(b)

Year of Q	Slope of PQ
1995	$\frac{20.1 - 2.7}{2000 - 1995} = 3.48$
1996	$\frac{20.1 - 4.8}{2000 - 1996} = 3.825$
1997	$\frac{20.1 - 7.8}{2000 - 1997} = 4.1$
1998	$\frac{20.1 - 11.2}{2000 - 1998} = 4.45$
1999	$\frac{20.1 - 15.2}{2000 - 1999} = 4.9$

(c) Approximately 5 billion dollars per year.

(d) $y = 0.3214x^2 - 1.3471x + 1.3857$

$$\lim_{h\to 0} \frac{y(10+h) - y(10)}{h} = \lim_{h\to 0} \frac{[0.3214(10+h)^2 - 1.3471(10+h) + 1.3857] - [0.3214(10)^2 - 1.3471(10) + 1.3857]}{h}$$

$$= \lim_{h\to 0} \frac{0.3214(20h + h^2) - 1.3471h}{h}$$

$$= 0.3214(20) - 1.3471$$

$$\approx 5.081$$

The predicted rate of change in 2000 is about 5.081 billion dollars per year.

Chapter 3
Derivatives

Section 3.1 Derivative of a Function (pp. 95–104)

Exploration 1 Reading the Graphs

1. The graph in Figure 3.3b represents the rate of change of the depth of the water in the ditch with respect to time. Since y is measured in inches and x is measured in days, the derivative $\frac{dy}{dx}$ would be measured in inches per day. Those are the units that should be used along the y-axis in Figure 3.3b.
2. The water in the ditch is 1 inch deep at the start of the first day and rising rapidly. It continues to rise, at a gradually decreasing rate, until the end of the second day, when it achieves a maximum depth of 5 inches. During days 3, 4, 5, and 6, the water level goes down, until it reaches a depth of 1 inch at the end of day 6. During the seventh day it rises again, almost to a depth of 2 inches.
3. The weather appears to have been wettest at the beginning of day 1 (when the water level was rising fastest) and driest at the end of day 4 (when the water level was declining the fastest).
4. The highest point on the graph of the derivative shows where the water is rising the fastest, while the lowest point (most negative) on the graph of the derivative shows where the water is declining the fastest.
5. The y-coordinate of point C gives the maximum depth of the water level in the ditch over the 7-day period, while the x-coordinate of C gives the time during the 7-day period that the maximum depth occurred. The derivative of the function changes sign from positive to negative at C', indicating that this is when the water level stops rising and begins falling.
6. Water continues to run down sides of hills and through underground streams long after the rain has stopped falling. Depending on how much high ground is located near the ditch, water from the first day's rain could still be flowing into the ditch several days later. Engineers responsible for flood control of major rivers must take this into consideration when they predict when floodwaters will "crest," and at what levels.

Quick Review 3.1

1. $\lim_{h\to 0} \frac{(2+h)^2 - 4}{4} = \lim_{h\to 0} \frac{(4+4h+h^2)-4}{h}$
$= \lim_{h\to 0} 4 + h$
$= 4 + 0 = 4$

2. $\lim_{x\to 2^+} \frac{x+3}{2} = \frac{2+3}{2} = \frac{5}{2}$

3. Since $\frac{|y|}{y} = -1$ for $y < 0$, $\lim_{y\to 0^-} \frac{|y|}{y} = -1$.

4. $\lim_{x\to 4} \frac{2x-8}{\sqrt{x}-2} = \lim_{x\to 4} \frac{2(\sqrt{x}+2)(\sqrt{x}-2)}{\sqrt{x}-2}$
$= \lim_{h\to 4} 2(\sqrt{x}+2) = 2(\sqrt{4}+2) = 8$

5. The vertex of the parabola is at (0, 1). The slope of the line through (0, 1) and another point (h, h^2+1) on the parabola is $\frac{(h^2+1)-1}{h-0} = h$. Since $\lim_{h\to 0} h = 0$, the slope of the line tangent to the parabola at its vertex is 0.

6. Use the graph of f in the window $[-6, 6]$ by $[-4, 4]$ to find that (0, 2) is the coordinate of the high point and $(2, -2)$ is the coordinate of the low point. Therefore, f is increasing on $(-\infty, 0]$ and $[2, \infty)$.

7. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1)^2 = (1-1)^2 = 0$
$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x+2) = 1 + 2 = 3$

8. $\lim_{h\to 0^+} f(1+h) = \lim_{x\to 1^+} f(x) = 0$

9. No, the two one-sided limits are different (see Exercise 7).

10. No, f is discontinuous at $x = 1$ because $\lim_{x\to 1} f(x)$ does not exist.

Section 3.1 Exercises

1. **(a)** The tangent line has slope 5 and passes through (2, 3).
$y = 5(x-2) + 3$
$y = 5x - 7$

(b) The normal line has slope $-\frac{1}{5}$ and passes through (2, 3).
$y = -\frac{1}{5}(x-2) + 3$
$y = -\frac{1}{5}x + \frac{17}{5}$

2. $f'(3) = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$
$= \lim_{h\to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h}$
$= \lim_{h\to 0} \frac{3-(3+h)}{3h(3+h)}$
$= \lim_{h\to 0} \frac{-1}{3(3+h)} = -\frac{1}{9}$

3. $f'(3) = \lim_{x\to 3} \frac{f(x)-f(3)}{x-3}$
$= \lim_{x\to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x-3}$
$= \lim_{x\to 3} \frac{3-x}{(x-3)(x)(3)}$
$= \lim_{x\to 3} -\frac{1}{3x} = -\frac{1}{9}$

4. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
$= \lim_{h\to 0} \frac{[3(x+h)-12]-(3x-12)}{h}$
$= \lim_{h\to 0} \frac{3h}{h} = \lim_{h\to 0} 3 = 3$

5. $\frac{dy}{dx} = \lim_{h\to 0} \frac{y(x+h)-y(x)}{h}$
$= \lim_{h\to 0} \frac{7(x+h)-7x}{h}$
$= \lim_{h\to 0} \frac{7h}{h} = \lim_{h\to 0} 7 = 7$

6. Let $f(x) = x^2$.
$\frac{d}{dx}(x^2) = f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$
$= \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h}$
$= \lim_{h\to 0} (2x+h) = 2x$

7. The graph of $y = f_1(x)$ is decreasing for $x < 0$ and increasing for $x > 0$, so its derivative is negative for $x < 0$ and positive for $x > 0$. (b)

8. The graph of $y = f_2(x)$ is always increasing, so its derivative is always ≥ 0. (a)

9. The graph of $y = f_3(x)$ oscillates between increasing and decreasing, so its derivative oscillates between positive and negative. (d)

10. The graph of $y = f_4(x)$ is decreasing, then increasing , then decreasing, and then increasing, so its derivative is negative, then positive, then negative, and then positive. (c)

11. $\frac{dy}{dx} = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h}$

$= \lim_{h\to 0} \frac{[2(x+h)^2 - 13(x+h) + 5] - (2x^2 - 13x + 5)}{h}$

$= \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 - 13x - 13h + 5 - 2x^2 + 13x - 5}{h}$

$= \lim_{h\to 0} \frac{4xh + 2h^2 - 13h}{h}$

$= \lim_{h\to 0} (4x + 2h - 13) = 4x - 13$

At $x = 3$, $\frac{dy}{dx} = 4(3) - 13 = -1$, so the tangent line has slope -1 and passes through $(3, y(3)) = (3, -16)$.

$y = -1(x - 3) - 16$

$y = -x - 13$

12. Let $f(x) = x^3$.

$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$

$= \lim_{h\to 0} \frac{(1+h)^3 - 1^3}{h}$

$= \lim_{h\to 0} \frac{1 + 3h + 3h^2 + h^3 - 1}{h}$

$= \lim_{h\to 0} (3 + 3h + h^2) = 3$

(a) The tangent line has slope 3 and passes through (1, 1). Its equation is $y = 3(x - 1) + 1$, or $y = 3x - 2$.

(b) The normal line has slope $-\frac{1}{3}$ and passes through (1, 1). Its equation is $y = -\frac{1}{3}(x - 1) + 1$, or $y = -\frac{1}{3}x + \frac{4}{3}$.

13. Since the graph of $y = x \ln x - x$ is decreasing for $0 < x < 1$ and increasing for $x > 1$, its derivative is negative for $0 < x < 1$ and positive for $x > 1$. The only one of the given functions with this property is $y = \ln x$. Note also that $y = \ln x$ is undefined for $x < 0$, which further agrees with the given graph. (ii)

14. Each of the functions $y = \sin x$, $y = x$, $y = \sqrt{x}$ has the property that $y(0) = 0$ but the graph has nonzero slope (or undefined slope) at $x = 0$, so none of these functions can be its own derivative. The function $y = x^2$ is not its own derivative because $y(1) = 1$ but

$y'(1) = \lim_{h\to 0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h\to 0} \frac{2h + h^2}{h}$

$= \lim_{h\to 0} (2 + h) = 2.$

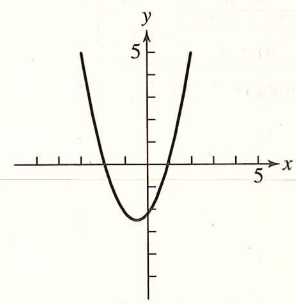
This leaves only e^x, which can plausibly be its own derivative because both the function value and the slope increase from very small positive values to very large values as we move from left to right along the graph. (iv)

15. (a) The amount of daylight is increasing at the fastest rate when the slope of the graph is largest. This occurs about one-fourth of the way through the year, sometime around April 1. The rate at this time is approximately $\frac{4 \text{ hours}}{24 \text{ days}}$ or $\frac{1}{6}$ hour per day.

(b) Yes, the rate of change is zero when the tangent to the graph is horizontal. This occurs near the beginning of the year and halfway through the year, around January 1 and July 1.

(c) Positive: January 1 through July 1
Negative: July 1 through December 31

16. The slope of the given graph is zero at $x \approx -2$ and at $x \approx 1$, so the derivative graph includes $(-2, 0)$ and $(1, 0)$. The slopes at $x = -3$ and at $x = 2$ are about 5 and the slope at $x = -0.5$ is about -2.5, so the derivative graph includes $(-3, 5)$, $(2, 5)$, and $(-0.5, -2.5)$. Connecting the points smoothly, we obtain the graph shown.



17. (a) Using Figure 3.10a, the number of rabbits is largest after 40 days and smallest from about 130 to 200 days. Using Figure 3.10b, the derivative is 0 at these times.

(b) Using Figure 3.10b, the derivative is largest after 20 days and smallest after about 63 days. Using Figure 3.10a, there were 1700 and about 1300 rabbits, respectively, at these times.

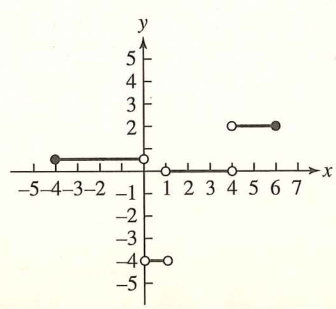
18. (a) The slope from $x = -4$ to $x = 0$ is $\frac{2 - 0}{0 - (-4)} = \frac{1}{2}$.

The slope from $x = 0$ to $x = 1$ is $\frac{-2 - 2}{1 - 0} = -4$.

The slope from $x = 1$ to $x = 4$ is $\frac{-2 - (-2)}{4 - 1} = 0$.

The slope from $x = 4$ to $x = 6$ is $\frac{2 - (-2)}{6 - 4} = 2$.

Note that the derivative is undefined at $x = 0$, $x = 1$, and $x = 4$. (The function is differentiable at $x = -4$ and at $x = 6$ because these are endpoints of the domain and the one-sided derivatives exist.) The graph of the derivative is shown.



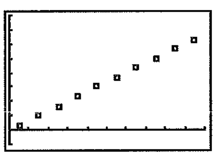
18. continued

(b) $x = 0, 1, 4$

19.

