85. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$. Then:

$$\begin{aligned}\lim_{t\to c}\left[\mathbf{r}(t) \times \mathbf{u}(t)\right] &= \lim_{t\to c}\left\{\left[y_1(t)z_2(t) - y_2(t)z_1(t)\right]\mathbf{i} - \left[x_1(t)z_2(t) - x_2(t)z_1(t)\right]\mathbf{j} + \left[x_1(t)y_2(t) - x_2(t)y_1(t)\right]\mathbf{k}\right\}\\ &= \left[\lim_{t\to c} y_1(t)\lim_{t\to c} z_2(t) - \lim_{t\to c} y_2(t)\lim_{t\to c} z_1(t)\right]\mathbf{i} - \left[\lim_{t\to c} x_1(t)\lim_{t\to c} z_2(t) - \lim_{t\to c} x_2(t)\lim_{t\to c} z_1(t)\right]\mathbf{j}\\ &\quad + \left[\lim_{t\to c} x_1(t)\lim_{t\to c} y_2(t) - \lim_{t\to c} x_2(t)\lim_{t\to c} y_1(t)\right]\mathbf{k}\\ &= \left[\lim_{t\to c} x_1(t)\mathbf{i} + \lim_{t\to c} y_1(t)\mathbf{j} + \lim_{t\to c} z_1(t)\mathbf{k}\right] \times \left[\lim_{t\to c} x_2(t)\mathbf{i} + \lim_{t\to c} y_2(t)\mathbf{j} + \lim_{t\to c} z_2(t)\mathbf{k}\right]\\ &= \lim_{t\to c}\mathbf{r}(t) \times \lim_{t\to c}\mathbf{u}(t)\end{aligned}$$

86. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$. Then:

$$\begin{aligned}\lim_{t\to c}\left[\mathbf{r}(t) \cdot \mathbf{u}(t)\right] &= \lim_{t\to c}\left[x_1(t)x_2(t) + y_1(t)y_2(t) + z_1(t)z_2(t)\right]\\ &= \lim_{t\to c} x_1(t)\lim_{t\to c} x_2(t) + \lim_{t\to c} y_1(t)\lim_{t\to c} y_2(t) + \lim_{t\to c} z_1(t)\lim_{t\to c} z_2(t)\\ &= \left[\lim_{t\to c} x_1(t)\mathbf{i} + \lim_{t\to c} y_1(t)\mathbf{j} + \lim_{t\to c} z_1(t)\mathbf{k}\right] \cdot \left[\lim_{t\to c} x_2(t)\mathbf{i} + \lim_{t\to c} y_2(t)\mathbf{j} + \lim_{t\to c} z_2(t)\mathbf{k}\right]\\ &= \lim_{t\to c}\mathbf{r}(t) \cdot \lim_{t\to c}\mathbf{u}(t)\end{aligned}$$

87. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Because $\mathbf{r}$ is continuous at $t = c$, then $\lim_{t\to c}\mathbf{r}(t) = \mathbf{r}(c)$.

$\mathbf{r}(c) = x(c)\mathbf{i} + y(c)\mathbf{j} + z(c)\mathbf{k} \Rightarrow x(c), y(c), z(c)$ are defined at c.

$$\|\mathbf{r}\| = \sqrt{(x(t))^2 + (y(t))^2 + (z(t))^2}$$

$$\lim_{t\to c}\|\mathbf{r}\| = \sqrt{(x(c))^2 + (y(c))^2 + (z(c))^2} = \|\mathbf{r}(c)\|$$

So, $\|\mathbf{r}\|$ is continuous at c

88. Let

$$f(t) = \begin{cases} 1, & \text{if } t \geq 0\\ -1, & \text{if } t < 0\end{cases}$$

and $\mathbf{r}(t) = f(t)\mathbf{i}$. Then $\mathbf{r}$ is not continuous at $c = 0$, whereas, $\|\mathbf{r}\| = 1$ is continuous for all t.

89. $\mathbf{r}(t) = t^2\mathbf{i} + (9t - 20)\mathbf{j} + t^2\mathbf{k}$

$\mathbf{u}(s) = (3s + 4)\mathbf{i} + s^2\mathbf{j} + (5s - 4)\mathbf{k}$.

Equating components:

$$\begin{aligned} t^2 &= 3s + 4\\ 9t - 20 &= s^2\\ t^2 &= 5s - 4\end{aligned}$$

So, $3s + 4 = 5s - 4 \Rightarrow s = 4$

$9t - 20 = s^2 = 16 \Rightarrow t = 4$.

The paths intersect at the same time $t = 4$ at the point $(16, 16, 16)$. The particles collide.

90. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

$\mathbf{u}(s) = (-2s + 3)\mathbf{i} + 8s\mathbf{j} + (12s + 2)\mathbf{k}$

Equating components

$$\begin{aligned} t &= -2s + 3\\ t^2 &= 8s\\ t^3 &= 12s + 2\\ (-2s + 3)^2 &= 8s\\ 4s^2 - 12s + 9 &= 8s\\ 4s^2 - 20s + 9 &= 0\\ (2s - 9)(2s - 1) &= 0\end{aligned}$$

For $s = \frac{1}{2}$, $t = -2\left(\frac{1}{2}\right) + 3 = 2$.

For $s = \frac{9}{2}$, $t = -2\left(\frac{9}{2}\right) + 3 = -6$ and $t^2 = 8\left(\frac{9}{2}\right) = 36$ and $t^3 = 12\left(\frac{9}{2}\right) = 54$. Impossible.

The paths intersect at $(2, 4, 8)$, but at different times $\left(t = 2 \text{ and } s = \frac{1}{2}\right)$. No collision.

91. No, not necessarily. See Exercise 90.

92. Yes. See Exercise 89.

93. True

94. False. The graph of $x = y = z = t^3$ represents a line.

95. True. See Exercises 89 and 90.

96. True. $y^2 + z^2 = t^2\sin^2 t + t^2\cos^2 t = t^2 = x$

Section 12.2 Differentiation and Integration of Vector-Valued Functions

1. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j},\ t_0 = 2$

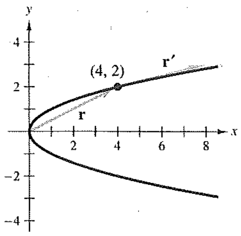
$x(t) = t^2,\ y(t) = t$

$x = y^2$

$\mathbf{r}(2) = 4\mathbf{i} + 2\mathbf{j}$

$\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(2) = 4\mathbf{i} + \mathbf{j}$



$\mathbf{r}'(t_0)$ is tangent to the curve at t_0.

2. $\mathbf{r}(t) = t\mathbf{i} + (t^2 - 1)\mathbf{j},\ t_0 = 1$

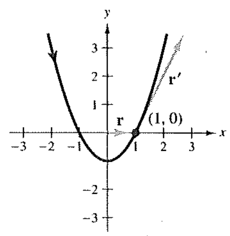
$x(t) = t,\ y(t) = t^2 - 1$

$y = x^2 - 1$

$\mathbf{r}(1) = \mathbf{i}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{r}'(1) = \mathbf{i} + 2\mathbf{j}$



$r'(t_0)$ is tangent to the curve at t_0.

3. $\mathbf{r}(t) = t^2\mathbf{i} + \frac{1}{t}\mathbf{j},\ t_0 = 2$

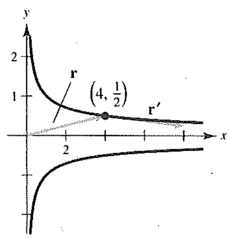
$x(t) = t^2,\ y(t) = \frac{1}{t}$

$x = \frac{1}{y^2}$

$\mathbf{r}(2) = 4\mathbf{i} + \frac{1}{2}\mathbf{j}$

$\mathbf{r}'(t) = 2t\mathbf{i} - \frac{1}{t^2}\mathbf{j}$

$\mathbf{r}'(2) = 4\mathbf{i} - \frac{1}{4}\mathbf{j}$



$\mathbf{r}'(t_0)$ is tangent to the curve at t_0.

