right) \\ &= a\left(\arccos\left(\frac{a-y}{a}\right) - \frac{\sqrt{2ay-y^2}}{a}\right) \end{aligned}$$

$$x = a \cdot \arccos\left(\frac{a-y}{a}\right) - \sqrt{2ay-y^2}, \ 0 \le y \le 2a$$



9. (a)



Generated by Mathematica

(b) $(-x, -y) = \left(-\int_0^t \cos\dfrac{\pi u^2}{2}\,du, -\int_0^t \sin\dfrac{\pi u^2}{2}\,du\right)$ is on the curve whenever (x, y) is on the curve.

(c) $x'(t) = \cos\dfrac{\pi t^2}{2}, y'(t) = \sin\dfrac{\pi t^2}{2},$

$x'(t)^2 + y'(t)^2 = 1$

So, $s = \int_0^a dt = a$.

On $[-\pi, \pi]$, $s = 2\pi$.

10. For $t = \dfrac{\pi}{2}, \dfrac{3}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}, \ldots$

$$y = \frac{2}{\pi}, \frac{-2}{3\pi}, \frac{2}{5\pi}, \frac{-2}{7\pi}, \ldots$$

So, the curve has length greater that

$$\begin{aligned} S &= \frac{2}{\pi} + \frac{2}{3\pi} + \frac{2}{5\pi} + \frac{2}{7\pi} + \cdots \\ &= \frac{2}{\pi}\left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\right) \\ &> \frac{2}{\pi}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots\right) \\ &= \infty. \text{ (Harmonic series)} \end{aligned}$$

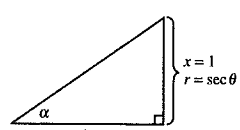
11. $r = \dfrac{ab}{a\sin\theta + b\cos\theta}, \quad 0 \le \theta \le \dfrac{\pi}{2}$

$$\begin{aligned} r(a\sin\theta + b\cos\theta) &= ab \\ ay + bx &= ab \\ \frac{y}{b} + \frac{x}{a} &= 1 \end{aligned}$$

Line segment

$\text{Area} = \dfrac{1}{2}ab$

12. (a) $$\begin{aligned} \text{Area} &= \int_0^\alpha \frac{1}{2}r^2\,d\theta \\ &= \frac{1}{2}\int_0^\alpha \sec^2\theta\,d\theta \end{aligned}$$



(b) $\tan\alpha = \dfrac{h}{1} \Rightarrow \text{Area} = \dfrac{1}{2}(1)\tan\alpha$

$\Rightarrow \tan\alpha = \int_0^\alpha \sec^2\theta\,d\theta$

(c) Differentiating, $\dfrac{d}{d\alpha}(\tan\alpha) = \sec^2\alpha$.

13. Let (r, θ) be on the graph.

$$\sqrt{r^2+1+2r\cos\theta}\sqrt{r^2+1-2r\cos\theta} = 1$$
$$(r^2+1)^2 - 4r^2\cos^2\theta = 1$$
$$r^4+2r^2+1-4r^2\cos^2\theta = 1$$
$$r^2(r^2-4\cos^2\theta+2) = 0$$
$$r^2 = 4\cos^2\theta - 2$$
$$r^2 = 2(2\cos^2\theta - 1)$$
$$r^2 = 2\cos 2\theta$$

14. If a dog is located at (r, θ) in the first quadrant, then its neighbor is at $\left(r, \theta + \frac{\pi}{2}\right)$:

$(x_1, y_1) = (r\cos\theta, r\sin\theta)$ and

$(x_2, y_2) = (-r\sin\theta, r\cos\theta)$.