Midpoint of Interval (x)	Slope $\left(\frac{\Delta y}{\Delta x}\right)$
0.5	$\frac{3.3 - 0}{1 - 0} = 3.3$
1.5	$\frac{13.3 - 3.3}{2 - 1} = 10.0$
2.5	$\frac{29.9 - 13.3}{3 - 2} = 16.6$
3.5	$\frac{53.2 - 29.9}{4 - 3} = 23.3$
4.5	$\frac{83.2 - 53.2}{5 - 4} = 30.0$
5.5	$\frac{119.8 - 83.2}{6 - 5} = 36.6$
6.5	$\frac{163.0 - 119.8}{7 - 6} = 43.2$
7.5	$\frac{212.9 - 163.0}{8 - 7} = 49.9$
8.5	$\frac{269.5 - 212.9}{9 - 8} = 56.6$
9.5	$\frac{332.7 - 269.5}{10 - 9} = 63.2$

A graph of the derivative data is shown.



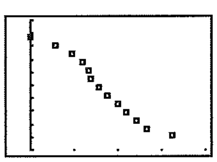
[0, 10] by[−10, 80]

(a) The derivative represents the speed of the skier.

(b) Since the distances are given in feet and the times are given in seconds, the units are feet per second.

(c) The graph appears to be approximately linear and passes through (0, 0) and (9.5, 63.2), so the slope is $\frac{63.2 - 0}{9.5 - 0} \approx 6.65$. The equation of the derivative is approximately $D = 6.65t$.

20. (a)

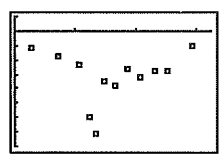


[−0.5, 4] by [700, 1700]

(b)

Midpoint of Interval (x)	Slope $\left(\frac{\Delta y}{\Delta x}\right)$
$\frac{0.00 + 0.56}{2} = 0.28$	$\frac{1512 - 1577}{0.56 - 0.00} \approx -116.07$
$\frac{0.56 + 0.92}{2} = 0.74$	$\frac{1448 - 1512}{0.92 - 0.56} \approx -177.78$
$\frac{0.92 + 1.19}{2} = 1.055$	$\frac{1384 - 1448}{1.19 - 0.92} \approx -237.04$
$\frac{1.19 + 1.30}{2} = 1.245$	$\frac{1319 - 1384}{1.30 - 1.19} \approx -590.91$
$\frac{1.30 + 1.39}{2} = 1.345$	$\frac{1255 - 1319}{1.39 - 1.30} \approx -711.11$
$\frac{1.39 + 1.57}{2} = 1.48$	$\frac{1191 - 1255}{1.57 - 1.39} \approx -355.56$
$\frac{1.57 + 1.74}{2} = 1.655$	$\frac{1126 - 1191}{1.74 - 1.57} \approx -382.35$
$\frac{1.74 + 1.98}{2} = 1.86$	$\frac{1062 - 1126}{1.98 - 1.74} \approx -266.67$
$\frac{1.98 + 2.18}{2} = 2.08$	$\frac{998 - 1062}{2.18 - 1.98} = -320.00$
$\frac{2.18 + 2.41}{2} = 2.295$	$\frac{933 - 998}{2.41 - 2.18} \approx -282.61$
$\frac{2.41 + 2.64}{2} = 2.525$	$\frac{869 - 933}{2.64 - 2.41} \approx -278.26$
$\frac{2.64 + 3.24}{2} = 2.94$	$\frac{805 - 869}{3.24 - 2.64} \approx -106.67$

A graph of the derivative data is shown.



[0, 3.24] by [−800, 100]

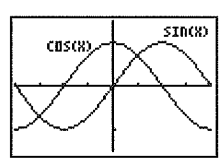
(c) Since the elevation y is given in feet and the distance x down river is given in miles, the units of the gradiant are feet per mile.

(d) Since the elevation y is given in feet and the distance x downriver is given in miles, the units of the distance $\frac{dy}{dx}$ are feet per mile.

(e) Look for the steepest part of the curve. This is where the elevation is dropping most rapidly, and therefore the most likely location for significant "rapids."

(f) Look for the lowest point on the graph. This is where the elevation is dropping most rapidly, and therefore the most likely location for significant "rapids."

21.



[$-\pi$, π] by [−1.5, 1.5]

The cosine function could be the derivative of the sine function. The values of the cosine are positive where the sine is increasing, zero where the sine has horizontal tangents, and negative where sine is decreasing.

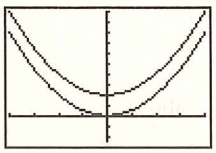
22. We show that the right-hand derivative at 1 does not exist.

$$\lim_{h\to0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^+}\frac{3(1+h)-(1)^3}{h}$$
$$=\lim_{h\to0^+}\frac{2+3h}{h}=\lim_{h\to0^+}\left(\frac{2}{h}+3\right)=\infty$$

23. $$\lim_{h\to0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to0^+}\frac{\sqrt{h}-\sqrt{0}}{h}=\lim_{h\to0^+}\frac{\sqrt{h}}{h}$$
$$=\lim_{h\to0^+}\frac{1}{\sqrt{h}}=\infty$$

Thus, the right-hand derivative at 0 does not exist.

24. Two parabolas are parallel if they have the same derivative at every value of x. This means that their tangent lines are parallel at each value of x.
Two such parabolas are given by $y = x^2$ and $y = x^2 + 4$. They are graphed below.

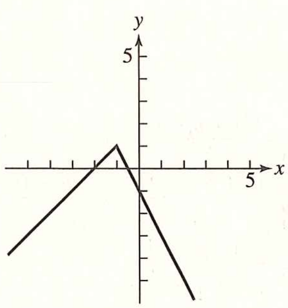


$[-4, 4]$ by $[-5, 20]$

The parabolas are "everywhere equidistant," as long as the distance between them is always measured along a vertical line.

25. For $x > -1$, the graph of $y = f(x)$ must lie on a line of slope -2 that passes through $(0, -1)$: $y = -2x - 1$. Then $y(-1) = -2(-1) - 1 = 1$, so for $x < -1$, the graph of $y = f(x)$ must lie on a line of slope 1 that passes through $(-1, 1)$: $y = 1(x + 1) + 1$ or $y = x + 2$.

Thus $f(x) = \begin{cases} x+2, & x<-1 \\ -2x-1, & x\ge -1 \end{cases}$



26. (a) $$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to0}\frac{2xh+h^2}{h}$$
$$=\lim_{h\to0}(2x+h)=2x$$

(b) $$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to0}\frac{2(x+h)-2x}{h}=\lim_{h\to0}2=2$$

(c) $\lim_{x\to1^-}f'(x)=\lim_{x\to1^-}2x=2(1)=2$

(d) $\lim_{x\to1^+}f'(x)=\lim_{x\to1^+}2=2$

(e) Yes, the one-sided limits exist and are the same, so $\lim_{x\to1}f'(x)=2$.

(f) $$\lim_{h\to0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^-}\frac{(1+h)^2-1^2}{h}$$
$$=\lim_{h\to0^-}\frac{2h+h^2}{h}=\lim_{h\to0^-}(2+h)=2$$

(g) $$\lim_{h\to0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^-}\frac{2(1+h)-1^2}{h}$$
$$=\lim_{h\to0^-}\frac{1+2h}{h}=\lim_{h\to0^-}\left(\frac{1}{h}+2\right)=-\infty$$

The right-hand derivative does not exist.

(h) It does not exist because the right-hand derivative does not exist.

27. The y-intercept of the derivative is $b - a$.

28. Since the function must be continuous at $x = 1$, we have $\lim_{x\to1^+}(3x+k)=f(1)=1$, so $3 + k = 1$, or $k = -2$.

This gives $f(x) = \begin{cases} x^3, & x\le 1 \\ 3x-2, & x>1. \end{cases}$

Now we confirm that $f(x)$ is differentiable at $x = 1$.

$$\lim_{h\to0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^-}\frac{(1+h)^3-(1)^3}{h}$$
$$=\lim_{h\to0}\frac{3h+3h^2+h^3}{h}$$
$$=\lim_{h\to0}(3+3h+h^2)=3$$

$$\lim_{h\to0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^+}\frac{[3(1+h)-2]-(1)^3]}{h}$$
$$=\lim_{h\to0^+}\frac{(1+3h)-1}{h}=\lim_{h\to0^+}3=3$$

Since the right-hand derivative equals the left-hand derivative at $x = 1$, the derivative exists (and is equal to 3) when $k = -2$.

29. (a) $1\cdot\frac{364}{365}\cdot\frac{363}{365}\approx0.992$

Alternate method: $\frac{_{365}P_3}{365^3}\approx0.992$

(b) Using the answer to part (a), the probability is about $1 - 0.992 = 0.008$.

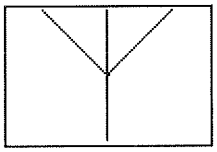
(c) Let P represent the answer to part (b), $P \approx 0.008$. Then the probability that three people all have different birthdays is $1 - P$. Adding a fourth person, the probability that all have different birthdays is $(1-P)\left(\frac{362}{365}\right)$, so the probability of a shared birthday is $1-(1-P)\left(\frac{362}{365}\right)\approx0.016$.

(d) No. Clearly February 29 is a much less likely birth date. Furthermore, census data do not support the assumption that the other 365 birth dates are equally likely. However, this simplifying assumption may still give us some insight into this problem even if the calculated probabilities aren't completely accurate.

Section 3.2 Differentiability (pp. 105–112)

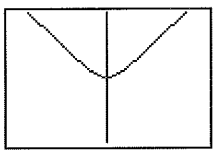
Exploration 1 Zooming in to "See" Differentiability

1. Zooming in on the graph of f at the point (0, 1) always produces a graph exactly like the one shown below, provided that a square window is used. The corner shows no sign of straightening out.



$[-0.25, 0.25]$ by $[0.836, 1.164]$

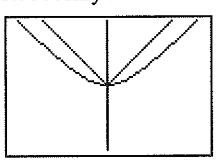
2. Zooming in on the graph of g at the point (0, 1) begins to reveal a smooth turning point. This graph shows the result of three zooms, each by a factor of 4 horizontally and vertically, starting with the window $[-4, 4]$ by $[-1.624, 3.624]$.



$[-0.0625, 0.0625]$ by $[0.959, 1.041]$

3. On our grapher, the graph became horizontal after 8 zooms. Results can vary on different machines.

4. As we zoom in on the graphs of f and g together, the differentiable function gradually straightens out to resemble its tangent line, while the nondifferentiable function stubbornly retains its same shape.



$[-0.03125, 0.03125]$ by $[0.9795, 1.0205]$

Exploration 2 Looking at the Symmetric Difference Quotient Analytically

1. $\dfrac{f(10+h)-f(10)}{h} = \dfrac{(10.01)^2 - 10^2}{0.01} = 20.01$

   $f'(10) = 2 \cdot 10 = 20$

   The difference quotient is 0.01 away from $f'(10)$.

2. $\dfrac{f(10+h)-f(10-h)}{2h} = \dfrac{(10.01)^2-(9.99)^2}{0.02} = 20$

   The symmetric difference quotient exactly equals $f'(10)$.

3. $\dfrac{f(10+h)-f(10)}{h} = \dfrac{(10.01)^3 - 10^3}{0.01} = 300.3001.$

   $f'(10) = 3 \cdot 10^2 = 300$

   The difference quotient is 0.3001 away from $f'(10)$.

   $\dfrac{f(10+h)-f(10-h)}{2h} = \dfrac{(10.01)^3-(9.99)^3}{0.02} = 300.0001.$

   The symmetric difference quotient is 0.0001 away from $f'(10)$.