4. (a) $\mathbf{r}(t) = (1 + t)\mathbf{i} + t^3\mathbf{j},\ t_0 = 1$

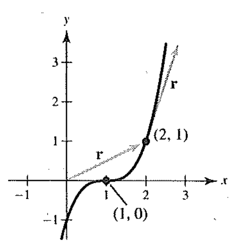
$x = 1 + t$

$y = t^3 = (x - 1)^3$

(b) $\mathbf{r}(1) = 2\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{r}'(1) = \mathbf{i} + 3\mathbf{j}$



$\mathbf{r}'(t_0)$ is tangent to the curve at t_0.

5. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j},\ t_0 = \frac{\pi}{2}$

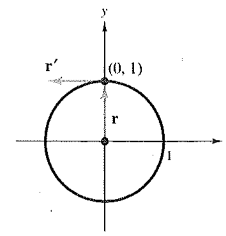
$x(t) = \cos t,\ y(t) = \sin t$

$x^2 + y^2 = 1$

$\mathbf{r}\left(\frac{\pi}{2}\right) = \mathbf{j}$

$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$\mathbf{r}'\left(\frac{\pi}{2}\right) = -\mathbf{i}$



$\mathbf{r}'(t_0)$ is tangent to the curve at t_0.

6. $\mathbf{r}(t) = 3\sin t\mathbf{i} + 4\cos t\mathbf{j},\ t_0 = \frac{\pi}{2}$

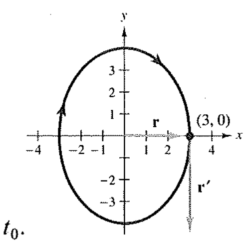
$x(t) = 3\sin t,\ y(t) = 4\cos t$

$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$, ellipse

$\mathbf{r}\left(\frac{\pi}{2}\right) = 3i$

$\mathbf{r}'(t) = 3\cos t\mathbf{i} - 4\sin t\mathbf{j}$

$\mathbf{r}'\left(\frac{\pi}{2}\right) = -4\mathbf{j}$



$\mathbf{r}'(t_0)$ is tangent to the curve at t_0.

7. $\mathbf{r}(t) = \langle e^t, e^{2t}\rangle,\ t_0 = 0$

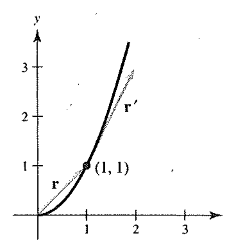
$x(t) = e^t,\ y(t) = e^{2t} = (e^t)^2$

$y = x^2,\ x > 0$

$\mathbf{r}(0) = \langle 1, 1\rangle$

$\mathbf{r}'(t) = \langle e^t, 2e^{2t}\rangle$

$\mathbf{r}'(0) = \langle 1, 2\rangle$



$\mathbf{r}'(t_0)$ is tangent to the curve at t_0.

8. $\mathbf{r}(t) = \langle e^{-t}, e^{t}\rangle, t_0 = 0$

$x(t) = e^{-t} = \dfrac{1}{e^t}, y(t) = e^t$

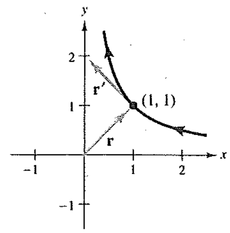
$y = \dfrac{1}{x}, x > 0$

$\mathbf{r}(0) = \langle 1, 1\rangle$

$\mathbf{r}'(t) = \langle -e^{-t}, e^{t}\rangle$

$\mathbf{r}'(0) = \langle -1, 1\rangle$

$\mathbf{r}'(t_0)$ is tangent to the curve at t_0.



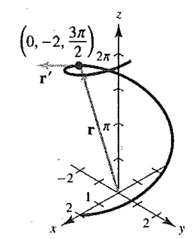
9. (a) and (b) $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}, t_0 = \dfrac{3\pi}{2}$

$x^2 + y^2 = 4, z = t$

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$

$\mathbf{r}\left(\dfrac{3\pi}{2}\right) = -2\mathbf{j} + \dfrac{3\pi}{2}\mathbf{k}$

$\mathbf{r}'\left(\dfrac{3\pi}{2}\right) = 2\mathbf{i} + \mathbf{k}$



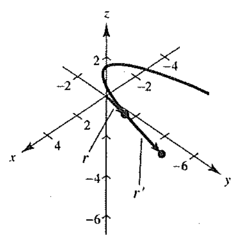
10. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{3}{2}\mathbf{k}, t_0 = 2$

$y = x^2, z = \frac{3}{2}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{r}(2) = 2\mathbf{i} + 4\mathbf{j} + \frac{3}{2}\mathbf{k}$

$\mathbf{r}'(2) = \mathbf{i} + 4\mathbf{j}$



11. $\mathbf{r}(t) = t^3\mathbf{i} - 3t\mathbf{j}$

$\mathbf{r}'(t) = 3t^2\mathbf{i} - 3\mathbf{j}$

12. $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (1 - t^3)\mathbf{j}$

$\mathbf{r}'(t) = \dfrac{1}{2\sqrt{t}}\mathbf{i} - 3t^2\mathbf{j}$

13. $\mathbf{r}(t) = \langle 2\cos t, 5\sin t\rangle$

$\mathbf{r}'(t) = \langle -2\sin t, 5\cos t\rangle$

14. $\mathbf{r}(t) = \langle t\cos t, -2\sin t\rangle$

$\mathbf{r}'(t) = \langle -t\sin t + \cos t, -2\cos t\rangle$

25. $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$

(a) $\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j}$

(b) $\mathbf{r}''(t) = -4\cos t\mathbf{i} - 4\sin t\mathbf{j}$

(c) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = (-4\sin t)(-4\cos t) + 4\cos t(-4\sin t) = 0$

15. $\mathbf{r}(t) = 6t\mathbf{i} - 7t^2\mathbf{j} + t^3\mathbf{k}$

$\mathbf{r}'(t) = 6\mathbf{i} - 14t\mathbf{j} + 3t^2\mathbf{k}$

16. $\mathbf{r}(t) = \dfrac{1}{t}\mathbf{i} + 16t\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}$

$\mathbf{r}'(t) = -\dfrac{1}{t^2}\mathbf{i} + 16\mathbf{j} + t\mathbf{k}$

17. $\mathbf{r}(t) = a\cos^3 t\mathbf{i} + a\sin^3 t\mathbf{j} + \mathbf{k}$

$\mathbf{r}'(t) = -3a\cos^2 t\sin t\mathbf{i} + 3a\sin^2 t\cos t\mathbf{j}$

18. $\mathbf{r}(t) = 4\sqrt{t}\mathbf{i} + t^2\sqrt{t}\mathbf{j} + \ln t^2\mathbf{k}$

$\mathbf{r}'(t) = \dfrac{2}{\sqrt{t}}\mathbf{i} + \left(2t\sqrt{t} + \dfrac{t^2}{2\sqrt{t}}\right)\mathbf{j} + \dfrac{2}{t}\mathbf{k}$

$= \dfrac{2}{\sqrt{t}}\mathbf{i} + \dfrac{5t^{3/2}}{2}\mathbf{j} + \dfrac{2}{t}\mathbf{k}$

19. $\mathbf{r}(t) = e^{-t}\mathbf{i} + 4\mathbf{j} + 5te^t\mathbf{k}$

$\mathbf{r}'(t) = -e^{-t}\mathbf{i} + (5e^t + 5te^t)\mathbf{k}$

20. $\mathbf{r}(t) = \langle t^3, \cos 3t, \sin 3t\rangle$

$\mathbf{r}'(t) = \langle 3t^2, -3\sin 3t, 3\cos 3t\rangle$

21. $\mathbf{r}(t) = \langle t\sin t, t\cos t, t\rangle$

$\mathbf{r}'(t) = \langle \sin t + t\cos t, \cos t - t\sin t, 1\rangle$

22. $\mathbf{r}(t) = \langle \arcsin t, \arccos t, 0\rangle$

$\mathbf{r}'(t) = \left\langle \dfrac{1}{\sqrt{1 - t^2}}, -\dfrac{1}{\sqrt{1 - t^2}}, 0\right\rangle$

