CHAPTER 3
Applications of Differentiation

Section 3.1 Extrema on an Interval

1. $f(x) = \dfrac{x^2}{x^2 + 4}$

$f'(x) = \dfrac{(x^2 + 4)(2x) - (x^2)(2x)}{(x^2 + 4)^2} = \dfrac{8x}{(x^2 + 4)^2}$

$f'(0) = 0$

2. $f(x) = \cos\dfrac{\pi x}{2}$

$f'(x) = -\dfrac{\pi}{2}\sin\dfrac{\pi x}{2}$

$f'(0) = 0$

$f'(2) = 0$

3. $f(x) = x + \dfrac{4}{x^2} = x + 4x^{-2}$

$f'(x) = 1 - 8x^{-3} = 1 - \dfrac{8}{x^3}$

$f'(2) = 0$

4. $f(x) = -3x\sqrt{x + 1}$

$f'(x) = -3x\left[\frac{1}{2}(x + 1)^{-1/2}\right] + \sqrt{x + 1}(-3)$

$= -\frac{3}{2}(x + 1)^{-1/2}\left[x + 2(x + 1)\right]$

$= -\frac{3}{2}(x + 1)^{-1/2}(3x + 2)$

$f'\left(-\frac{2}{3}\right) = 0$

5. $f(x) = (x + 2)^{2/3}$

$f'(x) = \frac{2}{3}(x + 2)^{-1/3}$

$f'(-2)$ is undefined.

6. Using the limit definition of the derivative,

$\displaystyle\lim_{x\to 0^-}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-}\frac{(4 - |x|) - 4}{x} = 1$

$\displaystyle\lim_{x\to 0^+}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+}\frac{(4 - |x|) - 4}{x - 0} = -1$

$f'(0)$ does not exist, because the one-sided derivatives are not equal.

7. Critical number: $x = 2$

$x = 2$: absolute maximum (and relative maximum)

8. Critical number: $x = 0$

$x = 0$: neither

9. Critical numbers: $x = 1, 2, 3$

$x = 1, 3$: absolute maxima (and relative maxima)

$x = 2$: absolute minimum (and relative minimum)

10. Critical numbers: $x = 2, 5$

$x = 2$: neither

$x = 5$: absolute maximum (and relative maximum)

11. $f(x) = x^3 - 3x^2$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

Critical numbers: $x = 0, 2$

12. $g(x) = x^4 - 4x^2$

$g'(x) = 4x^3 - 8x = 4x(x^2 - 2)$

Critical numbers: $x = 0, \pm\sqrt{2}$

13. $g(t) = t\sqrt{4 - t},\ t < 3$

$g'(t) = t\left[\dfrac{1}{2}(4 - t)^{-1/2}(-1)\right] + (4 - t)^{1/2}$

$= \dfrac{1}{2}(4 - t)^{-1/2}\left[-t + 2(4 - t)\right]$

$= \dfrac{8 - 3t}{2\sqrt{4 - t}}$

Critical number: $t = \dfrac{8}{3}$

14. $f(x) = \dfrac{4x}{x^2 + 1}$

$f'(x) = \dfrac{(x^2 + 1)(4) - (4x)(2x)}{(x^2 + 1)^2} = \dfrac{4(1 - x^2)}{(x^2 + 1)^2}$

Critical numbers: $x = \pm 1$

15. $h(x) = \sin^2 x + \cos x,\quad 0 < x < 2\pi$

$h'(x) = 2\sin x\cos x - \sin x = \sin x(2\cos x - 1)$

Critical numbers in $(0, 2\pi)$: $x = \dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$

16. $f(\theta) = 2\sec\theta + \tan\theta,\quad 0 < \theta < 2\pi$

$$f'(\theta) = 2\sec\theta\tan\theta + \sec^2\theta$$
$$= \sec\theta(2\tan\theta + \sec\theta)$$
$$= \sec\theta\left[2\left(\frac{\sin\theta}{\cos\theta}\right) + \frac{1}{\cos\theta}\right]$$
$$= \sec^2\theta(2\sin\theta + 1)$$

Critical numbers in $(0, 2\pi)$: $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

17. $f(x) = 3 - x,\quad [-1, 2]$

$f'(x) = -1 \Rightarrow$ no critical numbers

Left endpoint: $(-1, 4)$ Maximum

Right endpoint: $(2, 1)$ Minimum

18. $f(x) = \frac{2x + 5}{3},\quad [0, 5]$

$f'(x) = \frac{2}{3} \Rightarrow$ No critical numbers

Left endpoint: $\left(0, \frac{5}{3}\right)$ Minimum

Right endpoint: $(5, 5)$ Maximum

19. $g(x) = x^2 - 2x,\quad [0, 4]$

$g'(x) = 2x - 2 = 2(x - 1)$

Critical number: $x = 1$

Left endpoint: $(0, 0)$

Critical number: $(1, -1)$ Minimum

Right endpoint: $(4, 8)$ Maximum

20. $h(x) = -x^2 + 3x - 5,\quad [-2, 1]$

$h'(x) = -2x + 3$

Critical number: $x = \frac{3}{2}$ (not in interval)

Left endpoint: $(-2, -15)$ Minimum

Right endpoint: $(1, -3)$ Maximum

21. $f(x) = x^3 - \frac{3}{2}x^2,\quad [-1, 2]$

$f'(x) = 3x^2 - 3x = 3x(x - 1)$

Left endpoint: $\left(-1, -\frac{5}{2}\right)$ Minimum

Right endpoint: $(2, 2)$ Maximum

Critical number: $(0, 0)$

Critical number: $\left(1, -\frac{1}{2}\right)$

22. $f(x) = x^3 - 12x,\quad [0, 4]$

$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$

Left endpoint: $(0, 0)$

Critical number: $(2, -16)$ Minimum

Right endpoint: $(4, 16)$ Maximum

Note: $x = -2$ is not in the interval.

23. $f(x) = 3x^{2/3} - 2x,\quad [-1, 1]$

$$f'(x) = 2x^{-1/3} - 2 = \frac{2\left(1 - \sqrt[3]{x}\right)}{\sqrt[3]{x}}$$

Left endpoint: $(-1, 5)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $(1, 1)$

24. $g(x) = \sqrt[3]{x},\quad [-1, 1]$

$$g'(x) = \frac{1}{3x^{2/3}}$$

Left endpoint: $(-1, -1)$ Minimum

Critical number: $(0, 0)$

Right endpoint: $(1, 1)$ Maximum

25. $g(t) = \frac{t^2}{t^2 + 3},\quad [-1, 1]$

$$g'(t) = \frac{6t}{\left(t^2 + 3\right)^2}$$

Left endpoint: $\left(-1, \frac{1}{4}\right)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $\left(1, \frac{1}{4}\right)$ Maximum

26. $f(x) = \frac{2x}{x^2 + 1},\quad [-2, 2]$

$$f'(x) = \frac{\left(x^2 + 1\right)2 - 2x(2x)}{\left(x^2 + 1\right)^2} = \frac{2 - 2x^2}{\left(x^2 + 1\right)^2}$$

$$f'(x) = \frac{2\left(1 - x^2\right)}{\left(x^2 + 1\right)^2}$$

Left endpoint: $\left(-2, -\frac{4}{5}\right)$

Critical number: $(-1, -1)$ Minimum

Critical number: $(1, 1)$ Maximum

Right endpoint: $\left(2, \frac{4}{5}\right)$

27. $h(s) = \frac{1}{s-2}, \quad [0, 1]$

$h'(s) = \frac{-1}{(s-2)^2}$

Left endpoint: $\left(0, -\frac{1}{2}\right)$ Maximum

Right endpoint: $(1, -1)$ Minimum

28. $h(t) = \frac{t}{t-2}, \quad [3, 5]$

$h'(t) = \frac{-2}{(t-2)^2}$

Left endpoint: $(3, 3)$ Maximum

Right endpoint: $\left(5, \frac{5}{3}\right)$ Minimum

29. $y = 3 - |t - 3|, \quad [-1, 5]$

For $x < 3$, $y = 3 + (t - 3) = t$

and $y' = 1 \neq 0$ on $[-1, 3)$

For $x > 3$, $y = 3 - (t - 3) = 6 - t$

and $y' = -1 \neq 0$ on $(3, 5]$

So, $x = 3$ is the only critical number.