Quick Review 3.2

1. Yes
2. No (The $f(h)$ term in the numerator is incorrect.)
3. Yes
4. Yes
5. No (The denominator for this expression should be $2h$).
6. All reals
7. $[0, \infty)$
8. $[3, \infty)$
9. The equation is equivalent to $y = 3.2x + (3.2\pi + 5)$, so the slope is 3.2.
10. $$\frac{f(3+0.001)-f(3-0.001)}{0.002} = \frac{5(3+0.001)-5(3-0.001)}{0.002} = \frac{5(0.002)}{0.002} = 5$$

Section 3.2 Exercises

1. Left-hand derivative:

   $$\lim_{h\to0^-}\frac{f(0+h)-f(0)}{h} = \lim_{h\to0^-}\frac{h^2-0}{h} = \lim_{h\to0^-} h = 0$$

   Right-hand derivative:

   $$\lim_{h\to0^+}\frac{f(0+h)-f(0)}{h} = \lim_{h\to0^+}\frac{h-0}{h} = \lim_{h\to0^+} 1 = 1$$

   Since $0 \neq 1$, the function is not differentiable at the point P.

2. Left-hand derivative:

   $$\lim_{h\to0^-}\frac{f(1+h)-f(1)}{h} = \lim_{h\to0^-}\frac{2-2}{h} = \lim_{h\to0^-} 0 = 0$$

   Right-hand derivative:

   $$\lim_{h\to0^+}\frac{f(1+h)-f(1)}{h} = \lim_{h\to0^+}\frac{2(1+h)-2}{h} = \lim_{h\to0^+} 2 = 2$$

   Since $0 \neq 2$, the function is not differentiable at the point P.

3. Left-hand derivative:

   $$\lim_{h\to0^-}\frac{f(1+h)-f(1)}{h} = \lim_{h\to0^-}\frac{\sqrt{1+h}-1}{h}$$
   $$= \lim_{h\to0^-}\frac{(\sqrt{1+h}-1)(\sqrt{1+h}+1)}{h(\sqrt{1+h}+1)}$$
   $$= \lim_{h\to0^-}\frac{(1+h)-1}{h(\sqrt{1+h}+1)}$$
   $$= \lim_{h\to0^-}\frac{1}{\sqrt{1+h}+1} = \frac{1}{2}$$

   Right-hand derivative:

   $$\lim_{h\to0^+}\frac{f(1+h)-f(1)}{h} = \lim_{h\to0^+}\frac{[2(1+h)-1]-1}{h}$$
   $$= \lim_{h\to0^+}\frac{2h}{h}$$
   $$= \lim_{h\to0^+} 2 = 2$$

   Since $\frac{1}{2} \neq 2$, the function is not differentiable at the point P.

4. Left-hand derivative:

$$\lim_{h\to0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^-}\frac{(1+h)-1}{h}=\lim_{h\to0^-}1=1$$

Right-hand derivative:

$$\lim_{h\to0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^+}\frac{\frac{1}{1+h}-1}{h}$$
$$=\lim_{h\to0^+}\frac{1-(1+h)}{h(1+h)}$$
$$=\lim_{h\to0^+}\frac{-h}{h(1+h)}$$
$$=\lim_{h\to0^+}-\frac{1}{1+h}=-1$$

Since $1\neq-1$, the function is not differentiable at the point P.

5. **(a)** All points in $[-3, 2]$
(b) None
(c) None

6. **(a)** All points in $[-2, 3]$
(b) None
(c) None

7. **(a)** All points in $[-3, 3]$ except $x = 0$
(b) None
(c) $x = 0$

8. **(a)** All points in $[-2, 3]$ except $x = -1, 0, 2$
(b) $x = -1$
(c) $x = 0, x = 2$

9. **(a)** All points in $[-1, 2]$ except $x = 0$
(b) $x = 0$
(c) None

10. **(a)** All points in $[-3, 3]$ except $x = -2, 2$
(b) $x = -2, x = 2$
(c) None

11. Since $\lim_{x\to0}\tan^{-1}x=\tan^{-1}0=0\neq y(0)$, the problem is a discontinuity.

12. $$\lim_{h\to0^-}\frac{y(0+h)-y(0)}{h}=\lim_{h\to0^-}\frac{h^{4/5}}{h}=\lim_{h\to0^-}\frac{1}{h^{1/5}}=-\infty$$
$$\lim_{h\to0^+}\frac{y(0+h)-y(0)}{h}=\lim_{h\to0^+}\frac{h^{4/5}}{h}=\lim_{h\to0^+}\frac{1}{h^{1/5}}=\infty$$

The problem is a cusp.

13. Note that $y=x+\sqrt{x^2}+2=x+|x|+2$
$$=\begin{cases}2, & x\le0\\ 2x+2, & x>0.\end{cases}$$
$$\lim_{h\to0^-}\frac{y(0+h)-y(0)}{h}=\lim_{h\to0^-}\frac{2-2}{h}=\lim_{h\to0^-}0=0$$
$$\lim_{h\to0^+}\frac{y(0+h)-y(0)}{h}=\lim_{h\to0^+}\frac{(2h+2)-2}{h}=\lim_{h\to0^+}2=2$$

The problem is a corner.

14. $$\lim_{h\to0}\frac{y(0+h)-y(0)}{h}=\lim_{h\to0}\frac{(3-\sqrt[3]{h})-3}{h}=\lim_{h\to0}-\frac{\sqrt[3]{h}}{h}$$
$$=\lim_{h\to0}\left(-\frac{1}{h^{2/3}}\right)=-\infty$$

The problem is a vertical tangent.

15. Note that $y=3x-2|x|-1=\begin{cases}5x-1, & x\le0\\ x-1, & x>0\end{cases}$
$$\lim_{h\to0^-}\frac{y(0+h)-y(0)}{h}=\lim_{h\to0^-}\frac{(5h-1)-(-1)}{h}=\lim_{h\to0^-}5=5$$
$$\lim_{h\to0^+}\frac{y(0+h)-y(0)}{h}=\lim_{h\to0^+}\frac{(h-1)-(-1)}{h}=\lim_{h\to0^+}1=1$$

The problem is a corner.

16. $$\lim_{h\to0^-}\frac{y(0+h)-y(0)}{h}=\lim_{h\to0^-}\frac{\sqrt[3]{|h|}-0}{h}=\lim_{h\to0^-}\frac{-\sqrt[3]{h}}{h}$$
$$=\lim_{h\to0^-}\left(-\frac{1}{h^{2/3}}\right)=-\infty$$
$$\lim_{h\to0^+}\frac{y(0+h)-y(0)}{h}=\lim_{h\to0^+}\frac{\sqrt[3]{|h|}-0}{h}=\lim_{h\to0^+}\frac{\sqrt[3]{h}}{h}$$
$$=\lim_{h\to0^+}\frac{1}{h^{2/3}}=\infty$$

The problem is a cusp.

17. Find the zeros of the denominator.
$x^2-4x-5=0$
$(x+1)(x-5)=0$
$x=-1$ or $x=5$
The function is a rational function, so it is differentiable for all x in its domain: all reals except $x=-1, 5$.

18. The function is differentiable except possibly where $3x-6=0$, that is, at $x=2$. We check for differentiability at $x=2$, using k instead of the usual h, in order to avoid confusion with the function $h(x)$.

$$\lim_{k\to0}\frac{h(2+k)-h(2)}{k}=\lim_{k\to0}\frac{[\sqrt[3]{3(2+k)-6}+5]-5}{k}$$
$$=\lim_{k\to0}\frac{\sqrt[3]{3k}}{k}=\sqrt[3]{3}\lim_{k\to0}\frac{1}{k^{2/3}}=\infty$$

The function has a vertical tangent at $x=2$. It is differentiable for all reals except $x=2$.

19. Note that the sine function is odd, so
$$P(x)=\sin(|x|)-1=\begin{cases}-\sin x-1, & x<0\\ \sin x-1, & x\ge0.\end{cases}$$

The graph of $P(x)$ has a corner at $x=0$. The function is differentiable for all reals except $x=0$.

20. Since the cosine function is even, so $Q(x)=3\cos(|x|)=3\cos x$. The function is differentiable for all reals.

21. The function is piecewise-defined in terms of polynomials, so it is differentiable everywhere except possibly at $x=0$ and at $x=3$. Check $x=0$:

$$\lim_{h\to0^-}\frac{g(0+h)-g(0)}{h}=\lim_{h\to0^-}\frac{(h+1)^2-1}{h}=\lim_{h\to0^-}\frac{h^2+2h}{h}$$
$$=\lim_{h\to0^-}(h+2)=2$$
$$\lim_{h\to0^+}\frac{g(0+h)-g(0)}{h}=\lim_{h\to0^+}\frac{(2h+1)-1}{h}=\lim_{h\to0^+}2=2$$

The function is differentiable at $x=0$.

21. continued

Check $x = 3$:

Since $g(3) = (4 - 3)^2 = 1$ and

$\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-} (2x + 1) = 2(3) + 1 = 7$, the function is not continuous (and hence not differentiable) at $x = 3$.

The function is differentiable for all reals except $x = 3$.

22. Note that $C(x) = x|x| = \begin{cases} -x^2, & x < 0 \\ x^2, & x \ge 0 \end{cases}$, so it is differentiable for all x except possibly at $x = 0$.

Check $x = 0$:

$$\lim_{h\to 0} \frac{C(0 + h) - C(0)}{h} = \lim_{h\to 0} \frac{h|h| - 0}{h} = \lim_{h\to 0} |h| = 0$$

The function is differentiable for all reals.

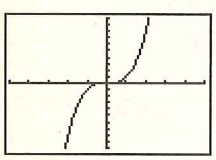
23. (a) $x = 0$ is not in their domains, or, they are both discontinuous at $x = 0$.

(b) For $\frac{1}{x}$: NDER $\left(\frac{1}{x}, 0\right) = 1{,}000{,}000$

For $\frac{1}{x^2}$: NDER $\left(\frac{1}{x^2}, 0\right) = 0$

(c) It returns an incorrect response because even though these functions are not defined at $x = 0$, they are defined at $x = \pm 0.001$. The responses differ from each other because $\frac{1}{x^2}$ is even (which automatically makes NDER $\left(\frac{1}{x^2}, 0\right) = 0$) and $\frac{1}{x}$ is odd.

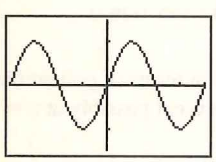
24.



$[-5, 5]$ by $[-10, 10]$

$\frac{dy}{dx} = x^3$

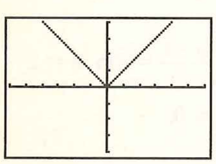
25.



$[-2\pi, 2\pi]$ by $[-1.5, 1.5]$

$\frac{dy}{dx} = \sin x$

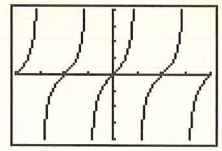
26.



$[-6, 6]$ by $[-4, 4]$

$\frac{dy}{dx} = \text{abs}(x)$ or $|x|$

27.

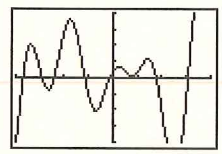


$[-2\pi, 2\pi]$ by $[-4, 4]$

$\frac{dy}{dx} = \tan x$

Note: Due to the way NDER is defined, the graph of $y = \text{NDER}(x)$ actually has two asymptotes for each asymptote of $y = \tan x$. The asymptotes of $y = \text{NDER}(x)$ occur at $x = \frac{\pi}{2} + k\pi \pm 0.001$, where k is an integer. A good window for viewing this behavior is $[1.566, 1.576]$ by $[-1000, 1000]$.

28.



$[-2\pi, 2\pi]$ by $[-20, 20]$

The graph of NDER (x) does not look like the graph of any basic function.

29. (a)

$$\lim_{x\to 1^-} f(x) = f(1)$$
$$\lim_{x\to 1^-} (3 - x) = a(1)^2 + b(1)$$
$$2 = a + b$$

The relationship is $a + b = 2$.