23. $\mathbf{r}(t) = t^3\mathbf{i} + \dfrac{1}{2}t^2\mathbf{j}$

(a) $\mathbf{r}'(t) = 3t^2\mathbf{i} + t\mathbf{j}$

(b) $\mathbf{r}''(t) = 6t\mathbf{i} + \mathbf{j}$

(c) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = 3t^2(6t) + t = 18t^3 + t$

24. $\mathbf{r}(t) = (t^2 + t)\mathbf{i} + (t^2 - t)\mathbf{j}$

(a) $\mathbf{r}'(\mathbf{t}) = (2t + 1)\mathbf{i} + (2t - 1)\mathbf{j}$

(b) $\mathbf{r}''(t) = 2\mathbf{i} + 2\mathbf{j}$

(c) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = (2t + 1)(2) + (2t - 1)(2) = 8t$

26. $\mathbf{r}(t) = 8\cos t\mathbf{i} + 3\sin t\mathbf{j}$

(a) $\mathbf{r}'(t) = -8\sin t\mathbf{i} + 3\cos t\mathbf{j}$

(b) $\mathbf{r}''(t) = -8\cos t\mathbf{i} - 3\sin t\mathbf{j}$

(c) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = (-8\sin t)(-8\cos t) + 3\cos t(-3\sin t) = 55\sin t\cos t$

27. $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - t\mathbf{j} + \frac{1}{6}t^3\mathbf{k}$

(a) $\mathbf{r}'(t) = t\mathbf{i} - \mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

(b) $\mathbf{r}''(t) = \mathbf{i} + t\mathbf{k}$

(c) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = t(1) - 1(0) + \frac{1}{2}t^2(t) = t + \frac{t^3}{2}$

28. $\mathbf{r}(t) = t\mathbf{i} + (2t+3)\mathbf{j} + (3t-5)\mathbf{k}$

(a) $\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

(b) $\mathbf{r}''(t) = 0$

(c) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = 0$

29. $\mathbf{r}(t) = \langle\cos t + t\sin t, \sin t - t\cos t, t\rangle$

(a) $\mathbf{r}'(t) = \langle -\sin t + \sin t + t\cos t, \cos t - \cos t + t\sin t, 1\rangle = \langle t\cos t, t\sin t, 1\rangle$

(b) $\mathbf{r}''(t) = \langle\cos t - t\sin t, \sin t + t\cos t, 0\rangle$

(c) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = (t\cos t)(\cos t - t\sin t) + (t\sin t)(\sin t + t\cos t) = t$

30. $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan(t)\rangle$

(a) $\mathbf{r}'(t) = \langle -e^{-t}, 2t, \sec^2 t\rangle$

(b) $\mathbf{r}''(t) = \langle e^{-t}, 2, 2\sec^2 t\tan t\rangle$

(c) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = -e^{-2t} + 4t + 2\sec^4 t\tan t$

31.

$$\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + t^2\mathbf{k},\ t_0 = -\frac{1}{4}$$

$$\mathbf{r}'(t) = -\pi\sin(\pi t)\mathbf{i} + \pi\cos(\pi t)\mathbf{j} + 2t\mathbf{k}$$

$$\mathbf{r}'\left(-\frac{1}{4}\right) = \frac{\sqrt{2}\pi}{2}\mathbf{i} + \frac{\sqrt{2}\pi}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$$

$$\left\|\mathbf{r}'\left(\frac{1}{4}\right)\right\| = \sqrt{\left(\frac{\sqrt{2}\pi}{2}\right)^2 + \left(\frac{\sqrt{2}\pi}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\pi^2 + \frac{1}{4}} = \frac{\sqrt{4\pi^2+1}}{2}$$

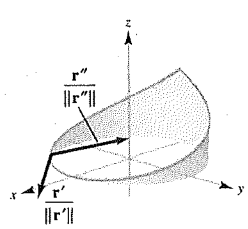
$$\frac{\mathbf{r}'(-1/4)}{\|\mathbf{r}'(-1/4)\|} = \frac{1}{\sqrt{4\pi^2+1}}\left(\sqrt{2}\pi\mathbf{i} + \sqrt{2}\pi\mathbf{j} - \mathbf{k}\right)$$

$$\mathbf{r}''(t) = -\pi^2\cos(\pi t)\mathbf{i} - \pi^2\sin(\pi t)\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{r}''\left(-\frac{1}{4}\right) = -\frac{\sqrt{2}\pi^2}{2}\mathbf{i} + \frac{\sqrt{2}\pi^2}{2}\mathbf{j} + 2\mathbf{k}$$

$$\left\|\mathbf{r}''\left(-\frac{1}{4}\right)\right\| = \sqrt{\left(-\frac{\sqrt{2}\pi^2}{2}\right)^2 + \left(\frac{\sqrt{2}\pi^2}{2}\right)^2 + (2)^2} = \sqrt{\pi^4+4}$$

$$\frac{\mathbf{r}''(-1/4)}{\|\mathbf{r}''(-1/4)\|} = \frac{1}{2\sqrt{\pi^4+4}}\left(-\sqrt{2}\pi^2\mathbf{i} + \sqrt{2}\pi^2\mathbf{j} + 4\mathbf{k}\right)$$



32. $\mathbf{r}(t) = \frac{3}{2}t\mathbf{i} + t^2\mathbf{j} + e^{-t}\mathbf{k},\ t_0 = \frac{1}{4}$

$\mathbf{r}'(t) = \frac{3}{2}\mathbf{i} + 2t\mathbf{j} - e^{-t}\mathbf{k},\ \mathbf{r}\left(\frac{1}{4}\right) = \frac{3}{8}\mathbf{i} + \frac{1}{16}\mathbf{j} + e^{-1/4}\mathbf{k}$

$\mathbf{r}'\left(\frac{1}{4}\right) = \frac{3}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - e^{-1/4}\mathbf{k}$

$\left\|r'\left(\frac{1}{4}\right)\right\| = \sqrt{\frac{9}{4} + \frac{1}{4} + e^{-1/2}} = \frac{1}{2}\sqrt{10 + 4e^{-1/2}}$

$\frac{r'(1/4)}{\|r'(1/4)\|} = \frac{3}{\sqrt{10 + 4e^{-1/2}}}\mathbf{i} + \frac{1}{\sqrt{10 + 4e^{-1/2}}}\mathbf{j} - \frac{2e^{-1/4}}{\sqrt{10 + 4e^{-1/2}}}\mathbf{k}$

$\mathbf{r}''(t) = 2\mathbf{j} + e^{-t}\mathbf{k},\ \mathbf{r}''\left(\frac{1}{4}\right) = 2\mathbf{j} + e^{-1/4}\mathbf{k}$

$\left\|\mathbf{r}''\left(\frac{1}{4}\right)\right\| = \sqrt{4 + e^{-1/2}}$

$\frac{\mathbf{r}''\left(\frac{1}{4}\right)}{\left\|\mathbf{r}''\left(\frac{1}{4}\right)\right\|} = \frac{2}{\sqrt{4 + e^{-1/2}}}\mathbf{j} + \frac{e^{-1/4}}{\sqrt{4 + e^{-1/2}}}\mathbf{k}$

33. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

$\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{r}'(0) = \mathbf{0}$

Smooth on $(-\infty, 0), (0, \infty)$

34. $\mathbf{r}(t) = \frac{1}{t-1}\mathbf{i} + 3t\mathbf{j}$

$\mathbf{r}'(t) = -\frac{1}{(t-1)^2}\mathbf{i} + 3\mathbf{j}$

Not continuous when $t = 1$

Smooth on $(-\infty, 1), (1, \infty)$

35. $\mathbf{r}(\theta) = 2\cos^3\theta\mathbf{i} + 3\sin^3\theta\mathbf{j}$

$\mathbf{r}'(\theta) = -6\cos^2\theta\sin\theta\mathbf{i} + 9\sin^2\theta\cos\theta\mathbf{j}$

$\mathbf{r}'\left(\frac{n\pi}{2}\right) = \mathbf{0}$

Smooth on $\left(\frac{n\pi}{2}, \frac{(n+1)\pi}{2}\right)$, n any integer.