Left endpoint: $(-1, -1)$ Minimum

Right endpoint: $(5, 1)$

Critical number: $(3, 3)$ Maximum

30. $g(x) = \frac{1}{1 + |x + 1|}, \quad [-3, 3]$

For $x < -1$, $g(x) = \frac{1}{1 - (x + 1)} = \frac{1}{-x} = -x^{-1}$

and $g'(x) = x^{-2} = \frac{1}{x^2} \neq 0$ on $[-3, -1)$

For $x > -1$, $g(x) = \frac{1}{1 + x + 1} = \frac{1}{2 + x} = (2 + x)^{-1}$

and $g'(x) = \frac{-1}{(2 + x)^2} \neq 0$ on $(-1, 3]$

So, $x = -1$ is the only critical number.

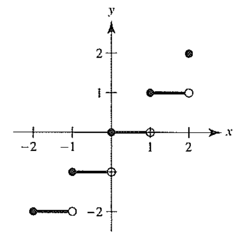
Left endpoint: $\left(-3, \frac{1}{3}\right)$

Critical number: $(-1, 1)$ Maximum

Right endpoint: $\left(3, \frac{1}{5}\right)$ Minimum

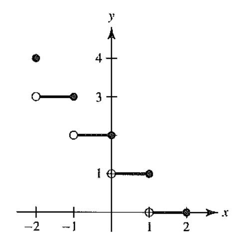
31. $f(x) = [\![x]\!], \quad [-2, 2]$

From the graph of f, you see that the maximum value of f is 2 for $x = 2$, and the minimum value is -2 for $-2 \le x < -1$.



32. $h(x) = [\![2 - x]\!], \quad [-2, 2]$

From the graph you see that the maximum value of h is 4 at $x = -2$, and the minimum value is 0 for $1 < x \le 2$.



33. $f(x) = \cos \pi x, \left[0, \frac{1}{6}\right]$

$f'(x) = -\pi \sin \pi x$

Left endpoint: $(0, 1)$ Maximum

Right endpoint: $\left(\frac{1}{6}, \frac{\sqrt{3}}{2}\right)$ Minimum

34. $g(x) = \sec x, \quad \left[-\frac{\pi}{6}, \frac{\pi}{3}\right]$

$g'(x) = \sec x \tan x$

Left endpoint: $\left(-\frac{\pi}{6}, \frac{2}{\sqrt{3}}\right) \approx \left(-\frac{\pi}{6}, 1.1547\right)$

Right endpoint: $\left(\frac{\pi}{3}, 2\right)$ Maximum

Critical number: $(0, 1)$ Minimum

35. $y = 3 \cos x, \quad [0, 2\pi]$

$y' = -3 \sin x$

Critical number in $(0, 2\pi)$: $x = \pi$

Left endpoint: $(0, 3)$ Maximum

Critical number: $(\pi, -3)$ Minimum

Right endpoint: $(2\pi, 3)$ Maximum

36. $y = \tan\left(\frac{\pi x}{8}\right), \quad [0, 2]$

$y' = \frac{\pi}{8}\sec^2\left(\frac{\pi x}{8}\right) \neq 0$

Left endpoint: $(0, 0)$ Minimum

Right endpoint: $(2, 1)$ Maximum

37. $f(x) = 2x - 3$

(a) Minimum: $(0, -3)$

Maximum: $(2, 1)$

(b) Minimum: $(0, -3)$

(c) Maximum: $(2, 1)$

(d) No extrema

38. $f(x) = 5 - x$

(a) Minimum: $(4, 1)$

Maximum: $(1, 4)$

(b) Maximum: $(1, 4)$

(c) Minimum: $(4, 1)$

(d) No extrema

39. $f(x) = x^2 - 2x$

(a) Minimum: $(1, -1)$

Maximum: $(-1, 3)$

(b) Maximum: $(3, 3)$

(c) Minimum: $(1, -1)$

(d) Minimum: $(1, -1)$

40. $f(x) = \sqrt{4 - x^2}$

(a) Minima: $(-2, 0)$ and $(2, 0)$

Maximum: $(0, 2)$

(b) Minimum: $(-2, 0)$

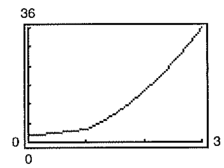
(c) Maximum: $(0, 2)$

(d) Maximum: $(1, \sqrt{3})$

41. $f(x) = \begin{cases} 2x + 2, & 0 \le x \le 1 \\ 4x^2, & 1 < x \le 3 \end{cases}$

Left endpoint: $(0, 2)$ Minimum

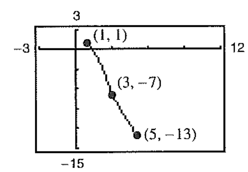
Right endpoint: $(3, 36)$ Maximum



42. $f(x) = \begin{cases} 2 - x^2, & 1 \le x < 3 \\ 2 - 3x, & 3 \le x \le 5 \end{cases}$

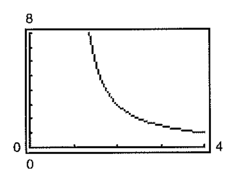
Left endpoint: $(1, 1)$ Maximum

Right endpoint: $(5, -13)$ Minimum



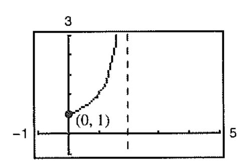
43. $f(x) = \frac{3}{x - 1}, \quad (1, 4]$

Right endpoint: $(4, 1)$ Minimum

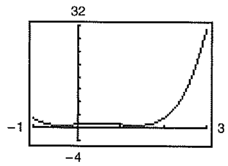


44. $f(x) = \frac{2}{2 - x}, \quad [0, 2)$

Left endpoint: $(0, 1)$ Minimum



45. $f(x) = x^4 - 2x^3 + x + 1, \quad [-1, 3]$



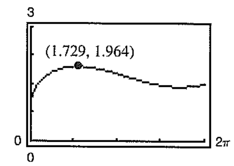
$f'(x) = 4x^3 - 6x^2 + 1 = (2x - 1)(2x^2 - 2x - 1) = 0$

$x = \frac{1}{2}, \frac{1 \pm \sqrt{3}}{2} \approx 0.5, -0.366, 1.366$

Right endpoint: $(3, 31)$ Maximum

Critical points: $\left(\frac{1 \pm \sqrt{3}}{2}, \frac{3}{4}\right)$ Minima

46. $f(x) = \sqrt{x} + \cos\frac{x}{2}, \quad [0, 2\pi]$

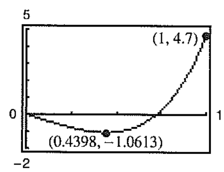


$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2}\sin\frac{x}{2}$

Left endpoint: $(0, 1)$ Minimum

Graphing utility: $(1.729, 1.964)$ Maximum

47. (a)



Minimum: $(0.4398, -1.0613)$

(b) $f(x) = 3.2x^5 + 5x^3 - 3.5x, \quad [0, 1]$

$f'(x) = 16x^4 + 15x^2 - 3.5$

$16x^4 + 15x^2 - 3.5 = 0$

$$x^2 = \frac{-15 \pm \sqrt{(15)^2 - 4(16)(-3.5)}}{2(16)} = \frac{-15 \pm \sqrt{449}}{32}$$

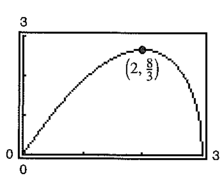
$$x = \sqrt{\frac{-15 + \sqrt{449}}{32}} \approx 0.4398$$

Left endpoint: $(0, 0)$

Critical point: $(0.4398, -1.0613)$ Minimum

Right endpoint: $(1, 4.7)$ Maximum

48. (a)