(b) Since the function needs to be continuous, we may assume that $a + b = 2$ and $f(1) = 2$.

$$\lim_{h\to 0^-} \frac{f(1 + h) - f(1)}{h} = \lim_{h\to 0^-} \frac{3 - (1 + h) - 2}{h} = \lim_{h\to 0^-} (-1) = -1$$

$$\begin{aligned}\lim_{h\to 0^+} \frac{f(1 + h) - f(1)}{h} &= \lim_{h\to 0^-} \frac{a(1 + h)^2 + b(1 + h) - 2}{h} \\ &= \lim_{h\to 0^-} \frac{a + 2ah + ah^2 + b + bh - 2}{h} \\ &= \lim_{h\to 0^-} \frac{2ah + ah^2 + bh + (a + b - 2)}{h} \\ &= \lim_{h\to 0^-} (2a + ah + b) \\ &= 2a + b\end{aligned}$$

Therefore, $2a + b = -1$. Substituting $2 - a$ for b gives $2a + (2 - a) = -1$, so $a = -3$.
Then $b = 2 - a = 2 - (-3) = 5$. The values are $a = -3$ and $b = 5$.

30. The function $f(x)$ does not have the intermediate value property. Choose some a in $(-1, 0)$ and b in $(0, 1)$. Then $f(a) = 0$ and $f(b) = 1$, but f does not take on any value between 0 and 1. Therefore, by the Intermediate Value Theorem for Derivatives, f cannot be the derivative of any function on $[-1, 1]$.

31. (a) Note that $-x \le \sin \frac{1}{x} \le x$, for all x,

so $\lim_{x \to 0} \left(x \sin \frac{1}{x} \right) = 0$ by the Sandwich Theorem.

Therefore, f is continuous at $x = 0$.

(b) $\frac{f(0+h) - f(0)}{h} = \frac{h \sin \frac{1}{h} - 0}{h} = \sin \frac{1}{h}$

(c) The limit does not exist because $\sin \frac{1}{h}$ oscillates between -1 and 1 an infinite number of times arbitrarily close to $h = 0$ (that is, for h in any open interval containing 0).

(d) No, because the limit in part (c) does not exist.

(e) $\frac{g(0+h) - g(0)}{h} = \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = h \sin \frac{1}{h}$

As noted in part (a), the limit of this as x approaches zero is 0, so $g'(0) = 0$.

Section 3.3 Rules for Differentiation (pp. 112–121)

Quick Review 3.3

1. $(x^2 - 2)(x^{-1} + 1) = x^2x^{-1} + x^2 \cdot 1 - 2x^{-1} - 2 \cdot 1$
$= x + x^2 - 2x^{-1} - 2$

2. $\left(\frac{x}{x^2+1}\right)^{-1} = \frac{x^2+1}{x} = \frac{x^2}{x} + \frac{1}{x} = x + x^{-1}$

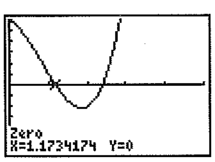
3. $3x^2 - \frac{2}{x} + \frac{5}{x^2} = 3x^2 - 2x^{-1} + 5x^{-2}$

4. $\frac{3x^4 - 2x^3 + 4}{2x^2} = \frac{3x^4}{2x^2} - \frac{2x^3}{2x^2} + \frac{4}{2x^2} = \frac{3}{2}x^2 - x + 2x^{-2}$

5. $(x^{-1} + 2)(x^{-2} + 1) = x^{-1}x^{-2} + x^{-1} \cdot 1 + 2x^{-2} + 2 \cdot 1$
$= x^{-3} + x^{-1} + 2x^{-2} + 2$

6. $\frac{x^{-1} + x^{-2}}{x^{-3}} = x^3(x^{-1} + x^{-2}) = x^2 + x$

7.



[0, 5] by [−6, 6]

At $x \approx 1.173$, $500x^6 \approx 1305$.
At $x \approx 2.394$, $500x^6 \approx 94{,}212$
After rounding, we have:
At $x \approx 1$, $500x^6 \approx 1305$.
At $x \approx 2$, $500x^6 \approx 94{,}212$.

8. (a) $f(10) = 7$

(b) $f(0) = 7$

(c) $f(x + h) = 7$

(d) $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{7 - 7}{x - a} = \lim_{x \to a} 0 = 0$

9. These are all constant functions, so the graph of each function is a horizontal line and the derivative of each function is 0.

10. (a) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{x+h}{\pi} - \frac{x}{\pi}}{h}$
$= \lim_{h \to 0} \frac{x + h - x}{\pi h} = \lim_{h \to 0} \frac{1}{\pi} = \frac{1}{\pi}$

(b) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{\pi}{x+h} - \frac{\pi}{x}}{h}$
$= \lim_{h \to 0} \frac{\pi x - \pi(x+h)}{hx(x+h)} = \lim_{h \to 0} \frac{-\pi h}{hx(x+h)}$
$= \lim_{h \to 0} -\frac{\pi}{x(x+h)} = -\frac{\pi}{x^2} = -\pi x^{-2}$

Section 3.3 Exercises

1. $\frac{dy}{dx} = \frac{d}{dx}(-x^2) + \frac{d}{dx}(3) = -2x + 0 = -2x$
$\frac{d^2y}{dx^2} = \frac{d}{dx}(-2x) = -2$

2. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{3}x^3\right) - \frac{d}{dx}(x) = x^2 - 1$
$\frac{d^2y}{dx^2} = \frac{d}{dx}(x^2) - \frac{d}{dx}(1) = 2x - 0 = 2x$

3. $\frac{dy}{dx} = \frac{d}{dx}(2x) + \frac{d}{dx}(1) = 2 + 0 = 2$
$\frac{d^2y}{dx^2} = \frac{d}{dx}(2) = 0$

4. $\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(x) + \frac{d}{dx}(1) = 2x + 1 + 0 = 2x + 1$
$\frac{d^2y}{dx^2} = \frac{d}{dx}(2x) + \frac{d}{dx}(1) = 2 + 0 = 2$

5. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{3}x^3\right) + \frac{d}{dx}\left(\frac{1}{2}x^2\right) + \frac{d}{dx}(x)$
$= x^2 + x + 1$
$\frac{d^2y}{dx^2} = \frac{d}{dx}(x^2) + \frac{d}{dx}(x) + \frac{d}{dx}(1) = 2x + 1 + 0 = 2x + 1$

6. $\frac{dy}{dx} = \frac{d}{dx}(1) - \frac{d}{dx}(x) + \frac{d}{dx}(x^2) - \frac{d}{dx}(x^3)$
$= 0 - 1 + 2x - 3x^2 = -1 + 2x - 3x^2$
$\frac{d^2y}{dx^2} = \frac{d}{dx}(-1) + \frac{d}{dx}(2x) - \frac{d}{dx}(3x^2)$
$= 0 + 2 - 6x = 2 - 6x$

7. $\frac{dy}{dx} = \frac{d}{dx}(x^4) - \frac{d}{dx}(7x^3) + \frac{d}{dx}(2x^2) + \frac{d}{dx}(15)$
$= 4x^3 - 21x^2 + 4x + 0 = 4x^3 - 21x^2 + 4x$
$\frac{d^2y}{dx^2} = \frac{d}{dx}(4x^3) - \frac{d}{dx}(21x^2) + \frac{d}{dx}(4x) = 12x^2 - 42x + 4$

8. $\frac{dy}{dx} = \frac{d}{dx}(5x^3) - \frac{d}{dx}(3x^5) = 15x^2 - 15x^4$
$\frac{d^2y}{dx^2} = \frac{d}{dx}(15x^2) - \frac{d}{dx}(15x^4) = 30x - 60x^3$

9. $\frac{dy}{dx} = \frac{d}{dx}(4x^{-2}) - \frac{d}{dx}(8x) + \frac{d}{dx}(1)$
$= -8x^{-3} - 8 + 0 = -8x^{-3} - 8$
$\frac{d^2y}{dx^2} = \frac{d}{dx}(-8x^{-3}) - \frac{d}{dx}(8) = 24x^{-4} - 0 = 24x^{-4}$

10. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{4}x^{-4}\right) - \frac{d}{dx}\left(\frac{1}{3}x^{-3}\right) + \frac{d}{dx}\left(\frac{1}{2}x^{-2}\right) - \frac{d}{dx}(x^{-1}) + \frac{d}{dx}(3)$

$= -x^{-5} + x^{-4} - x^{-3} + x^{-2} + 0$

$= -x^{-5} + x^{-4} - x^{-3} + x^{-2}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}(-x^{-5}) + \frac{d}{dx}(x^{-4}) - \frac{d}{dx}(x^{-3}) + \frac{d}{dx}(x^{-2})$

$= 5x^{-6} - 4x^{-5} + 3x^{-4} - 2x^{-3}$

11. (a) $\frac{dy}{dx} = \frac{d}{dx}[(x+1)(x^2+1)]$

$= (x+1)\frac{d}{dx}(x^2+1) + (x^2+1)\frac{d}{dx}(x+1)$

$= (x+1)(2x) + (x^2+1)(1)$

$= 2x^2 + 2x + x^2 + 1$

$= 3x^2 + 2x + 1$

(b) $\frac{dy}{dx} = \frac{d}{dx}[(x+1)(x^2+1)]$

$= \frac{d}{dx}(x^3 + x^2 + x + 1)$

$= 3x^2 + 2x + 1$

12. (a) $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+3}{x}\right)$

$= \frac{x\frac{d}{dx}(x^2+3) - (x^2+3)\frac{d}{dx}(x)}{x^2}$

$= \frac{x(2x) - (x^2+3)}{x^2}$

$= \frac{x^2-3}{x^2}$

(b) $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+3}{x}\right) = \frac{d}{dx}(x + 3x^{-1}) = 1 - 3x^{-2}$

$= 1 - \frac{3}{x^2}$

This is equivalent to the answer in part (a).

13. $\frac{dy}{dx} = \frac{d}{dx}\frac{2x+5}{3x-2} = \frac{(3x-2)(2) - (2x+5)(3)}{(3x-2)^2} = -\frac{19}{(3x-2)^2}$

14. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+5x-1}{x^2}\right) = \frac{d}{dx}(1 + 5x^{-1} - x^{-2})$

$= 0 - 5x^{-2} + 2x^{-3} = -\frac{5}{x^2} + \frac{2}{x^3}$

15. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{(x-1)(x^2+x+1)}{x^3}\right) = \frac{d}{dx}\left(\frac{x^3-1}{x^3}\right)$

$= \frac{d}{dx}(1 - x^{-3}) = 0 + 3x^{-4} = \frac{3}{x^4}$

16. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1-x}{1+x^2}\right) = \frac{(1+x^2)(-1) - (1-x)(2x)}{(1+x^2)^2}$

$= \frac{x^2-2x-1}{(1+x^2)^2}$

17. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2}{1-x^3}\right) = \frac{(1-x^3)(2x) - x^2(-3x^2)}{(1-x^3)^2} = \frac{x^4+2x}{(1-x^3)^2}$

18. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) = \frac{(\sqrt{x}+1)\frac{1}{2\sqrt{x}} - (\sqrt{x}-1)\frac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2}$

$= \frac{(\sqrt{x}+1) - (\sqrt{x}-1)}{2\sqrt{x}(\sqrt{x}+1)^2} = \frac{2}{2\sqrt{x}(\sqrt{x}+1)^2} = \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}$

19. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{(x+1)(x+2)}{(x-1)(x-2)}\right) = \frac{d}{dx}\left(\frac{x^2+3x+2}{x^2-3x+2}\right)$

$= \frac{(x^2-3x+2)(2x+3) - (x^2+3x+2)(2x-3)}{(x^2-3x+2)^2}$

$= \frac{(2x^3-3x^2-5x+6) - (2x^3+3x^2-5x-6)}{(x^2-3x+2)^2}$

$= \frac{12-6x^2}{(x^2-3x+2)^2}$

20. (a) Let $f(x) = x$.

$\frac{d}{dx}(x) = f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to0}\frac{(x+h)-x}{h}$

$= \lim_{h\to0}\frac{h}{h} = \lim_{h\to0}(1) = 1$

(b) Note that $u = u(x)$ is a function of x.