36. $\mathbf{r}(\theta) = (\theta + \sin\theta)\mathbf{i} + (1 - \cos\theta)\mathbf{j}$

$\mathbf{r}'(\theta) = (1 + \cos\theta)\mathbf{i} + \sin\theta\mathbf{j}$

$\mathbf{r}'((2n-1)\pi) = \mathbf{0}$, n any integer

Smooth on $((2n-1)\pi, (2n+1)\pi)$

37. $\mathbf{r}(\theta) = (\theta - 2\sin\theta)\mathbf{i} + (1 - 2\cos\theta)\mathbf{j}$

$\mathbf{r}'(\theta) = (1 - 2\cos\theta)\mathbf{i} + (2\sin\theta)\mathbf{j}$

$\mathbf{r}'(\theta) \neq \mathbf{0}$ for any value of θ

Smooth on $(-\infty, \infty)$

38. $\mathbf{r}(t) = \frac{2t}{8+t^3}\mathbf{i} + \frac{2t^2}{8+t^3}\mathbf{j}$

$\mathbf{r}'(t) = \frac{16 - 4t^3}{(t^3+8)^2}\mathbf{i} + \frac{32t - 2t^4}{(t^3+8)^2}\mathbf{j}$

$\mathbf{r}'(t) \neq \mathbf{0}$ for any value of t.

$\mathbf{r}$ is not continuous when $t = -2$.

Smooth on $(-\infty, -2), (-2, \infty)$

39. $\mathbf{r}(t) = (t-1)\mathbf{i} + \frac{1}{t}\mathbf{j} - t^2\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j} - 2t\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth or all $t \neq 0$: $(-\infty, 0), (0, \infty)$

40. $\mathbf{r}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + 3t\mathbf{k}$

$\mathbf{r}'(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + 3\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth for all t: $(-\infty, \infty)$

41. $\mathbf{r}(t) = t\mathbf{i} - 3t\mathbf{j} + \tan t\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} - 3\mathbf{j} + \sec^2 t\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth for all $t \neq \frac{\pi}{2} + n\pi = \frac{2n+1}{2}\pi$.

Smooth on intervals of form $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$,

n is an integer.

42. $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (t^2 - 1)\mathbf{j} + \frac{1}{4}t\mathbf{k}$

$\mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\mathbf{i} + 2t\mathbf{j} + \frac{1}{4}\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth for all $t > 0$: $(0, \infty)$

43. $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k},\ \mathbf{u}(t) = 4t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

(a) $\mathbf{r}'(t) = \mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}$

(b) $\mathbf{r}''(t) = 2\mathbf{k}$

(c) $\mathbf{r}(t) \cdot \mathbf{u}(t) = 4t^2 + 3t^3 + t^5$

$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 8t + 9t^2 + 5t^4$

(d) $3\mathbf{r}(t) - \mathbf{u}(t) = -t\mathbf{i} + (9t - t^2)\mathbf{j} + (3t^2 - t^3)\mathbf{k}$

$D_t[3\mathbf{r}(t) - \mathbf{u}(t)] = -\mathbf{i} + (9 - 2t)\mathbf{j} + (6t - 3t^2)\mathbf{k}$

(e) $\mathbf{r}(t) \times \mathbf{u}(t) = 2t^4\mathbf{i} - (t^4 - 4t^3)\mathbf{j} + (t^3 - 12t^2)\mathbf{k}$

$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = 8t^3\mathbf{i} + (12t^2 - 4t^3)\mathbf{j} + (3t^2 - 24t)\mathbf{k}$

(f) $\|\mathbf{r}(t)\| = \sqrt{10t^2 + t^4} = t\sqrt{10 + t^2}$

$$D_t[\|\mathbf{r}(t)\|] = \frac{10 + 2t^2}{\sqrt{10 + t^2}}$$

44. $\mathbf{r}(t) = t\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$

$\mathbf{u}(t) = \frac{1}{t}\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$

(a) $\mathbf{r}'(t) = \mathbf{i} + 2\cos t\mathbf{j} - 2\sin t\mathbf{k}$

(b) $\mathbf{r}''(t) = -2\sin t\mathbf{j} - 2\cos t\mathbf{k}$

(c) $\mathbf{r}(t) \cdot \mathbf{u}(t) = 1 + 4\sin^2 t + 4\cos^2 t = 5$

$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 0, t \neq 0$

(d) $3\mathbf{r}(t) - \mathbf{u}(t) = \left(3t - \frac{1}{t}\right)\mathbf{i} + 4\sin t\mathbf{j} + 4\cos t\mathbf{k}$

$$D_t[3\mathbf{r}(t) - \mathbf{u}(t)] = \left(3 - \frac{1}{t^2}\right)\mathbf{i} + 4\cos t\mathbf{j} - 4\sin t\mathbf{k}$$

(e) $$\mathbf{r}(t) \times \mathbf{u}(t) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2\sin t & 2\cos t \\ \frac{1}{t} & 2\sin t & 2\cos t \end{bmatrix} = 2\cos t\left(\frac{1}{t} - t\right)\mathbf{j} + 2\sin t\left(t - \frac{1}{t}\right)\mathbf{k}$$

$$D_t[\mathbf{r}(t) - \mathbf{u}(t)] = \left[-2\sin t\left(\frac{1}{t} - t\right) + 2\cos t\left(-\frac{1}{t^2} - 1\right)\right]\mathbf{j} + \left[2\cos t\left(t - \frac{1}{t}\right) + 2\sin t\left(1 + \frac{1}{t^2}\right)\right]\mathbf{k}$$

(f) $\|\mathbf{r}(t)\| = \sqrt{t^2 + 4}$

$$D_t(\|\mathbf{r}(t)\|) = \frac{1}{2}(t^2 + 4)^{-1/2}(2t) = \frac{t}{\sqrt{t^2 + 4}}$$

45. $\mathbf{r}(t) = t\mathbf{i} + 2t^2\mathbf{j} + t^3\mathbf{k}, \mathbf{u}(t) = t^4\mathbf{k}$

(a) $\mathbf{r}(t) \cdot \mathbf{u}(t) = t^7$

(i) $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 7t^6$

(ii) Alternate Solution:

$$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t) = (t\mathbf{i} + 2t^2\mathbf{j} + t^3\mathbf{k}) \cdot (4t^3\mathbf{k}) + (\mathbf{i} + 4t\mathbf{j} + 3t^2\mathbf{k}) \cdot (t^4\mathbf{k}) = 4t^6 + 3t^6 = 7t^6$$

(b) $\mathbf{r}(t) \times \mathbf{u}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2t^2 & t^3 \\ 0 & 0 & t^4 \end{vmatrix} = 2t^6\mathbf{i} - t^5\mathbf{j}$

(i) $D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = 12t^5\mathbf{i} - 5t^4\mathbf{j}$

(ii) Alternate Solution: $D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) \times \mathbf{r}'(t) \times \mathbf{u}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2t^2 & t^3 \\ 0 & 0 & 4t^3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4t & 3t^2 \\ 0 & 0 & t^4 \end{vmatrix} = 12t^5\mathbf{i} - 5t^4\mathbf{j}$

46. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, \ \mathbf{u}(t) = \mathbf{j} + t\mathbf{k}$

(a) $\mathbf{r}(t) \cdot \mathbf{u}(t) = \sin t + t^2$

(i) $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \cos t + 2t$

(ii) Alternate Solution:

$$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$$

$$= (\cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}) \cdot \mathbf{k} + (-\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}) \cdot (\mathbf{j} + t\mathbf{k}) = t + \cos t + t = 2t + \cos t$$

(b) $\mathbf{r}(t) \times \mathbf{u}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & t \\ 0 & 1 & t \end{vmatrix} = (t \sin t - t)\mathbf{i} - (t \cos t)\mathbf{j} + \cos t\mathbf{k}$

(i) $D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = (t \cos t + \sin t - 1)\mathbf{i} - (\cos t - t \sin t)\mathbf{j} - \sin t\mathbf{k}$

(ii) Alternate Solution:

$$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & t \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 1 \\ 0 & 1 & t \end{vmatrix} = (\sin t + t \cos t - 1)\mathbf{i} + (t \sin t - \cos t)\mathbf{j} - \sin t\mathbf{k}$$

47.