Maximum: $\left(2, \frac{8}{3}\right)$

(b) $f(x) = \frac{4}{3}x\sqrt{3 - x}, \quad [0, 3]$

$$f'(x) = \frac{4}{3}\left[x\left(\frac{1}{2}\right)(3 - x)^{-1/2}(-1) + (3 - x)^{1/2}(1)\right] = \frac{4}{3}(3 - x)^{-1/2}\left(\frac{1}{2}\right)\left[-x + 2(3 - x)\right] = \frac{2(6 - 3x)}{3\sqrt{3 - x}} = \frac{6(2 - x)}{3\sqrt{3 - x}} = \frac{2(2 - x)}{\sqrt{3 - x}}$$

Left endpoint: $(0, 0)$ Minimum

Critical point: $\left(2, \frac{8}{3}\right)$ Maximum

Right endpoint: $(3, 0)$ Minimum

49. $f(x) = (1 + x^3)^{1/2}, \quad [0, 2]$

$$f'(x) = \frac{3}{2}x^2(1 + x^3)^{-1/2}$$

$$f''(x) = \frac{3}{4}(x^4 + 4x)(1 + x^3)^{-3/2}$$

$$f'''(x) = -\frac{3}{8}(x^6 + 20x^3 - 8)(1 + x^3)^{-5/2}$$

Setting $f''' = 0$, you have $x^6 + 20x^3 - 8 = 0$.

$$x^3 = \frac{-20 \pm \sqrt{400 - 4(1)(-8)}}{2}$$

$$x = \sqrt[3]{-10 \pm \sqrt{108}} = \sqrt{3} - 1$$

In the interval $[0, 2]$, choose

$$x = \sqrt[3]{-10 \pm \sqrt{108}} = \sqrt{3} - 1 \approx 0.732.$$

$\left|f''\left(\sqrt[3]{-10 + \sqrt{108}}\right)\right| \approx 1.47$ is the maximum value.

50. $f(x) = \dfrac{1}{x^2 + 1}, \quad \left[\dfrac{1}{2}, 3\right]$

$$f'(x) = \frac{-2x}{(x^2 + 1)^2}$$

$$f''(x) = \frac{-2(1 - 3x^2)}{(x^2 + 1)^3}$$

$$f'''(x) = \frac{24x - 24x^3}{(x^2 + 1)^4}$$

Setting $f''' = 0$, you have $x = 0, \pm 1$.

$|f''(1)| = \dfrac{1}{2}$ is the maximum value.

51. $f(x) = (x + 1)^{2/3}, \quad [0, 2]$

$$f'(x) = \tfrac{2}{3}(x + 1)^{-1/3}$$

$$f''(x) = -\tfrac{2}{9}(x + 1)^{-4/3}$$

$$f'''(x) = \tfrac{8}{27}(x + 1)^{-7/3}$$

$$f^{(4)}(x) = -\tfrac{56}{81}(x + 1)^{-10/3}$$

$$f^{(5)}(x) = \tfrac{560}{243}(x + 1)^{-13/3}$$

$\left|f^{(4)}(0)\right| = \tfrac{56}{81}$ is the maximum value.

52. $f(x) = \dfrac{1}{x^2 + 1}, \quad [-1, 1]$

$$f'''(x) = \frac{24x - 24x^3}{(x^2 + 1)^4}$$

$$f^{(4)}(x) = \frac{24(5x^4 - 10x^2 + 1)}{(x^2 + 1)^5}$$

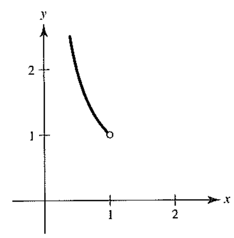
$$f^{(5)}(x) = \frac{-240x(3x^4 - 10x^2 + 3)}{(x^2 + 1)^6}$$

$\left|f^{(4)}(0)\right| = 24$ is the maximum value.

53. Answers will vary. *Sample answer*:

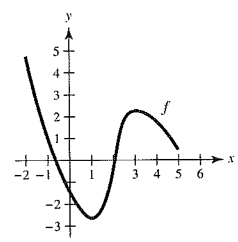
$y = \dfrac{1}{x}$ on the interval $(0, 1)$

There is no maximum or minimum value.

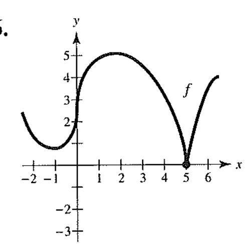


54. A: absolute minimum
B: relative maximum
C: neither
D: relative minimum
E: relative maximum
F: relative minimum
G: neither

55.



56.



57. (a) Yes
(b) No

58. (a) No

(b) Yes

59. (a) No

(b) Yes

60. (a) No

(b) Yes

61. $P = VI - RI^2 = 12I - 0.5I^2, 0 \le I \le 15$

$P = 0$ when $I = 0$.

$P = 67.5$ when $I = 15$.

$P' = 12 - I = 0$

Critical number: $I = 12$ amps

When $I = 12$ amps, $P = 72$, the maximum output.

No, a 20-amp fuse would not increase the power output. P is decreasing for $I > 12$.

62. $x = \dfrac{v^2 \sin 2\theta}{32}, \dfrac{\pi}{4} \le \theta \le \dfrac{3\pi}{4}$

$\dfrac{d\theta}{dt}$ is constant.

$\dfrac{dx}{dt} = \dfrac{dx}{d\theta}\dfrac{d\theta}{dt}$ (by the Chain Rule) $= \dfrac{v^2 \cos 2\theta}{16}\dfrac{d\theta}{dt}$

In the interval $[\pi/4, 3\pi/4]$, $\theta = \pi/4, 3\pi/4$ indicate minimums for dx/dt and $\theta = \pi/2$ indicates a maximum for dx/dt. This implies that the sprinkler waters longest when $\theta = \pi/4$ and $3\pi/4$. So, the lawn farthest from the sprinkler gets the most water.

63. $$S = 6hs + \frac{3s^2}{2}\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right), \frac{\pi}{6} \le \theta \le \frac{\pi}{2}$$

$$\frac{dS}{d\theta} = \frac{3s^2}{2}\left(-\sqrt{3}\csc\theta\cot\theta + \csc^2\theta\right)$$

$$= \frac{3s^2}{2}\csc\theta\left(-\sqrt{3}\cot\theta + \csc\theta\right) = 0$$

$$\csc\theta = \sqrt{3}\cot\theta$$

$$\sec\theta = \sqrt{3}$$

$$\theta = \operatorname{arcsec}\sqrt{3} \approx 0.9553 \text{ radians}$$

$$S\left(\frac{\pi}{6}\right) = 6hs + \frac{3s^2}{2}\left(\sqrt{3}\right)$$

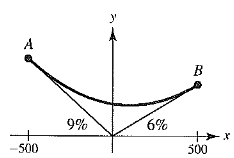
$$S\left(\frac{\pi}{6}\right) = 6hs + \frac{3s^2}{2}\left(\sqrt{3}\right)$$

$$S\left(\operatorname{arcsec}\sqrt{3}\right) = 6hs + \frac{3s^2}{2}\left(\sqrt{2}\right)$$

S is minimum when $\theta = \operatorname{arcsec}\sqrt{3} \approx 0.9553$ radian.

64. (a) Because the grade at A is 9%, $A(-500, 45)$

The grade at B is 6%, $B(500, 30)$.



(b) $y = ax^2 + bx + c$

$y' = 2ax + b$

At A: $2a(-500) + b = -0.09$

At B: $2a(500) + b = 0.06$

Solving these two equations, you obtain

$a = \dfrac{3}{40{,}000}$ and $b = -\dfrac{3}{200}$.