The slope joining these points is

$$\frac{r\cos\theta - r\sin\theta}{-r\sin\theta - r\cos\theta} = \frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$$
$$= \text{slope of tangent line at } (r, \theta).$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dr}}{\frac{dx}{dr}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$$
$$\Rightarrow \frac{dr}{d\theta} = -r$$
$$\frac{dr}{r} = -d\theta$$
$$\ln r = -\theta + C_1$$
$$r = e^{-\theta + C_1}$$
$$r = Ce^{-\theta}$$

$$r\left(\frac{\pi}{4}\right) = \frac{d}{\sqrt{2}} \Rightarrow r = Ce^{-\pi/4} = \frac{d}{\sqrt{2}} \Rightarrow C = \frac{d}{\sqrt{2}}e^{\pi/4}$$

Finally, $r = \frac{d}{\sqrt{2}}e^{((\pi/4)-\theta)}, \theta \ge \frac{\pi}{4}$.

15. (a) The first plane makes an angle of 70° with the positive x-axis, and is 150 miles from P:

$$x_1 = \cos 70°(150 - 375t)$$
$$y_1 = \sin 70°(150 - 375t)$$

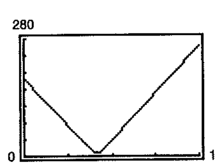
Similarly for the second plane,

$$x_2 = \cos 135°(190 - 450t)$$
$$= \cos 45°(-190 + 450t)$$
$$y_2 = \sin 135°(190 - 450t)$$
$$= \sin 45°(190 - 450t).$$

(b) $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \left[\left[\cos 45°(-190 + 450t) - \cos 70°(150 - 375t)\right]^2 + \left[\sin 45°(190 - 450t) - \sin 70°(150 - 375t)\right]^2\right]^{1/2}$$

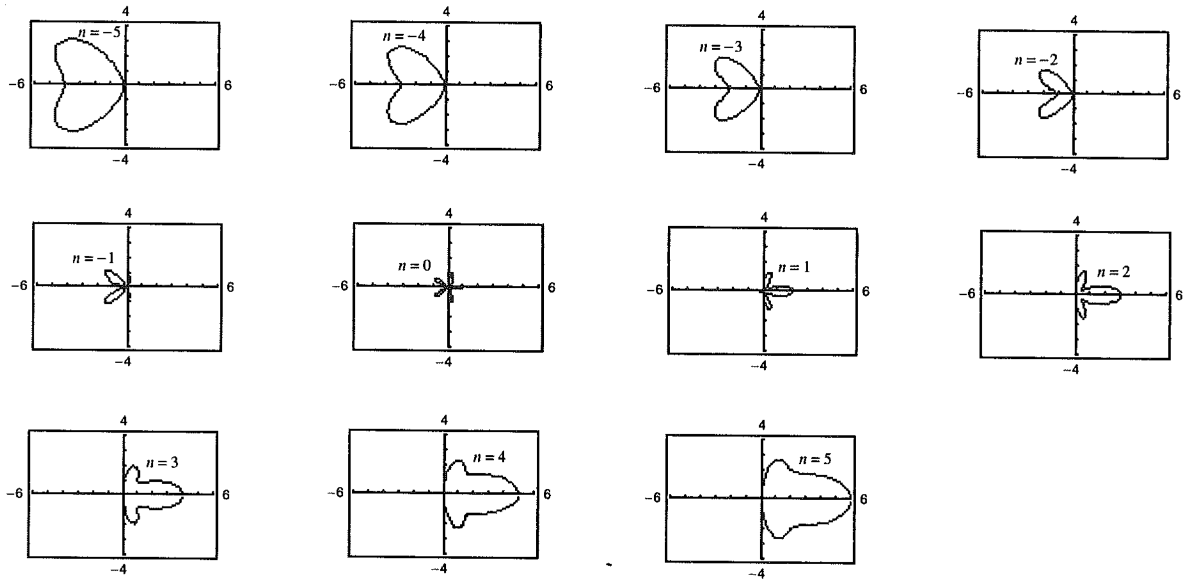
(c)



The minimum distance is 7.59 miles when $t = 0.4145$; Yes.

16. The curve is produced over the interval $0 \le \theta \le 10\pi$.

17.



$n = 1, 2, 3, 4, 5$ produce "bells"; $n = -1, -2, -3, -4, -5$ produce "hearts".

CHAPTER 11
Vectors and the Geometry of Space