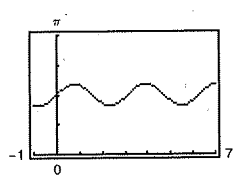
$$\mathbf{r}(t) = 3 \sin t\mathbf{i} + 4 \cos t\mathbf{j}$$

$$\mathbf{r}'(t) = 3 \cos t\mathbf{i} - 4 \sin t\mathbf{j}$$

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 9 \sin t \cos t - 16 \cos t \sin t = -7 \sin t \cos t$$

$$\cos \theta = \frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\| \|\mathbf{r}'(t)\|} = \frac{-7 \sin t \cos t}{\sqrt{9 \sin^2 t + 16 \cos^2 t}\sqrt{9 \cos^2 t + 16 \sin^2 t}}$$

$$\theta = \arccos\left[\frac{-7 \sin t \cos t}{\sqrt{(9 \sin^2 t + 16 \cos^2 t)}\sqrt{(9 \cos^2 t + 16 \sin^2 t)}}\right]$$



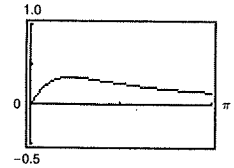
$\theta = 1.855$ maximum at $t = 3.927 = \left(\frac{5\pi}{4}\right)$ and $t = 0.785 = \left(\frac{\pi}{4}\right)$.

$\theta = 1.287$ minimum at $t = 2.356 = \left(\frac{3\pi}{4}\right)$ and $t = 5.498 = \left(\frac{7\pi}{4}\right)$.

$\theta = \frac{\pi}{2} = (1.571)$ for $t = \frac{n\pi}{2}, n = 0, 1, 2, 3, \ldots$

48.

$$\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$$
$$\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$$
$$\mathbf{r}(t)\cdot\mathbf{r}'(t) = 2t^3 + t$$
$$\|\mathbf{r}(t)\| = \sqrt{t^4 + t^2},\ \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 1}$$
$$\cos\theta = \frac{2t^3 + t}{\sqrt{t^4 + t^2}\sqrt{4t^2 + 1}}$$
$$\theta = \arccos\left[\frac{2t^3 + t}{\sqrt{t^4 + t^2}\sqrt{4t^2 + 1}}\right]$$



$\theta = 0.340(\approx 19.47^\circ)$ maximum at $t = 0.707 = \left(\frac{\sqrt{2}}{2}\right)$.

$\theta \neq \frac{\pi}{2}$ for any t.

49. $$\mathbf{r}'(t) = \lim_{\Delta t\to 0}\frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{[3(t+\Delta t) + 2]\mathbf{i} + \left[1 - (t+\Delta t)^2\right]\mathbf{j} - (3t+2)\mathbf{i} - (1 - t^2)\mathbf{j}}{\Delta t}$$
$$= \lim_{\Delta t\to 0}\frac{(3\Delta t)\mathbf{i} - \left(2t(\Delta t) + (\Delta t)^2\right)\mathbf{j}}{\Delta t} = \lim_{\Delta t\to 0} 3\mathbf{i} - (2t + \Delta t)\mathbf{j} = 3\mathbf{i} - 2t\mathbf{j}$$

50. $$\mathbf{r}'(t) = \lim_{\Delta t\to 0}\frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$
$$= \lim_{\Delta t\to 0}\frac{\left[\sqrt{t+\Delta t}\,\mathbf{i} + \frac{3}{t+\Delta t}\mathbf{j} - 2(t+\Delta t)\mathbf{k}\right] - \left[\sqrt{t}\,\mathbf{i} + \frac{3}{t}\mathbf{j} - 2t\mathbf{k}\right]}{\Delta t} = \lim_{\Delta t\to 0}\left[\frac{\sqrt{t+\Delta t} - \sqrt{t}}{\Delta t}\mathbf{i} + \frac{\frac{3}{t+\Delta t} - \frac{3}{t}}{\Delta t}\mathbf{j} - 2\mathbf{k}\right]$$
$$= \lim_{\Delta t\to 0}\left[\frac{\Delta t}{\Delta t\left(\sqrt{t+\Delta t} + \sqrt{t}\right)}\mathbf{i} + \frac{-3\Delta t}{(t+\Delta t)t(\Delta t)}\mathbf{j} - 2\mathbf{k}\right] = \lim_{\Delta t\to 0}\left[\frac{1}{\sqrt{t+\Delta t} + \sqrt{t}}\mathbf{i} - \frac{3}{(t+\Delta t)t}\mathbf{j} - 2\mathbf{k}\right] = \frac{1}{2\sqrt{t}}\mathbf{i} - \frac{3}{t^2}\mathbf{j} - 2\mathbf{k}$$

51. $$\mathbf{r}'(t) = \lim_{\Delta t\to 0}\frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$
$$= \lim_{\Delta t\to 0}\frac{\left\langle (t+\Delta t)^2, 0, 2(t+\Delta t)\right\rangle - \left\langle t^2, 0, 2t\right\rangle}{\Delta t} = \lim_{\Delta t\to 0}\frac{\left\langle 2t\Delta t + (\Delta t)^2, 0, 2\Delta t\right\rangle}{\Delta t} = \lim_{\Delta t\to 0}\langle 2t + \Delta t, 0, 2\rangle = \langle 2t, 0, 2\rangle$$

52. $$\mathbf{r}'(t) = \lim_{\Delta t\to 0}\frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{\langle 0, \sin(t+\Delta t), 4(t+\Delta t)\rangle - \langle 0, \sin t, 4t\rangle}{\Delta t}$$
$$= \lim_{\Delta t\to 0}\frac{\langle 0, \sin t\cdot\cos(\Delta t) + \sin(\Delta t)\cos t - \sin t, 4\Delta t\rangle}{\Delta t} = \lim_{\Delta t\to 0}\left\langle 0, \frac{\sin t(\cos(\Delta t) - 1)}{\Delta t} + \cos t\left(\frac{\sin(\Delta t)}{\Delta t}\right), 4\right\rangle$$
$$= \langle 0, 0 + \cos t, 4\rangle = \langle 0, \cos t, 4\rangle$$

53. $\int(2t\mathbf{i} + \mathbf{j} + \mathbf{k})\,dt = t^2\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

54. $\int\left(4t^3\mathbf{i} + 6t\mathbf{j} - 4\sqrt{t}\mathbf{k}\right)dt = t^4\mathbf{i} + 3t^2\mathbf{j} - \frac{8}{3}t^{3/2}\mathbf{k} + \mathbf{C}$

55. $\int\left(\frac{1}{t}\mathbf{i} + \mathbf{j} - t^{3/2}\mathbf{k}\right)dt = \ln t\mathbf{i} + t\mathbf{j} - \frac{2}{5}t^{5/2}\mathbf{k} + \mathbf{C}$

56. $\int\left[\ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + \mathbf{k}\right]dt = (t\ln t - t)\mathbf{i} + \ln t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

(Integration by parts)