Using the points $A(-500, 45)$ and $B(500, 30)$, you obtain

$$45 = \frac{3}{40{,}000}(-500)^2 + \left(-\frac{3}{200}\right)(-500) + C$$

$$30 = \frac{3}{40{,}000}(500)^2 + \left(-\frac{3}{200}\right)(500) + C.$$

In both cases, $C = 18.75 = \dfrac{75}{4}$. So, $y = \dfrac{3}{40{,}000}x^2 - \dfrac{3}{200}x + \dfrac{75}{4}$

(c)

x	−500	−400	−300	−200	−100	0	100	200	300	400	500
d	0	0.75	3	6.75	12	18.75	12	6.75	3	0.75	0

For $-500 \le x \le 0$, $d = (ax^2 + bx + c) - (-0.09x)$.

For $0 \le x \le 500$, $d = (ax^2 + bx + c) - (0.06x)$.

(d) $y' = \frac{3}{20{,}000}x - \frac{3}{200} = 0$

$$x = \frac{3}{200} \cdot \frac{20{,}000}{3} = 100$$

The lowest point on the highway is $(100, 18)$, which is not directly over the origin.

65. True. See Exercise 25.

66. True. This is stated in the Extreme Value Theorem.

67. True

68. False. Let $f(x) = x^2$. $x = 0$ is a critical number of f.

$g(x) = f(x - k) = (x - k)^2$

$x = k$ is a critical number of g.

69. If f has a maximum value at $x = c$, then $f(c) \ge f(x)$ for all x in I. So, $-f(c) \le -f(x)$ for all x in I. So, $-f$ has a minimum value at $x = c$.

70. $f(x) = ax^3 + bx^2 + cx + d, \quad a \ne 0$

$f'(x) = 3ax^2 + 2bx + c$

The quadratic polynomial can have zero, one, or two zeros.

$$x = \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$$

Zero critical numbers: $b^2 < 3ac$.

Example:
$(a = b = c = 1, d = 0)\, f(x) = x^3 + x^2 + x$ has no critical numbers.

One critical number: $b^2 = 3ac$.

Example: $(a = 1, b = c = d = 0)\, f(x) = x^3$ has one critical number, $x = 0$.

Two critical numbers: $b^2 > 3ac$.

Example:
$(a = c = 1, b = 2, d = 0)\, f(x) = x^3 + 2x^2 + x$ has two critical numbers: $x = -1, -\frac{1}{3}$.

71. First do an example: Let $a = 4$ and $f(x) = 4$. Then R is the square $0 \le x \le 4, 0 \le y \le 4$. Its area and perimeter are both $k = 16$.

Claim that all real numbers $a > 2$ work. On the one hand, if $a > 2$ is given, then let $f(x) = 2a/(a - 2)$. Then the rectangle

$$R = \left\{(x, y): 0 \le x \le a, 0 \le y \le \frac{2a}{a-2}\right\}$$

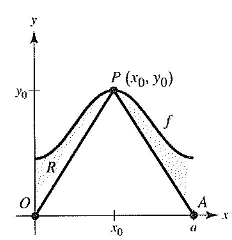
has $k = \frac{2a^2}{a-2}$:

$$\text{Area} = a\left(\frac{2a}{a-2}\right) = \frac{2a^2}{a-2}$$

$$\text{Perimeter} = 2a + 2\left(\frac{2a}{a-2}\right) = \frac{2a(a-2) + 2(2a)}{a-2} = \frac{2a^2}{a-2}.$$

To see that a must be greater than 2, consider $R = \{(x, y): 0 \le x \le a, 0 \le y \le f(x)\}$.

f attains its maximum value on $[0, a]$ at some point $P(x_0, y_0)$, as indicated in the figure.



Draw segments $\overline{OP}$ and $\overline{PA}$. The region R is bounded by the rectangle $0 \le x \le a, 0 \le y \le y_0$, so area$(R) = k \le ay_0$. Furthermore, from the figure, $y_0 < \overline{OP}$ and $y_0 < \overline{PA}$. So, $k = \text{Perimeter}(R) > \overline{OP} + \overline{PA} > 2y_0$. Combining, $2y_0 < k \le ay_0 \Rightarrow a > 2$.

Section 3.2 Rolle's Theorem and the Mean Value Theorem

1. $f(x) = \left|\frac{1}{x}\right|$

$f(-1) = f(1) = 1$. But, f is not continuous on $[-1, 1]$.

2. Rolle's Theorem does not apply to $f(x) = \cot(x/2)$ over $[\pi, 3\pi]$ because f is not continuous at $x = 2\pi$.

3. Rolle's Theorem does not apply to $f(x) = 1 - |x - 1|$ over $[0, 2]$ because f is not differentiable at $x = 1$.

4. $f(x) = \sqrt{(2 - x^{2/3})^3}$

$f(-1) = f(1) = 1$

$f'(x) = \dfrac{-\sqrt{(2 - x^{2/3})}}{x^{1/3}}$

f is not differentiable at $x = 0$.

5. $f(x) = x^2 - x - 2 = (x - 2)(x + 1)$

x-intercepts: $(-1, 0), (2, 0)$

$f'(x) = 2x - 1 = 0$ at $x = \frac{1}{2}$.

6. $f(x) = x(x - 3)$

x-intercepts: $(0, 0), (3, 0)$

$f'(x) = 2x - 3 = 0$ at $x = \frac{3}{2}$.

7. $f(x) = x\sqrt{x + 4}$

x-intercepts: $(-4, 0), (0, 0)$

$$f'(x) = x\frac{1}{2}(x + 4)^{-1/2} + (x + 4)^{1/2}$$

$$= (x + 4)^{-1/2}\left(\frac{x}{2} + (x + 4)\right)$$

$$f'(x) = \left(\frac{3}{2}x + 4\right)(x + 4)^{-1/2} = 0 \text{ at } x = -\frac{8}{3}$$

8. $f(x) = -3x\sqrt{x + 1}$

x-intercepts: $(-1, 0), (0, 0)$

$$f'(x) = -3x\frac{1}{2}(x + 1)^{-1/2} - 3(x + 1)^{1/2}$$

$$= -3(x + 1)^{-1/2}\left(\frac{x}{2} + (x + 1)\right)$$

$$f'(x) = -3(x + 1)^{-1/2}\left(\frac{3}{2}x + 1\right) = 0 \text{ at } x = -\frac{2}{3}$$

9. $f(x) = x^2 + 2x - 3 = (x + 3)(x - 1)$

$f(-3) = f(1) = 0$

$f'(x) = 2x + 2 = 0$ at $x = -1$

$c = -1$ and $f'(-1) = 0$.

10. $f(x) = \sin 2x$

$f(0) = f\left(\frac{\pi}{2}\right) = 0$

$f'(x) = 2\cos 2x = 0$ at $x = \frac{\pi}{4}$

$c = \frac{\pi}{4}$ and $f'\left(\frac{\pi}{4}\right) = 0$

11. $f(x) = -x^2 + 3x$, $[0, 3]$

$f(0) = f(3) = 0$

f is continuous on $[0, 3]$ and differentiable on $(0, 3)$. Rolle's Theorem applies.

$f'(x) = -2x + 3$

$-2x + 3 = 0 \Rightarrow x = \frac{3}{2}$

c-value: $\frac{3}{2}$

12. $f(x) = x^2 - 5x + 4, [1, 4]$

$f(1) = f(4) = 0$

f is continuous on $[1, 4]$. f is differentiable on $(1, 4)$. Rolle's Theorem applies.

$f'(x) = 2x - 5$

$2x - 5 = 0 \Rightarrow x = \frac{5}{2}$

c-value: $\frac{5}{2}$

13. $f(x) = (x - 1)(x - 2)(x - 3), [1, 3]$

$f(1) = f(3) = 0$

f is continuous on $[1, 3]$. f is differentiable on $(1, 3)$. Rolle's Theorem applies.

$$f(x) = x^3 - 6x^2 + 11x - 6$$
$$f'(x) = 3x^2 - 12x + 11$$

$3x^2 - 12x + 11 = 0 \Rightarrow x = \dfrac{6 \pm \sqrt{3}}{3}$

c-values: $\dfrac{6 - \sqrt{3}}{3}, \dfrac{6 + \sqrt{3}}{3}$

14. $f(x) = (x - 3)(x + 1)^2, [-1, 3]$

$f(-1) = f(3) = 0$

f is continuous on $[-1, 3]$. f is differentiable on $(-1, 3)$. Rolle's Theorem applies.

$$f'(x) = (x - 3)(2)(x + 1) + (x + 1)^2$$
$$= (x + 1)[2x - 6 + x + 1]$$
$$= (x + 1)(3x - 5)$$

c-values: $\frac{5}{3}$

15. $f(x) = x^{2/3} - 1, [-8, 8]$

$f(-8) = f(8) = 3$

f is continuous on $[-8, 8]$. f is not differentiable on $(-8, 8)$ because $f'(0)$ does not exist. Rolle's Theorem does not apply.