$\frac{d}{dx}(-u) = \lim_{h\to0}\frac{-u(x+h) - [-u(x)]}{h}$

$= \lim_{h\to0}\left(-\frac{u(x+h)-u(x)}{h}\right)$

$= -\lim_{h\to0}\frac{u(x+h)-u(x)}{h} = -\frac{du}{dx}$

21. $\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx}f(x) + f(x) \cdot \frac{d}{dx}(c)$

$= c \cdot \frac{d}{dx}f(x) + 0 = c \cdot \frac{d}{dx}f(x)$

22. $\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{f(x) \cdot 0 - 1 \cdot \frac{d}{dx}f(x)}{[f(x)]^2} = -\frac{f'(x)}{[f(x)]^2}$

23. (a) At $x = 0$, $\frac{d}{dx}(uv) = u(0)v'(0) + v(0)u'(0)$

$= (5)(2) + (-1)(-3) = 13$

(b) At $x = 0$, $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(0)u'(0) - u(0)v'(0)}{[v(0)]^2}$

$= \frac{(-1)(-3) - (5)(2)}{(-1)^2} = -7$

(c) At $x = 0$, $\frac{d}{dx}\left(\frac{v}{u}\right) = \frac{u(0)v'(0) - v(0)u'(0)}{[u(0)]^2}$

$= \frac{(5)(2) - (-1)(-3)}{(5)^2} = \frac{7}{25}$

(d) At $x = 0$, $\frac{d}{dx}(7v - 2u) = 7v'(0) - 2u'(0)$

$= 7(2) - 2(-3) = 20$

24. **(a)** At $x = 2$, $\frac{d}{dx}(uv) = u(2)v'(2) + v(2)u'(2)$

$= (3)(2) + (1)(-4) = 2$

(b) At $x = 2$, $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(2)u'(2) - u(2)v'(2)}{[v(2)]^2}$

$= \frac{(1)(-4) - (3)(2)}{(1)^2} = -10$

(c) At $x = 2$, $\frac{d}{dx}\left(\frac{v}{u}\right) = \frac{u(2)v'(2) - v(2)u'(2)}{[u(2)]^2}$

$= \frac{(3)(2) - (1)(-4)}{(3)^2} = \frac{10}{9}$

(d) Use the result from part (a) for $\frac{d}{dx}(uv)$.

At $x = 2$, $\frac{d}{dx}(3u - 2v + 2uv)$

$= 3u'(2) - 2v'(2) + 2\frac{d}{dx}(uv)$

$= 3(-4) - 2(2) + 2(2)$

$= -12$

25. $y'(x) = 2x + 5$
$y'(3) = 2(3) + 5 = 11$
The slope is 11. (iii)

26. The given equation is equivalent to $y = \frac{3}{2}x + 6$, so the slope is $\frac{3}{2}$. (iii)

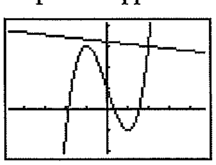
27. $y'(x) = 3x^2 - 3$
$y'(2) = 3(2)^2 - 3 = 9$

The tangent line has slope 9, so the perpendicular line has slope $-\frac{1}{9}$ and passes through (2, 3).

$y = -\frac{1}{9}(x - 2) + 3$

$y = -\frac{1}{9}x + \frac{29}{9}$

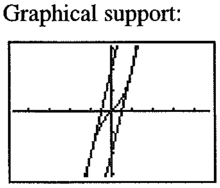
Graphical support:



[−4.7, 4.7] by [−2.1, 4.1]

28. $y'(x) = 3x^2 + 1$
The slope is 4 when $3x^2 + 1 = 4$, at $x = \pm 1$. The tangent at $x = -1$ has slope 4 and passes through $(-1, -2)$, so its equation is $y = 4(x + 1) - 2$, or $y = 4x + 2$. The tangent at $x = 1$ has slope 4 and passes through (1, 2), so its equation is $y = 4(x - 1) + 2$, or $y = 4x - 2$. The smallest slope occurs when $3x^2 + 1$ is minimized, so the smallest slope is 1 and occurs at $x = 0$.

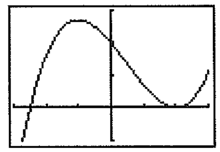
Graphical support:



[−4.7, 4.7] by [−3.1, 3.1]

29. $y'(x) = 6x^2 - 6x - 12$
$= 6(x^2 - x - 2)$
$= 6(x + 1)(x - 2)$
The tangent is parallel to the x-axis when $y' = 0$, at $x = -1$ and at $x = 2$. Since $y(-1) = 27$ and $y(2) = 0$, the two points where this occurs are $(-1, 27)$ and $(2, 0)$.

Graphical support:

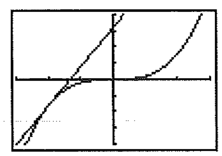


[−3, 3] by [−10, 30]

30. $y'(x) = 3x^2$
$y'(-2) = 12$

The tangent line has slope 12 and passes through $(-2, -8)$, so its equation is $y = 12(x + 2) - 8$, or $y = 12x + 16$. The x-intercept is $-\frac{4}{3}$ and the y-intercept is 16.

Graphical support:



[−3, 3] by [−20, 20]

31. $y'(x) = \frac{(x^2 + 1)(4) - 4x(2x)}{(x^2 + 1)^2} = \frac{-4x^2 + 4}{(x^2 + 1)^2}$

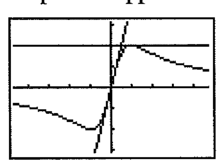
At the origin: $y'(0) = 4$

The tangent is $y = 4x$.

At (1, 2): $y'(1) = 0$

The tangent is $y = 2$.

Graphical support:



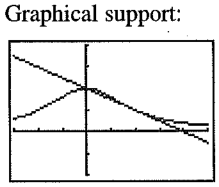
[−4.7, 4.7] by [−3.1, 3.1]

32. $y'(x) = \frac{(4 + x^2)(0) - 8(2x)}{(4 + x^2)^2} = -\frac{16x}{(4 + x^2)^2}$

$y'(2) = -\frac{1}{2}$

The tangent has slope $-\frac{1}{2}$ and passes through (2, 1). Its equation is $y = -\frac{1}{2}(x - 2) + 1$, or $y = -\frac{1}{2}x + 2$.

Graphical support:



[−3, 5] by [−2, 4]

33. $\frac{dP}{dV} = \frac{d}{dV}\left(\frac{nRT}{V-nb} - \frac{an^2}{V^2}\right)$

$= \frac{(V-nb)\frac{d}{dV}(nRT) - (nRT)\frac{d}{dV}(V-nb)}{(V-nb)^2} - \frac{d}{dV}(an^2V^{-2})$

$= \frac{0 - nRT}{(V-nb)^2} + 2an^2V^{-3}$

$= -\frac{nRT}{(V-nb)^2} + \frac{2an^2}{V^3}$

34. $\frac{ds}{dt} = \frac{d}{dt}(4.9t^2) = 9.8t$

$\frac{d^2s}{dt^2} = \frac{d}{dt}(9.8t) = 9.8$

35. $\frac{dR}{dM} = \frac{d}{dM}\left[M^2\left(\frac{C}{2} - \frac{M}{3}\right)\right]$

$= \frac{d}{dM}\left(\frac{C}{2}M^2 - \frac{1}{3}M^3\right)$

$= CM - M^2$

36. If the radius of a circle is changed by a very small amount Δr, the change in the area can be thought of as a very thin strip with length given by the circumference, $2\pi r$, and width Δr. Therefore, the change in the area can be thought of as $(2\pi r)(\Delta r)$, which means that the change in the area divided by the change in the radius is just $2\pi r$.

37. If the radius of a sphere is changed by a very small amount Δr, the change in the volume can be thought of as a very thin layer with an area given by the surface area, $4\pi r^2$, and a thickness given by Δr. Therefore, the change in the volume can be thought of as $(4\pi r^2)(\Delta r)$, which means that the change in the volume divided by the change in the radius is just $4\pi r^2$.

38. Let $t(x)$ be the number of trees and $y(x)$ be the yield per tree x years from now. Then $t(0) = 156$, $y(0) = 12$, $t'(0) = 13$, and $y'(0) = 1.5$. The rate of increase of production is

$\frac{d}{dx}(ty) = t(0)y'(0) + y(0)t'(0) = (156)(1.5) + (12)(13)$

$= 390$ bushels of annual production per year.

39. Let $m(x)$ be the number of members and $c(x)$ be the pavillion cost x years from now. Then $m(0) = 65$, $c(0) = 250$, $m'(0) = 6$, and $c'(0) = 10$. The rate of change of each member's share is $\frac{d}{dx}\left(\frac{c}{m}\right) = \frac{m(0)c'(0) - c(0)m'(0)}{[m(0)]^2}$

$= \frac{(65)(10) - (250)(6)}{(65)^2} \approx -0.201$ dollars per year. Each member's share of the cost is decreasing by approximately 20 cents per year.

40. (a) It is insignificant in the limiting case and can be treated as zero (and removed from the expression).

(b) It was "rejected" because it is incomparably smaller than the other terms: $v\,du$ and $u\,dv$.

(c) $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$. This is equivalent to the product rule given in the text.

(d) Because dx is 'infinitely small," and this could be thought of as dividing by zero.

(e) $d\left(\frac{u}{v}\right) = \frac{u + du}{v + dv} - \frac{u}{v}$

$= \frac{(u + du)(v) - (u)(v + dv)}{(v + dv)(v)}$

$= \frac{uv + vdu - uv - udv}{v^2 + vdv}$

$= \frac{vdu - udv}{v^2}$

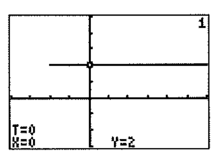
■ Section 3.4 Velocity and Other Rates of Change (pp. 122–133)

Exploration 1 Growth Rings on a Tree

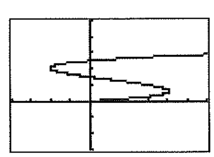
1. Figure 3.22 is a better model, as it shows rings of equal *area* as opposed to rings of equal *width*. It is not likely that a tree could sustain increased growth year after year, although climate conditions do produce some years of greater growth than others.
2. Rings of equal area suggest that the tree adds approximately the same amount of wood to its girth each year. With access to approximately the same raw materials from which to make the wood each year, this is how most trees actually grow.
3. Since change in area is constant, so also is $\frac{\text{change in area}}{2\pi}$. If we denote this latter constant by k, we have $\frac{k}{\text{change in radius}} = r$, which means that r varies inversely as the change in the radius. In other words, the change in radius must get smaller when r gets bigger, and vice-versa.

Exploration 2 Modeling Horizontal Motion

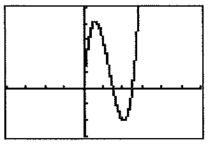
1. The particle reverses direction at about $t = 0.61$ and $t = 2.06$.



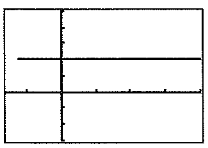
2. When the trace cursor is moving to the right the particle is moving to the right, and when the cursor is moving to the left the particle is moving to the left. Again we find the particle reverses direction at about $t = 0.61$ and $t = 2.06$.



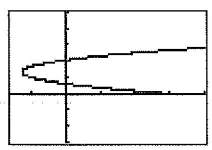
3. When the trace cursor is moving upward the particle is moving to the right, and when the cursor is moving downward the particle is moving to the left. Again we find the same values of t for when the particle reverses direction.



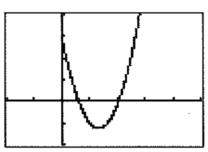
4. We can represent the velocity by graphing the parametric equations
$x_4(t) = x_1'(t) = 12t^2 - 32t + 15, y_4(t) = 2$ (part 1),
$x_5(t) = x_1'(t) = 12t^2 - 32t + 15, y_5(t) = t$ (part 2),
$x_6(t) = t, y_6(t) = x_1'(t) = 12t^2 - 32t + 15$ (part 3)



[−8, 20] by [−3, 5]
(x_4, y_4)



[−8, 20] by [−3, 5]
(x_5, y_5)



[−2, 5] by [−10, 20]
(x_6, y_6)

For (x_4, y_4) and (x_5, y_5), the particle is moving to the right when the x-coordinate of the graph (velocity) is positive, moving to the left when the x-coordinate of the graph (velocity) is negative, and is stopped when the x-coordinate of the graph (velocity) is 0. For (x_6, y_6), the particle is moving to the right when the y-coordinate of the graph (velocity) is positive, moving to the left when the y-coordinate of the graph (velocity) is negative, and is stopped when the y-coordinate of the graph (velocity) is 0.