57. $\int\left[(2t-1)\mathbf{i}+4t^3\mathbf{j}+3\sqrt{t}\mathbf{k}\right]dt = (t^2-t)\mathbf{i}+t^4\mathbf{j}+2t^{3/2}\mathbf{k}+\mathbf{C}$

58. $\int\left[e^t\mathbf{i}+\sin t\mathbf{j}+\cos t\mathbf{k}\right]dt = e^t\mathbf{i}-\cos t\mathbf{j}+\sin t\mathbf{k}+\mathbf{C}$

59. $\int\left[\sec^2 t\mathbf{i}+\dfrac{1}{1+t^2}\mathbf{j}\right]dt = \tan t\mathbf{i}+\arctan t\mathbf{j}+\mathbf{C}$

60. $\int\left[e^{-t}\sin t\mathbf{i}+e^{-t}\cos t\mathbf{j}\right]dt = \dfrac{e^{-t}}{2}(-\sin t-\cos t)\mathbf{i}+\dfrac{e^{-t}}{2}(-\cos t+\sin t)\mathbf{j}+\mathbf{C}$

61. $\int_0^1(8t\mathbf{i}+t\mathbf{j}-\mathbf{k})\,dt = \left[4t^2\mathbf{i}\right]_0^1+\left[\dfrac{t^2}{2}\mathbf{j}\right]_0^1-\left[t\mathbf{k}\right]_0^1 = 4\mathbf{i}+\dfrac{1}{2}\mathbf{j}-\mathbf{k}$

62. $\int_{-1}^1\left(t\mathbf{i}+t^3\mathbf{j}+\sqrt[3]{t}\mathbf{k}\right)dt = \left[\dfrac{t^2}{2}\mathbf{i}\right]_{-1}^1+\left[\dfrac{t^4}{4}\mathbf{j}\right]_{-1}^1+\left[\dfrac{3}{4}t^{4/3}\mathbf{k}\right]_{-1}^1 = \mathbf{0}$

63. $\int_0^{\pi/2}\left[(a\cos t)\mathbf{i}+(a\sin t)\mathbf{j}+\mathbf{k}\right]dt = \left[a\sin t\mathbf{i}\right]_0^{\pi/2}-\left[a\cos t\mathbf{j}\right]_0^{\pi/2}+\left[t\mathbf{k}\right]_0^{\pi/2} = a\mathbf{i}+a\mathbf{j}+\dfrac{\pi}{2}\mathbf{k}$

64. $\int_0^{\pi/4}\left[(\sec t\tan t)\mathbf{i}+(\tan t)\mathbf{j}+(2\sin t\cos t)\mathbf{k}\right]dt = \left[\sec t\mathbf{i}+\ln|\sec t|\mathbf{j}+\sin^2 t\mathbf{k}\right]_0^{\pi/4} = \left(\sqrt{2}-1\right)\mathbf{i}+\ln\sqrt{2}\mathbf{j}+\frac{1}{2}\mathbf{k}$

65. $\int_0^2\left(t\mathbf{i}+e^t\mathbf{j}-te^t\mathbf{k}\right)dt = \left[\dfrac{t^2}{2}\mathbf{i}\right]_0^2+\left[e^t\mathbf{j}\right]_0^2-\left[(t-1)e^t\mathbf{k}\right]_0^2 = 2\mathbf{i}+\left(e^2-1\right)\mathbf{j}-\left(e^2+1\right)\mathbf{k}$

66. $\left\|t\mathbf{i}+t^2\mathbf{j}\right\| = \sqrt{t^2+t^4} = t\sqrt{1+t^2}$ for $t\ge 0$

$\int_0^3\left\|t\mathbf{i}+t^2\mathbf{j}\right\|dt = \int_0^3 t\sqrt{1+t^2}\,dt = \left[\frac{1}{3}\left(1+t^2\right)^{3/2}\right]_0^3 = \frac{1}{3}\left(10^{3/2}-1\right)$

67. $\mathbf{r}(t) = \int\left(4e^{2t}\mathbf{i}+3e^t\mathbf{j}\right)dt = 2e^{2t}\mathbf{i}+3e^t\mathbf{j}+\mathbf{C}$

$\mathbf{r}(0) = 2\mathbf{i}+3\mathbf{j}+\mathbf{C} = 2\mathbf{i} \Rightarrow \mathbf{C} = -3\mathbf{j}$

$\mathbf{r}(t) = 2e^{2t}\mathbf{i}+3\left(e^t-1\right)\mathbf{j}$

68. $\mathbf{r}(t) = \int\left(3t^2\mathbf{j}+6\sqrt{t}\mathbf{k}\right)dt = t^3\mathbf{j}+4t^{3/2}\mathbf{k}+\mathbf{C}$

$\mathbf{r}(0) = C = \mathbf{i}+2\mathbf{j}$

$\mathbf{r}(t) = \mathbf{i}+\left(2+t^3\right)\mathbf{j}+4t^{3/2}\mathbf{k}$

69. $\mathbf{r}'(t) = \int -32\mathbf{j}\,dt = -32t\mathbf{j}+\mathbf{C}_1$

$\mathbf{r}'(0) = \mathbf{C}_1 = 600\sqrt{3}\mathbf{i}+600\mathbf{j}$

$\mathbf{r}'(t) = 600\sqrt{3}\mathbf{i}+(600-32t)\mathbf{j}$

$\mathbf{r}(t) = \int\left[600\sqrt{3}\mathbf{i}+(600-32t)\mathbf{j}\right]dt$

$= 600\sqrt{3}t\mathbf{i}+\left(600t-16t^2\right)\mathbf{j}+\mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = \mathbf{0}$

$\mathbf{r}(t) = 600\sqrt{3}t\mathbf{i}+\left(600t-16t^2\right)\mathbf{j}$

70. $\mathbf{r}''(t) = -4\cos t\mathbf{j}-3\sin t\mathbf{k}$

$\mathbf{r}'(t) = -4\sin t\mathbf{j}+3\cos t\mathbf{k}+\mathbf{C}_1$

$\mathbf{r}'(0) = 3\mathbf{k} = 3\mathbf{k}+\mathbf{C}_1 \Rightarrow \mathbf{C}_1 = \mathbf{0}$

$\mathbf{r}(t) = 4\cos t\mathbf{j}+3\sin t\mathbf{k}+\mathbf{C}_2$

$\mathbf{r}(0) = 4j+\mathbf{C}_2 = 4\mathbf{j} \Rightarrow \mathbf{C}_2 = 0$

$\mathbf{r}(t) = 4\cos t\mathbf{j}+3\sin t\mathbf{k}$

71. $\mathbf{r}(t) = \int\left(te^{-t^2}\mathbf{i}-e^{-t}\mathbf{j}+\mathbf{k}\right)dt = -\dfrac{1}{2}e^{-t^2}\mathbf{i}+e^{-t}\mathbf{j}+t\mathbf{k}+\mathbf{C}$

$\mathbf{r}(0) = -\dfrac{1}{2}\mathbf{i}+\mathbf{j}+\mathbf{C} = \dfrac{1}{2}\mathbf{i}-\mathbf{j}+\mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i}-2\mathbf{j}+\mathbf{k}$

$\mathbf{r}(t) = \left(1-\dfrac{1}{2}e^{-t^2}\right)\mathbf{i}+\left(e^{-t}-2\right)\mathbf{j}+(t+1)\mathbf{k}$

$= \left(\dfrac{2-e^{-t^2}}{2}\right)\mathbf{i}+\left(e^{-t}-2\right)\mathbf{j}+(t+1)\mathbf{k}$

72. $\mathbf{r}(t) = \int\left[\frac{1}{1+t^2}\mathbf{i} + \frac{1}{t^2}\mathbf{j} + \frac{1}{t}\mathbf{k}\right]dt$

$= \arctan t\mathbf{i} - \frac{1}{t}\mathbf{j} + \ln t\mathbf{k} + \mathbf{C}$

$\mathbf{r}(1) = \frac{\pi}{4}\mathbf{i} - \mathbf{j} + \mathbf{C} = 2\mathbf{i} \Rightarrow \mathbf{C} = \left(2 - \frac{\pi}{4}\right)\mathbf{i} + \mathbf{j}$

$\mathbf{r}(t) = \left[2 - \frac{\pi}{4} + \arctan t\right]\mathbf{i} + \left(1 - \frac{1}{t}\right)\mathbf{j} + \ln t\mathbf{k}$

73. See "Definition of the Derivative of a Vector-Valued Function" and Figure 12.8 on page 842.

74. To find the integral of a vector-valued function, you integrate each component function separately. The constant of integration $\mathbf{C}$ is a constant vector.