16. $f(x) = 3 - |x - 3|, [0, 6]$

$f(0) = f(6) = 0$

f is continuous on $[0, 6]$. f is not differentiable on $(0, 6)$ because $f'(3)$ does not exist. Rolle's Theorem does not apply.

17. $f(x) = \dfrac{x^2 - 2x - 3}{x + 2}, [-1, 3]$

$f(-1) = f(3) = 0$

f is continuous on $[-1, 3]$. (**Note:** the discontinuity, $x = -2$, is not in the interval.) f is differentiable on $(-1, 3)$. Rolle's Theorem applies.

$$f'(x) = \frac{(x + 2)(2x - 2) - (x^2 - 2x - 3)(1)}{(x + 2)^2} = 0$$
$$\frac{x^2 + 4x - 1}{(x + 2)^2} = 0$$
$$x = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$$

c-value: $-2 + \sqrt{5}$

18. $f(x) = \dfrac{x^2 - 1}{x}, [-1, 1]$

$f(-1) = f(1) = 0$

f is not continuous on $[-1, 1]$ because $f(0)$ does not exist. Rolle's Theorem does not apply.

19. $f(x) = \sin x, [0, 2\pi]$

$f(0) = f(2\pi) = 0$

f is continuous on $[0, 2\pi]$. f is differentiable on $(0, 2\pi)$. Rolle's Theorem applies.

$f'(x) = \cos x$

c-values: $\dfrac{\pi}{2}, \dfrac{3\pi}{2}$

20. $f(x) = \cos x, [0, 2\pi]$

$f(0) = f(2\pi) = 1$

f is continuous on $[0, 2\pi]$. f is differentiable on $(0, 2\pi)$. Rolle's Theorem applies.

$f'(x) = -\sin x$

c-value: π

21. $f(x) = \dfrac{6x}{\pi} - 4\sin^2 x, \left[0, \dfrac{\pi}{6}\right]$

$f(0) = f\left(\dfrac{\pi}{6}\right) = 0$

f is continuous on $[0, \pi/6]$. f is differentiable on $(0, \pi/6)$. Rolle's Theorem applies.

$$f'(x) = \frac{6}{\pi} - 8\sin x\cos x = 0$$
$$\frac{6}{\pi} = 8\sin x\cos x$$
$$\frac{6}{\pi} = 4\sin 2x$$
$$\frac{3}{2\pi} = \sin 2x$$
$$\frac{1}{2}\arcsin\left(\frac{3}{2\pi}\right) = x$$
$$x \approx 0.2489$$

c-value: 0.2489

22. $f(x) = \cos 2x, [-\pi, \pi]$

$f(-\pi) = f(\pi) = 1$

f is continuous on $[-\pi, \pi]$ and differentiable on $(-\pi, \pi)$. Rolle's Theorem applies.

$$f'(x) = -2\sin 2x$$
$$-2\sin 2x = 0$$
$$\sin 2x = 0$$
$$x = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$$

c-values: $-\frac{\pi}{2}, 0, \frac{\pi}{2}$

23. $f(x) = \tan x, [0, \pi]$

$f(0) = f(\pi) = 0$

f is not continuous on $[0, \pi]$ because $f(\pi/2)$ does not exist. Rolle's Theorem does not apply.

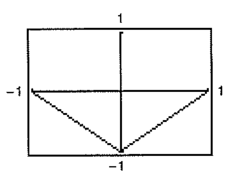
24. $f(x) = \sec x, [\pi, 2\pi]$

f is not continuous on $[\pi, 2\pi]$ because $f(3\pi/2) = \sec(3\pi/2)$ does not exist. Rolle's Theorem does not apply.

25. $f(x) = |x| - 1, [-1, 1]$

$f(-1) = f(1) = 0$

f is continuous on $[-1, 1]$. f is not differentiable on $(-1, 1)$ because $f'(0)$ does not exist. Rolle's Theorem does not apply.



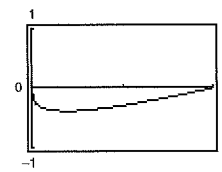
26. $f(x) = x - x^{-1/3}, [0, 1]$

$f(0) = f(1) = 0$

f is continuous on $[0, 1]$. f is differentiable on $(0, 1)$. (**Note:** f is not differentiable at $x = 0$.) Rolle's Theorem applies.

$$f'(x) = 1 - \frac{1}{3\sqrt[3]{x^2}} = 0$$
$$1 = \frac{1}{3\sqrt[3]{x^2}}$$
$$\sqrt[3]{x^2} = \frac{1}{3}$$
$$x^2 = \frac{1}{27}$$
$$x = \sqrt{\frac{1}{27}} = \frac{\sqrt{3}}{9}$$

c-value: $\frac{\sqrt{3}}{9} \approx 0.1925$

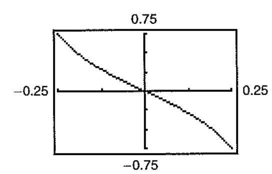


27. $f(x) = x - \tan \pi x, \left[-\frac{1}{4}, \frac{1}{4}\right]$

$f\left(-\frac{1}{4}\right) = -\frac{1}{4} + 1 = \frac{3}{4}$

$f\left(\frac{1}{4}\right) = \frac{1}{4} - 1 = -\frac{3}{4}$

Rolle's Theorem does not apply



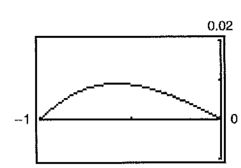
28. $f(x) = \frac{x}{2} - \sin\frac{\pi x}{6}, [-1, 0]$

$f(-1) = f(0) = 0$

f is continuous on $[-1, 0]$. f is differentiable on $(-1, 0)$. Rolle's Theorem applies.

$$f'(x) = \frac{1}{2} - \frac{\pi}{6}\cos\frac{\pi x}{6} = 0$$
$$\cos\frac{\pi x}{6} = \frac{3}{\pi}$$
$$x = -\frac{6}{\pi}\arccos\frac{3}{\pi} \quad \text{[Value needed in } (-1, 0).]$$
$$\approx -0.5756 \text{ radian}$$

c-value: -0.5756



29. $f(t) = -16t^2 + 48t + 6$

(a) $f(1) = f(2) = 38$

(b) $v = f'(t)$ must be 0 at some time in $(1, 2)$.

$$f'(t) = -32t + 48 = 0$$
$$t = \frac{3}{2} \text{ sec}$$

30. $C(x) = 10\left(\frac{1}{x} + \frac{x}{x + 3}\right)$

(a) $C(3) = C(6) = \frac{25}{3}$

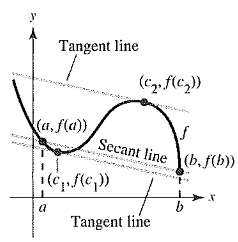
(b) $C'(x) = 10\left(-\frac{1}{x^2} + \frac{3}{(x + 3)^2}\right) = 0$

$$\frac{3}{x^2 + 6x + 9} = \frac{1}{x^2}$$
$$2x^2 - 6x - 9 = 0$$
$$x = \frac{6 \pm \sqrt{108}}{4} = \frac{6 \pm 6\sqrt{3}}{4} = \frac{3 \pm 3\sqrt{3}}{2}$$

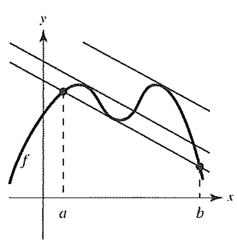
In the interval

$(3, 6)$: $c = \frac{3 + 3\sqrt{3}}{2} \approx 4.098 \approx 410$ components

31.