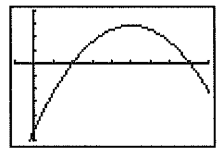
Exploration 3 Seeing Motion on a Graphing Calculator

1. Let tMin = 0 and tMax = 10.
2. Since the rock achieves a maximum height of 400 feet, set yMax to be slightly greater than 400, for example yMax = 420.
4. The grapher proceeds with constant increments of t (time), so pixels appear on the screen at regular time intervals. When the rock is moving more slowly, the pixels appear closer together. When the rock is moving faster, the pixels appear farther apart. We observe faster motion when the pixels are farther apart.

Quick Review 3.4

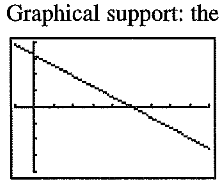
1. The coefficient of x^2 is negative, so the parabola opens downward.

   Graphical support:



[−1, 9] by [−300, 200]

2. The y-intercept is $f(0) = -256$.
   See the solution to Exercise 1 for graphical support.
3. The x-intercepts occur when $f(x) = 0$.
$$-16x^2 + 160x - 256 = 0$$
$$-16(x^2 - 10x + 16) = 0$$
$$-16(x - 2)(x - 8) = 0$$
$$x = 2 \text{ or } x = 8$$
   The x-intercepts are 2 and 8. See the solution to Exercise 1 for graphical support.
4. Since $f(x) = -16(x^2 - 10x + 16)$
$= -16(x^2 - 10x + 25 - 9) = -16(x - 5)^2 + 144$,
   the range is $(-\infty, 144]$.
   See the solution to Exercise 1 for graphical support.
5. Since $f(x) = -16(x^2 - 10x + 16)$
$= -16(x^2 - 10x + 25 - 9) = -16(x - 5)^2 + 144$,
   the vertex is at (5, 144). See the solution to Exercise 1 for graphical support.
6.
$$f(x) = 80$$
$$-16x^2 + 160x - 256 = 80$$
$$-16x^2 + 160x - 336 = 0$$
$$-16(x^2 - 10x + 21) = 0$$
$$-16(x - 3)(x - 7) = 0$$
$$x = 3 \text{ or } x = 7$$
   $f(x) = 80$ at $x = 3$ and at $x = 7$.
   See the solution to Exercise 1 for graphical support.
7.
$$\frac{dy}{dx} = 100$$
$$-32x + 160 = 100$$
$$60 = 32x$$
$$x = \frac{15}{8}$$
   $\frac{dy}{dx} = 100$ at $x = \frac{15}{8}$
   Graphical support: the graph of NDER $f(x)$ is shown.



[−1, 9] by [−200, 200]

8.
$$\frac{dy}{dx} > 0$$
$$-32x + 160 > 0$$
$$-32x > -160$$
$$x < 5$$
   $\frac{dy}{dx} > 0$ when $x < 5$.
   See the solution to Exercise 7 for graphical support.

9. Note that $f'(x) = -32x + 160$.

$$\lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = f'(3) = -32(3) + 160 = 64$$

For graphical support, use the graph shown in the solution to Exercise 7 and observe that NDER $(f(x), 3) \approx 64$.

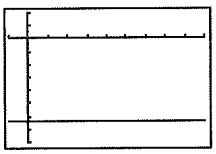
10. $f'(x) = -32x + 160$

$f''(x) = -32$

At $x = 7$ (and, in fact, at any other of x),

$\frac{d^2y}{dx^2} = -32.$

Graphical support: the graph of NDER(NDER $f(x)$) is shown.



$[-1, 9]$ by $[-40, 10]$

Section 3.4 Exercises

1. Since $V = s^3$, the instantaneous rate of change is $\frac{dV}{ds} = 3s^2$.

2. **(a)** Displacement $= s(5) - s(0) = 12 - 2 = 10$ m

(b) Average velocity $= \frac{10 \text{ m}}{5 \text{ sec}} = 2$ m/sec

(c) Velocity $= s'(t) = 2t - 3$
At $t = 4$, velocity $= s'(4) = 2(4) - 3 = 5$ m/sec

(d) Acceleration $= s''(t) = 2$ m/sec^2

(e) The particle changes direction when

$s'(t) = 2t - 3 = 0$, so $t = \frac{3}{2}$ sec.

(f) Since the acceleration is always positive, the position s is at a minimum when the particle changes direction, at $t = \frac{3}{2}$ sec. Its position at this time is $s\left(\frac{3}{2}\right) = -\frac{1}{4}$ m.

3. **(a)** Velocity: $v(t) = \frac{ds}{dt} = \frac{d}{dt}(24t - 0.8t^2) = 24 - 1.6t$ m/sec

Acceleration: $a(t) = \frac{dv}{dt} = \frac{d}{dt}(24 - 1.6t) = -1.6$ m/sec^2

(b) The rock reaches its highest point when $v(t) = 24 - 1.6t = 0$, at $t = 15$. It took 15 seconds.

(c) The maximum height was $s(15) = 180$ meters.

(d)
$$s(t) = \frac{1}{2}(180)$$
$$24t - 0.8t^2 = 90$$
$$0 = 0.8t^2 - 24t + 90$$
$$t = \frac{24 \pm\sqrt{(-24)^2 - 4(0.8)(90)}}{2(0.8)}$$
$$\approx 4.393, 25.607$$

It took about 4.393 seconds to reach half its maximum height.

(e)
$$s(t) = 0$$
$$24t - 0.8t^2 = 0$$
$$0.8t(30 - t) = 0$$
$$t = 0 \text{ or } t = 30$$

The rock was aloft from $t = 0$ to $t = 30$, so it was aloft for 30 seconds.

4. On Mars:

Velocity $= \frac{ds}{dt} = \frac{d}{dt}(1.86t^2) = 3.72t$

Solving $3.72t = 16.6$, the downward velocity reaches 16.6 m/sec after about 4.462 sec.

On Jupiter:

Velocity $= \frac{ds}{dt} = \frac{d}{dt}(11.44t^2) = 22.88t$

Solving $22.88t = 16.6$, the downward velocity reaches 16.6 m/sec after about 0.726 sec.

5. The rock reaches its maximum height when the velocity $s'(t) = 24 - 9.8t = 0$, at $t \approx 2.449$. Its maximum height is about $s(2.449) \approx 29.388$ meters.

6. Moon:
$$s(t) = 0$$
$$832t - 2.6t^2 = 0$$
$$2.6t(320 - t) = 0$$
$$t = 0 \text{ or } t = 320$$

It takes 320 seconds to return.

Earth:
$$s(t) = 0$$
$$832t - 16t^2 = 0$$
$$16t(52 - t) = 0$$
$$t = 0 \text{ or } t = 52$$

It takes 52 seconds to return.

7. The following is one way to simulate the problem situation.

For the moon:

$x_1(t) = 3(t < 160) + 3.1(t \geq 160)$

$y_1(t) = 832t - 2.6t^2$

t-values: 0 to 320

window: $[0, 6]$ by $[-10{,}000, 70{,}000]$

For the earth:

$x_1(t) = 3(t < 26) + 3.1(t \geq 26)$

$y_1(t) = 832t - 16t^2$

t-values: 0 to 52

window: $[0, 6]$ by $[-1000, 11{,}000]$

8. The growth rate is given by

$b'(t) = 10^4 - 2 \cdot 10^3 t = 10{,}000 - 2000t$.

At $t = 0$: $b'(0) = 10{,}000$ bacteria/hour

At $t = 5$: $b'(5) = 0$ bacteria/hour

At $t = 10$: $b'(10) = -10{,}000$ bacteria/hour

9. $Q(t) = 200(30 - t)^2 = 200(900 - 60t + t^2)$
$= 180{,}000 - 12{,}000t + 200t^2$
$Q'(t) = -12{,}000 + 400t$
The rate of change of the amount of water in the tank after 10 minutes is $Q'(10) = -8000$ gallons per minute.
Note that $Q'(10) < 0$, so the rate at which the water is running *out* is positive. The water is running out at the rate of 8000 gallons per minute.

The average rate for the first 10 minutes is

$\dfrac{Q(10) - Q(0)}{10 - 0} = \dfrac{80{,}000 - 180{,}000}{10} = -10{,}000$ gallons per minute.

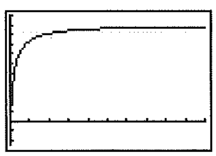
The water is flowing out at an average rate of 10,000 gallons per minute over the first 10 min.

10. (a) Average cost $= \dfrac{c(100)}{100} = \dfrac{11{,}000}{100} = \$110$ per machine

(b) $c'(x) = 100 - 0.2x$
Marginal cost $= c'(100) = \$80$ per machine

(c) Actual cost of 101st machine is
$c(101) - c(100) = \$79.90$, which is very close to the marginal cost calculated in part (b).

11. (a)



[0, 50] by [−500, 2200]
The values of x which make sense are the whole numbers, $x \geq 0$.

(b) Marginal revenue $= r'(x) = \dfrac{d}{dx}\left[2000\left(1 - \dfrac{1}{x+1}\right)\right]$

$= \dfrac{d}{dx}\left(2000 - \dfrac{2000}{x+1}\right)$

$= 0 - \dfrac{(x+1)(0) - (2000)(1)}{(x+1)^2} = \dfrac{2000}{(x+1)^2}$

(c) $r'(5) = \dfrac{2000}{(5+1)^2} = \dfrac{2000}{36} \approx 55.56$

The increase in revenue is approximately \$55.56.

(d) The limit is 0. This means that as x gets large, one reaches a point where very little extra revenue can be expected from selling more desks.

12. $v(t) = s'(t) = 3t^2 - 12t + 9$
$a(t) = v'(t) = 6t - 12$
Find when velocity is zero.
$3t^2 - 12t + 9 = 0$
$3(t^2 - 4t + 3) = 0$
$3(t - 1)(t - 3) = 0$
$t = 1$ or $t = 3$
At $t = 1$, the acceleration is $a(1) = -6$ m/sec^2
At $t = 3$, the acceleration is $a(3) = 6$ m/sec^2

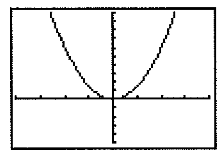
13. $a(t) = v'(t) = 6t^2 - 18t + 12$
Find when acceleration is zero.
$6t^2 - 18t + 12 = 0$
$6(t^2 - 3t + 2) = 0$
$6(t - 1)(t - 2) = 0$
$t = 1$ or $t = 2$
At $t = 1$, the speed is $|v(1)| = |0| = 0$ m/sec.
At $t = 2$, the speed is $|v(2)| = |-1| = 1$ m/sec.

14. (a) $g'(x) = \dfrac{d}{dx}(x^3) = 3x^2$

$h'(x) = \dfrac{d}{dx}(x^3 - 2) = 3x^2$

$t'(x) = \dfrac{d}{dx}(x^3 + 3) = 3x^2$

(b) The graphs of NDER $g(x)$, NDER $h(x)$, and NDER $t(x)$ are all the same, as shown.



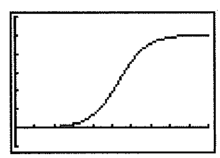
[−4, 4] by [−10, 20]

(c) $f(x)$ must be of the form $f(x) = x^3 + c$, where c is a constant.

(d) Yes. $f(x) = x^3$

(e) Yes. $f(x) = x^3 + 3$

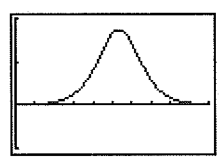
15. (a)



[0, 200] by [−2, 12]

(b) The values of x which make sense are the whole numbers, $x \geq 0$.

(c)



[0, 200] by [−0.1, 0.2]
P is most sensitive to changes in x when $|P'(x)|$ is largest. It is relatively sensitive to changes in x between approximately $x = 60$ and $x = 160$.