75. At $t = t_0$, the graph of $\mathbf{u}(t)$ is increasing in the x, y, and z directions simultaneously.

76. The graph of $\mathbf{u}(t)$ does not change position relative to the xy-plane.

77. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then

$c\mathbf{r}(t) = cx(t)\mathbf{i} + cy(t)\mathbf{j} + cz(t)\mathbf{k}$ and

$D_t[c\mathbf{r}(t)] = cx'(t)\mathbf{i} + cy'(t)\mathbf{j} + cz'(t)\mathbf{k}$

$= c[x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}] = c\mathbf{r}'(t).$

78. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$.

$$\mathbf{r}(t) \pm \mathbf{u}(t) = [x_1(t) \pm x_2(t)]\mathbf{i} + [y_1(t) \pm y_2(t)]\mathbf{j} + [z_1(t) \pm z_2(t)]\mathbf{k}$$

$$D_t[\mathbf{r}(t) \pm \mathbf{u}(t)] = [x_1'(t) \pm x_2'(t)]\mathbf{i} + [y_1'(t) \pm y_2'(t)]\mathbf{j} + [z_1'(t) \pm z_2'(t)]\mathbf{k}$$

$$= [x_1'(t)\mathbf{i} + y_1'(t)\mathbf{j} + z_1'(t)\mathbf{k}] \pm [x_2'(t)\mathbf{i} + y_2'(t)\mathbf{j} + z_2'(t)\mathbf{k}] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$$

79. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then $w(t)\mathbf{r}(t) = w(t)x(t)\mathbf{i} + w(t)y(t)\mathbf{j} + w(t)z(t)\mathbf{k}$.

$$D_t[w(t)\mathbf{r}(t)] = [w(t)x'(t) + w'(t)x(t)]\mathbf{i} + [w(t)y'(t) + w'(t)y(t)]\mathbf{j} + [w(t)z'(t) + w'(t)z(t)]\mathbf{k}$$

$$= w(t)[x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}] + w'(t)[x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}] = w(t)\mathbf{r}'(t) + w'(t)\mathbf{r}(t)$$

80. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$.

$$\mathbf{r}(t) \times \mathbf{u}(t) = [y_1(t)z_2(t) - z_1(t)y_2(t)]\mathbf{i} - [x_1(t)z_2(t) - z_1(t)x_2(t)]\mathbf{j} + [x_1(t)y_2(t) - y_1(t)x_2(t)]\mathbf{k}$$

$$D_1[\mathbf{r}(t) \times \mathbf{u}(t)] = [y_1(t)z_2'(t) + y_1'(t)z_2(t) - z_1(t)y_2'(t) - z_1'(t)y_2(t)]\mathbf{i} - [x_1(t)z_2'(t) + x_1'(t)z_2(t) - z_1(t)x_2'(t) - z_1'(t)x_2(t)]\mathbf{j}$$

$$+ [x_1(t)y_2'(t) + x_1'(t)y_2(t) - y_1(t)x_2'(t) - y_1'(t)x_2(t)]\mathbf{k}$$

$$= \{[y_1(t)z_2'(t) - z_1(t)y_2'(t)]\mathbf{i} - [x_1(t)z_2'(t) - z_1(t)x_2'(t)]\mathbf{j} + [x_1(t)y_2'(t) - y_1(t)x_2'(t)]\mathbf{k}\}$$

$$+ \{[y_1'(t)z_2(t) - z_1'(t)y_2(t)]\mathbf{i} - [x_1'(t)z_2(t) - z_1'(t)x_2(t)]\mathbf{j} + [x_1'(t)y_2(t) - y_1'(t)x_2(t)]\mathbf{k}\}$$

$$= \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$$

81. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $\mathbf{r}(w(t)) = x(w(t))\mathbf{i} + y(w(t))\mathbf{j} + z(w(t))\mathbf{k}$ and

$$D_t[\mathbf{r}(w(t))] = x'(w(t))w'(t)\mathbf{i} + y'(w(t))w'(t)\mathbf{j} + z'(w(t))w'(t)\mathbf{k} \quad \text{(Chain Rule)}$$

$$= w'(t)[x'(w(t))\mathbf{i} + y'(w(t))\mathbf{j} + z'(w(t))\mathbf{k}] = w'(t)\mathbf{r}'(w(t)).$$

82. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

$$\mathbf{r}(t) \times \mathbf{r}'(t) = [y(t)z'(t) - z(t)y'(t)]\mathbf{i} - [x(t)z'(t) - z(t)x'(t)]\mathbf{j} + [x(t)y'(t) - y(t)x'(t)]\mathbf{k}$$

$$D_t[\mathbf{r}(t) \times \mathbf{r}'(t)] = [y(t)z''(t) + y'(t)z'(t) - z(t)y''(t) - z'(t)y'(t)]\mathbf{i} - [x(t)z''(t) + x'(t)z'(t) - z(t)x''(t) - z'(t)x'(t)]\mathbf{j}$$

$$+ [x(t)y''(t) + x'(t)y'(t) - y(t)x''(t) - y'(t)x'(t)]\mathbf{k}$$

$$= [y(t)z''(t) - z(t)y''(t)]\mathbf{i} - [x(t)z''(t) - z(t)x''(t)]\mathbf{j} + [x(t)y''(t) - y(t)x''(t)]\mathbf{k} = \mathbf{r}(t) \times \mathbf{r}''(t)$$

83. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$, $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$, and $\mathbf{v}(t) = x_3(t)\mathbf{i} + y_3(t)\mathbf{j} + z_3(t)\mathbf{k}$. Then:

$$\mathbf{r}(t)\cdot\left[\mathbf{u}(t)\times\mathbf{v}(t)\right] = x_1(t)\left[y_2(t)z_3(t) - z_2(t)y_3(t)\right] - y_1(t)\left[x_2(t)z_3(t) - z_2(t)x_3(t)\right] + z_1(t)\left[x_2(t)y_3(t) - y_2(t)x_3(t)\right]$$

$$\begin{aligned}
D_t\left[\mathbf{r}(t)\cdot\left(\mathbf{u}(t)\times\mathbf{v}(t)\right)\right] &= x_1(t)y_2(t)z_3'(t) + x_1(t)y_2'(t)z_3(t) + x_1'(t)y_2(t)z_3(t) - x_1(t)y_3(t)z_2'(t) \\
&\quad - x_1(t)y_3'(t)z_2(t) - x_1'(t)y_3(t)z_2(t) - y_1(t)x_2(t)z_3'(t) - y_1(t)x_2'(t)z_3(t) - y_1'(t)x_2(t)z_3(t) \\
&\quad + y_1(t)z_2(t)x_3'(t) + y_1(t)z_2'(t)x_3(t) + y_1'(t)z_2(t)x_3(t) + z_1(t)x_2(t)y_3'(t) + z_1(t)x_2'(t)y_3(t) \\
&\quad + z_1'(t)x_2(t)y_3(t) - z_1(t)y_2(t)x_3'(t) - z_1(t)y_2'(t)x_3(t) - z_1'(t)y_2(t)x_3(t) \\
&= \left\{x_1'(t)\left[y_2(t)z_3(t) - y_3(t)z_2(t)\right] + y_1'(t)\left[-x_2(t)z_3(t) + z_2(t)x_3(t)\right] + z_1'(t)\left[x_2(t)y_3(t) - y_2(t)x_3(t)\right]\right\} \\
&\quad + \left\{x_1(t)\left[y_2'(t)z_3(t) - y_3(t)z_2'(t)\right] + y_1(t)\left[-x_2'(t)z_3(t) + z_2'(t)x_3(t)\right] + z_1(t)\left[x_2'(t)y_3(t) - y_2'(t)x_3(t)\right]\right\} \\
&\quad + \left\{x_1(t)\left[y_2(t)z_3'(t) - y_3'(t)z_2(t)\right] + y_1(t)\left[-x_2(t)z_3'(t) + z_2(t)x_3'(t)\right] + z_1(t)\left[x_2(t)y_3'(t) - y_2(t)x_3'(t)\right]\right\} \\
&= \mathbf{r}'(t)\cdot\left[\mathbf{u}(t)\times\mathbf{v}(t)\right] + \mathbf{r}(t)\cdot\left[\mathbf{u}'(t)\times\mathbf{v}(t)\right] + \mathbf{r}(t)\cdot\left[\mathbf{u}(t)\times\mathbf{v}'(t)\right]
\end{aligned}$$