32.



33. f is not continuous on the interval $[0, 6]$. (f is not continuous at $x = 2$.)

34. f is not differentiable at $x = 2$. The graph of f is not smooth at $x = 2$.

35. $f(x) = \frac{1}{x - 3}$, $[0, 6]$

f has a discontinuity at $x = 3$.

36. $f(x) = |x - 3|$, $[0, 6]$

f is not differentiable at $x = 3$.

37. $f(x) = -x^2 + 5$

(a) Slope $= \frac{1 - 4}{2 + 1} = -1$

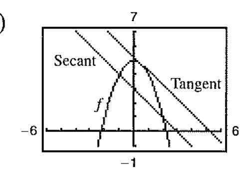
Secant line: $y - 4 = -(x + 1)$
$$y = -x + 3$$
$$x + y - 3 = 0$$

(b) $f'(x) = -2x = -1 \Rightarrow x = c = \frac{1}{2}$

(c) $f(c) = f\left(\frac{1}{2}\right) = -\frac{1}{4} + 5 = \frac{19}{4}$

Tangent line: $y - \frac{19}{4} = -\left(x - \frac{1}{2}\right)$
$$4y - 19 = -4x + 2$$
$$4x + 4y - 21 = 0$$

(d)



38. $f(x) = x^2 - x - 12$

(a) Slope $= \frac{-6 - 0}{-2 - 4} = 1$

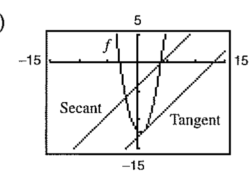
Secant line: $y - 0 = x - 4$
$$x - y - 4 = 0$$

(b) $f'(x) = 2x - 1 = 1 \Rightarrow x = c = 1$

(c) $f(c) = f(1) = -12$

Tangent line: $y + 12 = x - 1$
$$x - y - 13 = 0$$

(d)



39. $f(x) = x^2$ is continuous on $[-2, 1]$ and differentiable on $(-2, 1)$.

$$\frac{f(1) - f(-2)}{1 - (-2)} = \frac{1 - 4}{3} = -1$$

$$f'(x) = 2x = -1$$

$$x = -\frac{1}{2}$$

$$c = -\frac{1}{2}$$

40. $f(x) = x^3$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

$$\frac{f(1) - f(0)}{1 - 0} = \frac{1 - 0}{1} = 1$$

$$f'(x) = 3x^2 = 1$$

$$x = \pm\frac{\sqrt{3}}{3}$$

In the interval $(0, 1)$: $c = \frac{\sqrt{3}}{3}$

41. $f(x) = x^3 + 2x$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$.

$$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{3 - (-3)}{2} = 3$$

$$f'(x) = 3x^2 + 2 = 3$$

$$3x^2 = 1$$

$$x = \pm\frac{1}{\sqrt{3}}$$

$$c = \pm\frac{\sqrt{3}}{3}$$

42. $f(x) = x^4 - 8x$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$.

$$\frac{f(2) - f(0)}{2 - 0} = \frac{0 - 0}{2} = 0$$

$$f'(x) = 4x^3 - 8 = 4(x^3 - 2) = 0$$

$$x^3 = 2$$

$$x = \sqrt[3]{2}$$

$$c = \sqrt[3]{2}$$

43. $f(x) = x^{2/3}$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

$$\frac{f(1) - f(0)}{1 - 0} = 1$$

$$f'(x) = \frac{2}{3}x^{-1/3} = 1$$

$$x = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

$$c = \frac{8}{27}$$

44. $f(x) = \frac{x + 1}{x}$ is not continuous at $x = 0$. The Mean Value Theorem does not apply.

45. $f(x) = |2x + 1|$ is not differentiable at $x = -1/2$. The Mean Value Theorem does not apply.

46. $f(x) = \sqrt{2 - x}$ is continuous on $[-7, 2]$ and differentiable on $(-7, 2)$.

$$\frac{f(2) - f(-7)}{2 - (-7)} = \frac{0 - 3}{9} = -\frac{1}{3}$$

$$f'(x) = \frac{-1}{2\sqrt{2 - x}} = -\frac{1}{3}$$

$$2\sqrt{2 - x} = 3$$

$$\sqrt{2 - x} = \frac{3}{2}$$

$$2 - x = \frac{9}{4}$$

$$x = -\frac{1}{4}$$

$$c = -\frac{1}{4}$$

47. $f(x) = \sin x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$$

$$f'(x) = \cos x = 0$$

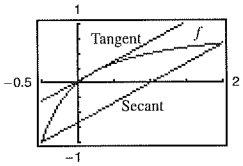
$$x = \pi/2$$

$$c = \frac{\pi}{2}$$

48. $f(x) = \cos x + \tan x$ is not continuous at $x = \pi/2$. The Mean Value Theorem does not apply.

49. $f(x) = \dfrac{x}{x+1}, \left[-\dfrac{1}{2}, 2\right]$

(a)–(c)



(b) Secant line:

$$\text{slope} = \frac{f(2) - f(-1/2)}{2 - (-1/2)} = \frac{2/3 - (-1)}{5/2} = \frac{2}{3}$$

$$y - \frac{2}{3} = \frac{2}{3}(x - 2)$$

$$y = \frac{2}{3}(x - 1)$$

(c) $$f'(x) = \frac{1}{(x+1)^2} = \frac{2}{3}$$

$$(x+1)^2 = \frac{3}{2}$$

$$x = -1 \pm \sqrt{\frac{3}{2}} = -1 \pm \frac{\sqrt{6}}{2}$$

In the interval $[-1/2, 2]$: $c = -1 + \left(\sqrt{6}/2\right)$

$$f(c) = \frac{-1 + \left(\sqrt{6}/2\right)}{\left[-1 + \left(\sqrt{6}/2\right)\right] + 1} = \frac{-2 + \sqrt{6}}{\sqrt{6}} = \frac{-2}{\sqrt{6}} + 1$$

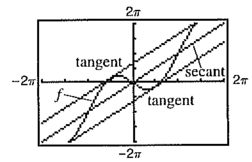
Tangent line: $$y - 1 + \frac{2}{\sqrt{6}} = \frac{2}{3}\left(x - \frac{\sqrt{6}}{2} + 1\right)$$

$$y - 1 + \frac{\sqrt{6}}{3} = \frac{2}{3}x - \frac{\sqrt{6}}{3} + \frac{2}{3}$$

$$y = \frac{1}{3}\left(2x + 5 - 2\sqrt{6}\right)$$

50. $f(x) = x - 2\sin x, [-\pi, \pi]$

(a)–(c)



(b) Secant line:

$$\text{slope} = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)} = \frac{\pi - (-\pi)}{2\pi} = 1$$

$$y - \pi = 1(x - \pi)$$

$$y = x$$

(c) $$f'(x) = 1 - 2\cos x = 1$$

$$\cos x = 0$$

$$x = c = \pm\frac{\pi}{2}$$

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 2$$

$$f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} + 2$$

Tangent lines: $$y - \left(\frac{\pi}{2} - 2\right) = 1\left(x - \frac{\pi}{2}\right)$$

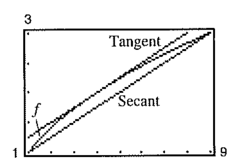
$$y = x - 2$$

$$y - \left(-\frac{\pi}{2} + 2\right) = 1\left(x + \frac{\pi}{2}\right)$$

$$y = x + 2$$

51. $f(x) = \sqrt{x}, [1, 9]$

(a)–(c)