(d) The marginal profit, $P'(x)$, is greatest at $x \approx 106.44$. Since x must be an integer,
$P(106) \approx 4.924$ thousand dollars or \$4924.

(e) $P'(50) \approx 0.013$, or \$13 per package sold
$P'(100) \approx 0.165$, or \$165 per package sold
$P'(125) \approx 0.118$, or \$118 per package sold
$P'(150) \approx 0.031$, or \$31 per package sold
$P'(175) \approx 0.006$, or \$6 per package sold
$P'(300) \approx 10^{-6}$, or \$0.001 per package sold

(f) The limit is 10. The maximum possible profit is \$10,000 monthly.

(g) Yes. In order to sell more and more packages, the company might need to lower the price to a point where they won't make any additional profit.

16. **(a)** 190 ft/sec

(b) 2 seconds

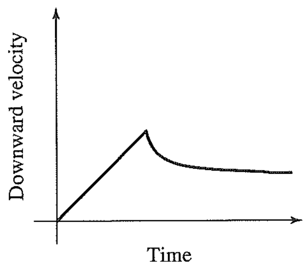
(c) After 8 seconds, and its velocity was 0 ft/sec then

(d) After about 11 seconds, and it was falling 90 ft/sec then

(e) About 3 seconds (from the rocket's highest point)

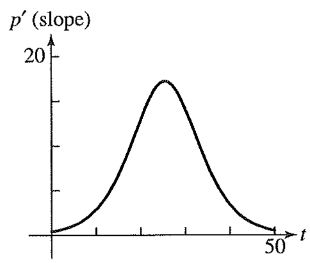
(f) The acceleration was greatest just before the engine stopped. The acceleration was constant from $t = 2$ to $t = 11$, while the rocket was in free fall.

17. Note that "downward velocity" is positive when McCarthy is falling downward. His downward velocity increases steadily until the parachute opens, and then decreases to a constant downward velocity. One possible sketch:



18. **(a)** We estimate the slopes at several points as follows, then connect the points to create a smooth curve.

t (days)	0	10	20	30	40	50
Slope (flies/day)	0.5	3.0	13.0	14.0	3.5	0.5



Horizontal axis: Days
Vertical axis: Flies per day

(b) Fastest: Around the 25th day
Slowest: Day 50 or day 0

19. The particle is at (5, 2) when $4t^3 - 16t^2 + 15t = 5$, which occurs at $t \approx 2.83$.

20. The motion can be simulated in parametric mode using $x_1(t) = 2t^3 - 13t^2 + 22t - 5$ and $y_1(t) = 2$ in $[-6, 8]$ by $[-3, 5]$.

(a) It begins at the point $(-5, 2)$ moving in the positive direction. After a little more than one second, it has moved a bit past (6, 2) and it turns back in the negative direction for approximately 2 seconds. At the end of that time, it is near $(-2, 2)$ and it turns back again in the positive direction. After that, it continues moving in the positive direction indefinitely, speeding up as it goes.

(b) The particle speeds up when its *speed* is increasing, which occurs during the approximate intervals $1.153 \le t \le 2.167$ and $t \ge 3.180$. It slows down during the approximate intervals $0 \le t \le 1.153$ and $2.167 \le t \le 3.180$. One way to determine the endpoints of these intervals is to use a grapher to find the minimums and maximums for the speed, $|\text{NDER } x(t)|$, using function mode in the window $[0, 5]$ by $[0, 10]$.

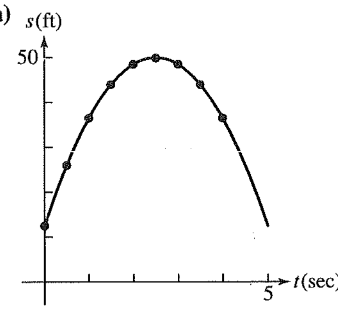
(c) The particle changes direction at $t \approx 1.153$ sec and at $t \approx 3.180$ sec.

(d) The particle is at rest "instantaneously" at $t \approx 1.153$ sec and at $t \approx 3.180$ sec.

(e) The velocity starts out positive but decreasing, it becomes negative, then starts to increase, and becomes positive again and continues to increase.
The speed is decreasing, reaches 0 at $t \approx 1.15$ sec, then increases until $t \approx 2.17$ sec, decreases until $t \approx 3.18$ sec when it is 0 again, and then increases after that.

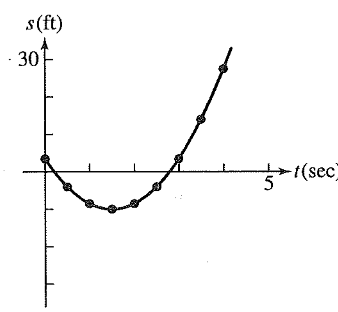
(f) The particle is at (5, 2) when $2t^3 - 13t^2 + 22t - 5 = 5$ at $t \approx 0.745$ sec, $t \approx 1.626$ sec, and at $t \approx 4.129$ sec.

21. **(a)**



(b) $s'(1) = 18$, $s'(2.5) = 0$, $s'(3.5) = -12$

22. **(a)**

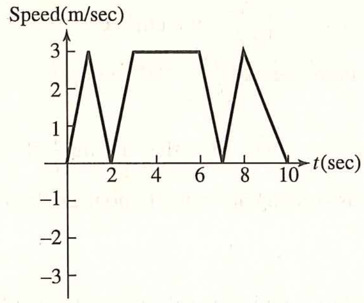


(b) $s'(1) = -6$, $s'(2.5) = 12$, $s'(3.5) = 24$

23. **(a)** The body reverses direction when v changes sign, at $t = 2$ and at $t = 7$.

(b) The body is moving at a constant speed, $|v| = 3$ m/sec, between $t = 3$ and $t = 6$.

(c) The speed graph is obtained by reflecting the negative portion of the velocity graph, $2 < t < 7$, about the x-axis.



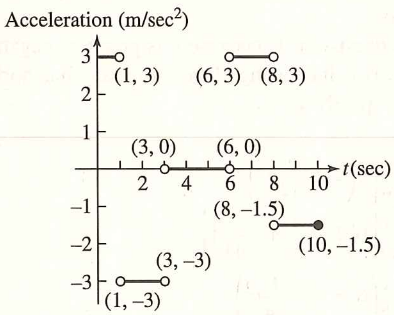
(d) For $0 \le t < 1$: $a = \dfrac{3-0}{1-0} = 3 \text{ m/sec}^2$

For $1 < t < 3$: $a = \dfrac{-3-3}{3-1} = -3 \text{ m/sec}^2$

For $3 < t < 6$: $a = \dfrac{-3-(-3)}{6-3} = 0 \text{ m/sec}^2$

For $6 < t < 8$: $a = \dfrac{3-(-3)}{8-6} = 3 \text{ m/sec}^2$

For $8 < t \le 10$: $a = \dfrac{0-3}{10-8} = -1.5 \text{ m/sec}^2$



24. **(a)** The particle is moving left when the graph of s has negative slope, for $2 < t < 3$ and for $5 < t \le 6$.
The particle is moving right when the graph of s has positive slope, for $0 \le t < 1$.
The particle is standing still when the graph of s is horizontal, for $1 < t < 2$ and for $3 < t < 5$.

(b) For $0 \le t < 1$: $v = \dfrac{2-0}{1-0} = 2$ cm/sec

Speed $= |v| = 2$ cm/sec

For $1 < t < 2$: $v = \dfrac{2-2}{2-1} = 0$ cm/sec

Speed $= |v| = 0$ cm/sec

For $2 < t < 3$: $v = \dfrac{-2-2}{3-2} = -4$ cm/sec

Speed $= |v| = 4$ cm/sec

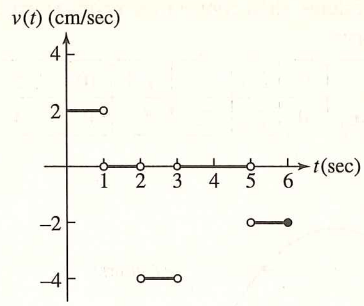
For $3 < t < 5$: $v = \dfrac{-2-(-2)}{5-3} = 0$ cm/sec

Speed $= |v| = 0$ cm/sec

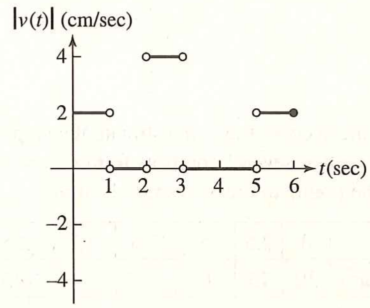
For $5 < t \le 6$: $v = \dfrac{-4-(-2)}{6-5} = -2$ cm/sec

Speed $= |v| = 2$ cm/sec

Velocity graph:



Speed graph:



25. **(a)** The particle moves forward when $v > 0$, for $0 \le t < 1$ and for $5 < t < 7$.
The particle moves backward when $v < 0$, for $1 < t < 5$.
The particle speeds up when v is negative and decreasing, for $1 < t < 2$, and when v is positive and increasing, for $5 < t < 6$.
The particle slows down when v is positive and decreasing, for $0 \le t < 1$ and for $6 < t < 7$, and when v is negative and increasing, for $3 < t < 5$.

(b) Note that the acceleration $a = \dfrac{dv}{dt}$ is undefined at $t = 2$, $t = 3$, and $t = 6$.

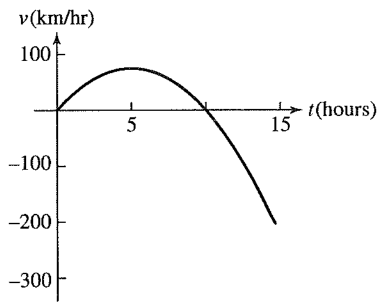
The acceleration is positive when v is increasing, for $3 < t < 6$.
The acceleration is negative when v is decreasing, for $0 \le t < 2$ and for $6 < t < 7$.
The acceleration is zero when v is constant, for $2 < t < 3$ and for $7 < t \le 9$.

(c) The particle moves at its greatest speed when $|v|$ is maximized, at $t = 0$ and for $2 < t < 3$.

(d) The particle stands still for more than an instant when v stays at zero, for $7 < t \le 9$.

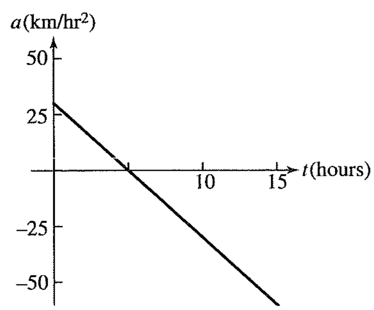
26. **(a)** To graph the velocity, we estimate the slopes at several points as follows, then connect the points to create a smooth curve.

t (hours)	0	2.5	5	7.5	10	12.5	15
v (km/hour)	0	56	75	56	0	−94	−225

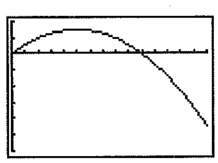


To graph the acceleration, we estimate the slope of the velocity graph at several points as follows, and then connect the points to create a smooth curve.

t (hours)	0	2.5	5	7.5	10	12.5	15
a (km/hour²)	30	15	0	−15	−30	−45	−60

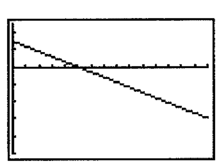


(b) $\frac{ds}{dt} = 30t - 3t^2$



[0, 15] by [−300, 100]

$\frac{d^2s}{dt^2} = 30 - 6t$



[0, 15] by [−100, 50]

The graphs are very similar.

27. **(a)** Solving $160 = 490t^2$ gives $t = \pm\frac{4}{7}$.

It took $\frac{4}{7}$ of a second. The average velocity was $\frac{160 \text{ cm}}{\left(\frac{4}{7}\right) \text{ sec}} = 280$ cm/sec.

(b) $v = s'(t) = 980t$

$a = s''(t) = 980$

The velocity was $s'\left(\frac{4}{7}\right) = 560$ cm/sec.