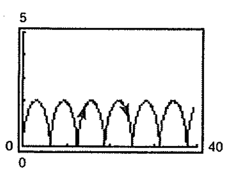
84. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. If $\mathbf{r}(t)\cdot\mathbf{r}(t)$ is constant, then:

$$\begin{aligned}
x^2(t) + y^2(t) + z^2(t) &= C \\
D_t\left[x^2(t) + y^2(t) + z^2(t)\right] &= D_t[C] \\
2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t) &= 0 \\
2\left[x(t)x'(t) + y(t)y'(t) + z(t)z'(t)\right] &= 0 \\
2\left[\mathbf{r}(t)\cdot\mathbf{r}'(t)\right] &= 0.
\end{aligned}$$

So, $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$.

85. $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$

(a)



The curve is a cycloid.

(b) $\mathbf{r}'(t) = (1 - \cos t)\mathbf{i} + \sin t\mathbf{j}$

$\mathbf{r}''(t) = \sin t\mathbf{i} + \cos t\mathbf{j}$

$$\begin{aligned}\left\|\mathbf{r}'(t)\right\| &= \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} \\ &= \sqrt{2 - 2\cos t}\end{aligned}$$

Minimum of $\left\|\mathbf{r}'(t)\right\|$ is 0, $(t = 0)$.

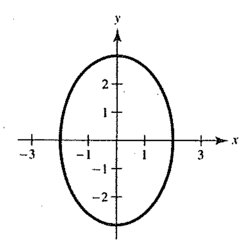
Maximum of $\left\|\mathbf{r}'(t)\right\|$ is 2, $(t = \pi)$.

$\left\|\mathbf{r}''(t)\right\| = \sqrt{\sin^2 t + \cos^2 t} = 1$

Minimum and maximum of $\left\|\mathbf{r}'(t)\right\|$ is 1.

86. $r(t) = 2\cos t\mathbf{i} + 3\sin t\mathbf{j}$

(a) Ellipse



(b) $\mathbf{r}'(t) = -2\sin t\mathbf{i} + 3\cos t\mathbf{j}$

$\mathbf{r}''(t) = -2\cos t\mathbf{i} - 3\sin t\mathbf{j}$

$\left\|\mathbf{r}'(t)\right\| = \sqrt{4\sin^2 t + 9\cos^2 t}$

Minimum of $\left\|\mathbf{r}'(t)\right\|$ is 2, $(t = \pi/2)$.

Maximum of $\left\|\mathbf{r}'(t)\right\|$ is 3, $(t = 0)$.

87. $\mathbf{r}(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{j}$

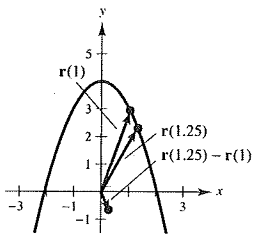
$\mathbf{r}'(t) = (e^t \cos t + e^t \sin t)\mathbf{i} + (e^t \cos t - e^t \sin t)\mathbf{j}$

$\mathbf{r}''(t) = (-e^t \sin t + e^t \cos t + e^t \sin t + e^t \cos t)\mathbf{i} + (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t)\mathbf{j} = 2e^t \cos t\mathbf{i} - 2e^t \sin t\mathbf{j}$

$\mathbf{r}(t) \cdot \mathbf{r}''(t) = 2e^{2t} \sin t \cos t - 2e^{2t} \sin t \cos t = 0$

So, $\mathbf{r}(t)$ is always perpendicular to $\mathbf{r}''(t)$.

88. (a) $\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j}$



(b) $\mathbf{r}(1) = \mathbf{i} + 3\mathbf{j}$

$\mathbf{r}(1.25) = 1.25\mathbf{i} + 2.4375\mathbf{j}$

$\mathbf{r}(1.25) - \mathbf{r}(1) = 0.25\mathbf{i} - 0.5625\mathbf{j}$

(c) $\mathbf{r}'(t) = \mathbf{i} - 2t\mathbf{j}$

$\mathbf{r}'(1) = \mathbf{i} - 2\mathbf{j}$

$$\frac{\mathbf{r}(1.25) - \mathbf{r}(1)}{1.25 - 1} = \frac{0.25\mathbf{i} - 0.5625\mathbf{j}}{0.25} = \mathbf{i} - 2.25\mathbf{j}$$

This vector approximates $\mathbf{r}'(1)$.

89. True

90. False. The definite integral is a vector, not a real number.

91. False. Let $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$.

$\|\mathbf{r}(t)\| = \sqrt{2}$

$\frac{d}{dt}[\|\mathbf{r}(t)\|] = 0$

$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = 1$

92. False.

$D[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$

(See Theorem 2.2, part 4)

Section 12.3 Velocity and Acceleration

1. $\mathbf{r}(t) = 3t\mathbf{i} + (t - 1)\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}$

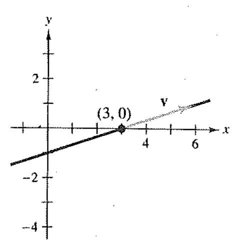
$\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{0}$

$x = 3t,\ y = t - 1,$

$y = \frac{x}{3} - 1$

At $(3, 0), t = 1$.

$\mathbf{v}(1) = 3\mathbf{i} + \mathbf{j}, \mathbf{a}(1) = \mathbf{0}$

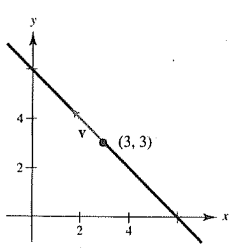


2. $\mathbf{r}(t) = (6 - t)\mathbf{i} + t\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = -\mathbf{i} + \mathbf{j}$

$\mathbf{a}(t) = \mathbf{r}''(t) = 0$

$x = 6 - t, y = t, y = 6 - x$



3. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

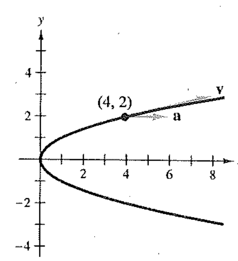
$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{i}$

$x = t^2, y = t, x = y^2$

At $(4, 2), t = 2$.

$\mathbf{v}(2) = 4\mathbf{i} + \mathbf{j}$

$\mathbf{a}(2) = 2\mathbf{i}$



4. $\mathbf{r}(t) = t\mathbf{i} + (-t^2 + 4)\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} - 2t\mathbf{j}$

$a(t) = \mathbf{r}''(t) = -2\mathbf{j}$

$x = t, y = -t^2 + 4 = 4 - x^2$

At $(1, 3)$, $t = 1$, $\mathbf{v}(1) = \mathbf{i} - 2\mathbf{j}$, $\mathbf{a}(1) = -2\mathbf{j}$