(b) Secant line:

$$\text{slope} = \frac{f(9) - f(1)}{9 - 1} = \frac{3 - 1}{8} = \frac{1}{4}$$

$$y - 1 = \frac{1}{4}(x - 1)$$

$$y = \frac{1}{4}x + \frac{3}{4}$$

(c) $$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{4}$$

$$x = c = 4$$

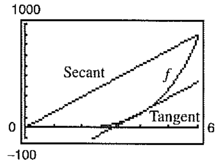
$$f(4) = 2$$

Tangent line: $$y - 2 = \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x + 1$$

52. $f(x) = x^4 - 2x^3 + x^2, [0, 6]$

(a)–(c)



(b) Secant line:

$$\text{slope} = \frac{f(6) - f(0)}{6 - 0} = \frac{900 - 0}{6} = 150$$

$$y - 0 = 150(x - 0)$$

$$y = 150x$$

(c) $f'(x) = 4x^3 - 6x^2 + 2x = 150$

Using a graphing utility, there is one solution in $(0, 6)$, $x = c \approx 3.8721$ and $f(c) \approx 123.6721$

Tangent line: $y - 123.6721 = 150(x - 3.8721)$

$$y = 150x - 457.143$$

53. $s(t) = -4.9t^2 + 300$

(a) $v_{\text{avg}} = \dfrac{s(3) - s(0)}{3 - 0} = \dfrac{255.9 - 300}{3} = -14.7$ m/sec

(b) $s(t)$ is continuous on $[0, 3]$ and differentiable on $(0, 3)$. Therefore, the Mean Value Theorem applies.

$v(t) = s'(t) = -9.8t = -14.7$ m/sec

$t = \dfrac{-14.7}{-9.8} = 1.5$ sec

54. $S(t) = 200\left(5 - \dfrac{9}{2 + t}\right)$

(a) $\dfrac{S(12) - S(0)}{12 - 0} = \dfrac{200[5 - (9/14)] - 200[5 - (9/2)]}{12}$

$= \dfrac{450}{7}$

(b) $S'(t) = 200\left(\dfrac{9}{(2 + t)^2}\right) = \dfrac{450}{7}$

$\dfrac{1}{(2 + t)^2} = \dfrac{1}{28}$

$2 + t = 2\sqrt{7}$

$t = 2\sqrt{7} - 2 \approx 3.2915$ months

$S'(t)$ is equal to the average value in April.

55. No. Let $f(x) = x^2$ on $[-1, 2]$.

$f'(x) = 2x$

$f'(0) = 0$ and zero is in the interval $(-1, 2)$ but $f(-1) \neq f(2)$.

56. $f(a) = f(b)$ and $f'(c) = 0$ where c is in the interval (a, b).

(a) $g(x) = f(x) + k$

$g(a) = g(b) = f(a) + k$

$g'(x) = f'(x) \Rightarrow g'(c) = 0$

Interval: $[a, b]$

Critical number of g: c

(b) $g(x) = f(x - k)$

$g(a + k) = g(b + k) = f(a)$

$g'(x) = f'(x - k)$

$g'(c + k) = f'(c) = 0$

Interval: $[a + k, b + k]$

Critical number of g: $c + k$

(c) $g(x) = f(kx)$

$g\left(\dfrac{a}{k}\right) = g\left(\dfrac{b}{k}\right) = f(a)$

$g'(x) = kf'(kx)$

$g'\left(\dfrac{c}{k}\right) = kf'(c) = 0$

Interval: $\left[\dfrac{a}{k}, \dfrac{b}{k}\right]$

Critical number of g: $\dfrac{c}{k}$

57. $f(x) = \begin{cases} 0, & x = 0 \\ 1 - x, & 0 < x \le 1 \end{cases}$

No, this does not contradict Rolle's Theorem. f is not continuous on $[0, 1]$.

58. No. If such a function existed, then the Mean Value Theorem would say that there exists $c \in (-2, 2)$ such that

$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{6 + 2}{4} = 2.$$

But, $f'(x) < 1$ for all x.

59. Let $S(t)$ be the position function of the plane. If $t = 0$ corresponds to 2 P.M., $S(0) = 0$, $S(5.5) = 2500$ and the Mean Value Theorem says that there exists a time t_0, $0 < t_0 < 5.5$, such that

$$S'(t_0) = v(t_0) = \frac{2500 - 0}{5.5 - 0} \approx 454.54.$$

Applying the Intermediate Value Theorem to the velocity function on the intervals $[0, t_0]$ and $[t_0, 5.5]$, you see that there are at least two times during the flight when the speed was 400 miles per hour. $(0 < 400 < 454.54)$

60. Let $T(t)$ be the temperature of the object. Then $T(0) = 1500°$ and $T(5) = 390°$. The average temperature over the interval $[0, 5]$ is

$$\frac{390 - 1500}{5 - 0} = -222° \text{ F/h}.$$

By the Mean Value Theorem, there exist a time t_0, $0 < t_0 < 5$, such that $T'(t_0) = -222°$F/h.

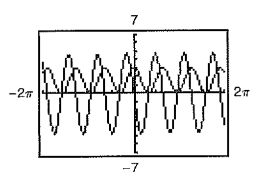
61. Let $S(t)$ be the difference in the positions of the 2 bicyclists, $S(t) = S_1(t) - S_2(t)$. Because $S(0) = S(2.25) = 0$, there must exist a time $t_0 \in (0, 2.25)$ such that $S'(t_0) = v(t_0) = 0$.
At this time, $v_1(t_0) = v_2(t_0)$.

62. Let $t = 0$ correspond to 9:13 A.M. By the Mean Value Theorem, there exists t_0 in $\left(0, \frac{1}{30}\right)$ such that

$$v'(t_0) = a(t_0) = \frac{85 - 35}{1/30} = 1500 \text{ mi/h}^2.$$

63. $f(x) = 3\cos^2\left(\frac{\pi x}{2}\right)$, $f'(x) = 6\cos\left(\frac{\pi x}{2}\right)\left(-\sin\left(\frac{\pi x}{2}\right)\right)\left(\frac{\pi}{2}\right)$

$$= -3\pi\cos\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi x}{2}\right)$$

(a) 

(b) f and f' are both continuous on the entire real line.

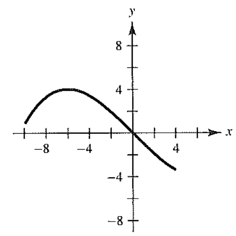
(c) Because $f(-1) = f(1) = 0$, Rolle's Theorem applies on $[-1, 1]$. Because $f(1) = 0$ and $f(2) = 3$, Rolle's Theorem does not apply on $[1, 2]$.

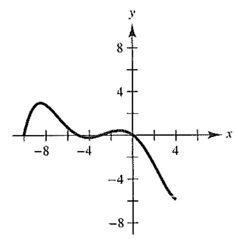
(d) $\lim_{x \to 3^-} f'(x) = 0$

$\lim_{x \to 3^+} f'(x) = 0$

64. (a) f is continuous on $[-10, 4]$ and changes sign, $(f(-8) > 0, f(3) < 0)$. By the Intermediate Value Theorem, there exists at least one value of x in $[-10, 4]$ satisfying $f(x) = 0$.

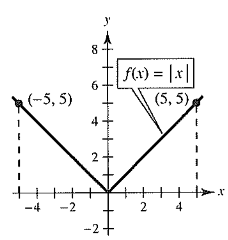
(b) There exist real numbers a and b such that $-10 < a < b < 4$ and $f(a) = f(b) = 2$. Therefore, by Rolle's Theorem there exists at least one number c in $(-10, 4)$ such that $f'(c) = 0$. This is called a critical number.

(c) 

(d) 

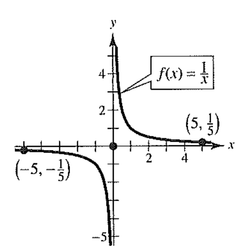
(e) No, f' did not have to be continuous on $[-10, 4]$.

65. f is continuous on $[-5, 5]$ and does not satisfy the conditions of the Mean Value Theorem. $\Rightarrow$ f is not differentiable on $(-5, 5)$. Example: $f(x) = |x|$



66. f is not continuous on $[-5, 5]$.