The acceleration was $s''\left(\frac{4}{7}\right) = 980$ cm/sec^2.

(c) Since there were about 16 flashes during $\frac{4}{7}$ of a second, the light was flashing at a rate of about 28 flashes per second.

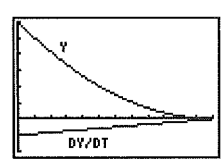
28. Graph C is position, graph A is velocity, and graph B is acceleration.
A is the derivative of C because it is positive, negative, and zero where C is increasing, decreasing, and has horizontal tangents, respectively. The relationship between B and A is similar.

29. Graph C is position, graph B is velocity, and graph A is acceleration.
B is the derivative of C because it is negative and zero where C is decreasing and has horizontal tangents, respectively.
A is the derivative of B because it is positive, negative, and zero where B is increasing, decreasing, and has horizontal tangents, respectively.

30. **(a)**
$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt}\left[6\left(1 - \frac{t}{12}\right)^2\right] \\ &= \frac{d}{dt}\left[6\left(1 - \frac{t}{6} + \frac{t^2}{144}\right)\right] \\ &= \frac{d}{dt}\left(6 - t + \frac{1}{24}t^2\right) \\ &= 0 - 1 + \frac{t}{12} = \frac{t}{12} - 1\end{aligned}$$

(b) The fluid level is falling fastest when $\frac{dy}{dt}$ is the most negative, at $t = 0$, when $\frac{dy}{dt} = -1$. The fluid level is falling slowest at $t = 12$, when $\frac{dy}{dt} = 0$.

(c)



[0, 12] by [−2, 6]

y is decreasing and $\frac{dy}{dt}$ is negative over the entire interval. y decreases more rapidly early in the interval, and the magnitude of $\frac{dy}{dt}$ is larger then. $\frac{dy}{dt}$ is 0 at $t = 12$, where the graph of y seems to have a horizontal tangent.

31. **(a)** $\frac{dV}{dr} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$

When $r = 2$, $\frac{dV}{dr} = 4\pi(2)^2 = 16\pi$ cubic feet of volume per foot of radius.

(b) The increase in the volume is

$\frac{4}{3}\pi(2.2)^3 - \frac{4}{3}\pi(2)^3 \approx 11.092$ cubic feet.

32. For $t > 0$, the speed of the aircraft in meters per second after t seconds is $\frac{20}{9}t$. Multiplying by $\frac{3600 \text{ sec}}{1 \text{ h}} \cdot \frac{1 \text{ km}}{1000 \text{ m}}$, we find that this is equivalent to $8t$ kilometers per hour. Solving $8t = 200$ gives $t = 25$ seconds. The aircraft takes 25 seconds to become airborne, and the distance it travels during this time is $D(25) \approx 694.444$ meters.

33. Let v_0 be the exit velocity of a particle of lava. Then $s(t) = v_0 t - 16t^2$ feet, so the velocity is $\frac{ds}{dt} = v_0 - 32t$. Solving $\frac{ds}{dt} = 0$ gives $t = \frac{v_0}{32}$. Then the maximum height, in feet, is $s\left(\frac{v_0}{32}\right) = v_0\left(\frac{v_0}{32}\right) - 16\left(\frac{v_0}{32}\right)^2 = \frac{v_0^2}{64}$. Solving $\frac{v_0^2}{64} = 1900$ gives $v_0 \approx \pm 348.712$. The exit velocity was about 348.712 ft/sec. Multiplying by $\frac{3600 \text{ sec}}{1 \text{ h}} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}}$, we find that this is equivalent to about 237.758 mi/h.

34. By estimating the slope of the velocity graph at that point.

35. Since profit = revenue − cost, the Sum and Difference Rule gives $\frac{d}{dx}(\text{profit}) = \frac{d}{dx}(\text{revenue}) - \frac{d}{dx}(\text{cost})$, where x is the number of units produced. This means that marginal profit = marginal revenue − marginal cost.

36. **(a)** It takes 135 seconds.

(b) Average speed $= \frac{\Delta F}{\Delta t} = \frac{5-0}{73-0} = \frac{5}{73}$

≈ 0.068 furlongs/sec.

(c) Using a symmetric difference quotient, the horse's speed is approximately

$\frac{\Delta F}{\Delta t} = \frac{4-2}{59-33} = \frac{2}{26} = \frac{1}{13} \approx 0.077$ furlongs/sec.

(d) The horse is running the fastest during the last furlong (between 9th and 10th furlong markers). This furlong takes only 11 seconds to run, which is the least amount of time for a furlong.

(e) The horse accelerates the fastest during the first furlong (between markers 0 and 1).

37. **(a)** Assume that f is even. Then,

$$\begin{aligned} f'(-x) &= \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h} \\ &= \lim_{h\to 0}\frac{f(x-h)-f(x)}{h}, \text{ and substituting } k=-h, \\ &= \lim_{k\to 0}\frac{f(x+k)-f(x)}{-k} \\ &= -\lim_{k\to 0}\frac{f(x+k)-f(x)}{k} = -f'(x) \end{aligned}$$

So, f' is an odd function.

(b) Assume that f is odd. Then,

$$\begin{aligned} f'(-x) &= \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h} \\ &= \lim_{h\to 0}\frac{-f(x-h)+f(x)}{h}, \end{aligned}$$

and substituting $k = -h$,

$$\begin{aligned} &= \lim_{k\to 0}\frac{-f(x+k)+f(x)}{-k} \\ &= \lim_{k\to 0}\frac{f(x+k)-f(x)}{k} = f'(x) \end{aligned}$$

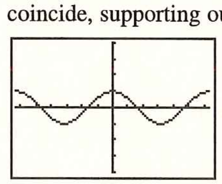
So, f' is an even function.

38. $$\begin{aligned} \frac{d}{dx}(fgh) &= \frac{d}{dx}[f(gh)] = f\cdot\frac{d}{dx}(gh) + gh\cdot\frac{d}{dx}(f) \\ &= f\left(g\cdot\frac{dh}{dx} + h\cdot\frac{dg}{dx}\right) + gh\cdot\frac{df}{dx} \\ &= \left(\frac{df}{dx}\right)gh + f\left(\frac{dg}{dx}\right)h + fg\left(\frac{dh}{dx}\right) \end{aligned}$$

Section 3.5 Derivatives of Trigonometric Functions (pp. 134–141)

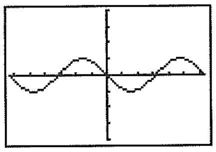
Exploration 1 Making a Conjecture with NDER

1. When the graph of $\sin x$ is increasing, the graph of NDER ($\sin x$) is positive (above the x-axis).
2. When the graph of $\sin x$ is decreasing, the graph of NDER ($\sin x$) is negative (below the x-axis).
3. When the graph of $\sin x$ stops increasing and starts decreasing, the graph of NDER ($\sin x$) crosses the x-axis from above to below.
4. The slope of the graph of $\sin x$ matches the value of NDER ($\sin x$) at these points.
5. We conjecture that NDER ($\sin x$) = $\cos x$. The graphs coincide, supporting our conjecture.



$[-2\pi, 2\pi]$ by $[-4, 4]$

6. When the graph of $\cos x$ is increasing, the graph of NDER $(\cos x)$ is positive (above the x-axis).
When the graph of $\cos x$ is decreasing, the graph of NDER $(\cos x)$ is negative (below the x-axis).
When the graph of $\cos x$ stops increasing and starts decreasing, the graph of NDER $(\cos x)$ crosses the x-axis from above to below.
The slope of the graph of $\cos x$ matches the value of NDER $(\cos x)$ at these points.
We conjecture that NDER $(\cos x) = -\sin x$. The graphs coincide, supporting our conjecture.



$[-2\pi, 2\pi]$ by $[-4, 4]$

Quick Review 3.5

1. $135° \cdot \frac{\pi}{180°} = \frac{3\pi}{4} \approx 2.356$

2. $1.7 \cdot \frac{180°}{\pi} = \left(\frac{306}{\pi}\right)^{\circ} \approx 97.403°$

3. $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

4. Domain: All reals
Range: $[-1, 1]$

5. Domain: $x \neq \frac{k\pi}{2}$ for odd integers k
Range: All reals

6. $\cos a = \pm\sqrt{1 - \sin^2 a} = \pm\sqrt{1 - (-1)^2} = \pm\sqrt{0} = 0$

7. If $\tan a = -1$, then $a = \frac{3\pi}{4} + k\pi$ for some integer k, so $\sin a = \pm\frac{1}{\sqrt{2}}$.

8. $\frac{1 - \cos h}{h} = \frac{(1 - \cos h)(1 + \cos h)}{h(1 + \cos h)} = \frac{1 - \cos^2 h}{h(1 + \cos h)}$
$= \frac{\sin^2 h}{h(1 + \cos h)}$

9. $y'(x) = 6x^2 - 14x$
$y'(3) = 12$
The tangent line has slope 12 and passes through (3, 1), so its equation is $y = 12(x - 3) + 1$, or $y = 12x - 35$.

10. $a(t) = v'(t) = 6t^2 - 14t$
$a(3) = 12$

Section 3.5 Exercises

1. $\frac{d}{dx}(1 + x - \cos x) = 0 + 1 - (-\sin x) = 1 + \sin x$

2. $\frac{d}{dx}(2 \sin x - \tan x) = 2 \cos x - \sec^2 x$

3. $\frac{d}{dx}\left(\frac{1}{x} + 5 \sin x\right) = -\frac{1}{x^2} + 5 \cos x$

4. $\frac{d}{dx}(x \sec x) = x\frac{d}{dx}(\sec x) + \sec x\frac{d}{dx}(x)$
$= x \sec x \tan x + \sec x$

5. $\frac{d}{dx}(4 - x^2 \sin x) = \frac{d}{dx}(4) - \left[x^2\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(x^2)\right]$
$= 0 - [x^2 \cos x + (\sin x)(2x)]$
$= -x^2 \cos x - 2x \sin x$

6. $\frac{d}{dx}(3x + x \tan x) = \frac{d}{dx}(3x) + \left[x\frac{d}{dx}(\tan x) + (\tan x)\frac{d}{dx}(x)\right]$
$= 3 + x \sec^2 x + \tan x$

7. $\frac{d}{dx}\left(\frac{4}{\cos x}\right) = \frac{d}{dx}(4 \sec x) = 4 \sec x \tan x$

8. $\frac{d}{dx}\frac{x}{1 + \cos x} = \frac{(1 + \cos x)\frac{d}{dx}(x) - x\frac{d}{dx}(1 + \cos x)}{(1 + \cos x)^2}$
$= \frac{1 + \cos x + x \sin x}{(1 + \cos x)^2}$

9. $\frac{d}{dx}\frac{\cot x}{1 + \cot x} = \frac{(1 + \cot x)\frac{d}{dx}(\cot x) - (\cot x)\frac{d}{dx}(1 + \cot x)}{(1 + \cot x)^2}$
$= \frac{(1 + \cot x)(-\csc^2 x) - (\cot x)(-\csc^2 x)}{(1 + \cot x)^2}$
$= -\frac{\csc^2 x}{(1 + \cot x)^2} = -\frac{\csc^2 x \sin^2 x}{(1 + \cot x)^2 \sin^2 x} = -\frac{1}{(\sin x + \cos x)^2}$

10. $\frac{d}{dx}\frac{\cos x}{1 + \sin x} = \frac{(1 + \sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(1 + \sin x)}{(1 + \sin x)^2}$
$= \frac{(1 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(1 + \sin x)^2}$
$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2}$
$= \frac{-(1 + \sin x)}{(1 + \sin x)^2}$
$= -\frac{1}{1 + \sin x}$

11. $y'(x) = \frac{d}{dx}(\sin x + 3) = \cos x$
$y'(\pi) = \cos \pi = -1$
The tangent line has slope -1 and passes through $(\pi, \sin \pi + 3) = (\pi, 3)$.
Its equation is $y = -1(x - \pi) + 3$, or $y = -x + \pi + 3$.