Example: $f(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$



67. $f(x) = x^5 + x^3 + x + 1$

f is differentiable for all x.

$f(-1) = -2$ and $f(0) = 1$, so the Intermediate Value Theorem implies that f has at least one zero c in $[-1, 0]$, $f(c) = 0$.

Suppose f had 2 zeros, $f(c_1) = f(c_2) = 0$. Then Rolle's Theorem would guarantee the existence of a number a such that

$f'(a) = f(c_2) - f(c_1) = 0.$

But, $f'(x) = 5x^4 + 3x^2 + 1 > 0$ for all x. So, f has exactly one real solution.

68. $f(x) = 2x^5 + 7x - 1$

f is differentiable for all x.

$f(0) = -1$ and $f(1) = 8$, so the Intermediate Value Theorem implies that f has at least one zero c in $[0, 1]$, $f(c) = 0$.

Suppose f had 2 zeros, $f(c_1) = f(c_2) = 0$. Then Rolle's Theorem would guarantee the existence of a number a such that

$f'(a) = f(c_2) - f(c_1) = 0.$

But $f'(x) = 10x^4 + 7 > 0$ for all x. So, $f(x) = 0$ has exactly one real solution.

69. $f(x) = 3x + 1 - \sin x$

f is differentiable for all x.

$f(-\pi) = -3\pi + 1 < 0$ and $f(0) = 1 > 0$, so the Intermediate Value Theorem implies that f has at least one zero c in $[-\pi, 0]$, $f(c) = 0$.

Suppose f had 2 zeros, $f(c_1) = f(c_2) = 0$. Then Rolle's Theorem would guarantee the existence of a number a such that

$f'(a) = f(c_2) - f(c_1) = 0.$

But $f'(x) = 3 - \cos x > 0$ for all x. So, $f(x) = 0$ has exactly one real solution.

70. $f(x) = 2x - 2 - \cos x$

$f(0) = -3$, $f(\pi) = 2\pi - 2 + 1 = 2\pi - 1 > 0$. By the Intermediate Value Theorem, f has at least one zero.

Suppose f had 2 zeros, $f(c_1) = f(c_2) = 0$. Then Rolle's Theorem would guarantee the existence of a number a such that

$f'(a) = f(c_2) - f(c_1) = 0.$

But, $f'(x) = 2 + \sin x \ge 1$ for all x. So, f has exactly one real solution.

71. f continuous at $x = 0$: $1 = b$

f continuous at $x = 1$: $a + 1 = 5 + c$

f differentiable at $x = 1$: $a = 2 + 4 = 6$. So, $c = 2$.

$$f(x) = \begin{cases} 1, & x = 0 \\ 6x + 1, & 0 < x \le 1 \\ x^2 + 4x + 2, & 1 < x \le 3 \end{cases}$$

$$= \begin{cases} 6x + 1, & 0 \le x \le 1 \\ x^2 + 4x + 2, & 1 < x \le 3 \end{cases}$$

72. f continuous at $x = -1$: $a = 2$

f continuous at $x = 0$: $2 = c$

f continuous at $x = 1$: $b + 2 = d + 4 \Rightarrow b = d + 2$

f differentiable at $x = 0$: $0 = 0$

f differentiable at $x = 1$: $2b = d$

So, $b = -2$ and $d = -4$.

73. $f'(x) = 0$

$f(x) = c$

$f(2) = 5$

So, $f(x) = 5$.

74. $f'(x) = 4$

$f(x) = 4x + c$

$f(0) = 1 \Rightarrow c = 1$

So, $f(x) = 4x + 1$.

75. $f'(x) = 2x$

$f(x) = x^2 + c$

$f(1) = 0 \Rightarrow 0 = 1 + c \Rightarrow c = -1$

So, $f(x) = x^2 - 1$.

76. $f'(x) = 2x + 3$

$f(x) = x^2 + 3x + c$

$f(1) = 0 \Rightarrow 0 = 1 + 3 + c \Rightarrow c = -4$

So, $f(x) = x^2 + 3x - 4$.

77. False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

78. False. f must also be continuous *and* differentiable on each interval. Let

$$f(x) = \frac{x^3 - 4x}{x^2 - 1}.$$

79. True. A polynomial is continuous and differentiable everywhere.

80. True

81. Suppose that $p(x) = x^{2n+1} + ax + b$ has two real roots x_1 and x_2. Then by Rolle's Theorem, because $p(x_1) = p(x_2) = 0$, there exists c in (x_1, x_2) such that $p'(c) = 0$. But $p'(x) = (2n + 1)x^{2n} + a \neq 0$, because $n > 0, a > 0$. Therefore, $p(x)$ cannot have two real roots.

82. Suppose $f(x)$ is not constant on (a, b). Then there exists x_1 and x_2 in (a, b) such that $f(x_1) \neq f(x_2)$. Then by the Mean Value Theorem, there exists c in (a, b) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0.$$

This contradicts the fact that $f'(x) = 0$ for all x in (a, b).

83. If $p(x) = Ax^2 + Bx + C$, then

$$\begin{aligned} p'(x) = 2Ax + B = \frac{f(b) - f(a)}{b - a} &= \frac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{b - a} \\ &= \frac{A(b^2 - a^2) + B(b - a)}{b - a} \\ &= \frac{(b - a)[A(b + a) + B]}{b - a} \\ &= A(b + a) + B. \end{aligned}$$

So, $2Ax = A(b + a)$ and $x = (b + a)/2$ which is the midpoint of $[a, b]$.

84. (a) $f(x) = x^2, g(x) = -x^3 + x^2 + 3x + 2$

$f(-1) = g(-1) = 1, f(2) = g(2) = 4$

Let $h(x) = f(x) - g(x)$. Then, $h(-1) = h(2) = 0$. So, by Rolle's Theorem these exists $c \in (-1, 2)$ such that

$h'(c) = f'(c) - g'(c) = 0.$

So, at $x = c$, the tangent line to f is parallel to the tangent line to g.

$h(x) = x^3 - 3x - 2, h'(x) = 3x^2 - 3 = 0 \Rightarrow x = c = 1$

(b) Let $h(x) = f(x) - g(x)$. Then $h(a) = h(b) = 0$ by Rolle's Theorem, there exists c in (a, b) such that

$h'(c) = f'(c) - g'(c) = 0.$

So, at $x = c$, the tangent line to f is parallel to the tangent line to g.

85. Suppose $f(x)$ has two fixed points c_1 and c_2. Then, by the Mean Value Theorem, there exists c such that

$$f'(c) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = \frac{c_2 - c_1}{c_2 - c_1} = 1.$$

This contradicts the fact that $f'(x) < 1$ for all x.

86. $f(x) = \frac{1}{2}\cos x$ differentiable on $(-\infty, \infty)$.

$f'(x) = -\frac{1}{2}\sin x$

$-\frac{1}{2} \leq f'(x) \leq \frac{1}{2} \Rightarrow f'(x) < 1$ for all real numbers.

So, from Exercise 62, f has, at most, one fixed point.
$(x \approx 0.4502)$

87. Let $f(x) = \cos x$. f is continuous and differentiable for all real numbers. By the Mean Value Theorem, for any interval $[a, b]$, there exists c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\frac{\cos b - \cos a}{b - a} = -\sin c$$

$$\cos b - \cos a = (-\sin c)(b - a)$$

$$|\cos b - \cos a| = |-\sin c||b - a|$$

$$|\cos b - \cos a| \leq |b - a| \text{ since } |-\sin c| \leq 1.$$

88. Let $f(x) = \sin x. f$ is continuous and differentiable for all real numbers. By the Mean Value Theorem, for any interval $[a, b]$, there exists c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$
$$\sin(b) - \sin(a) = (b - a)\cos(c)$$
$$|\sin(b) - \sin(a)| = |b - a||\cos(c)|$$
$$|\sin a - \sin b| \le |a - b|.$$

89. Let $0 < a < b$. $f(x) = \sqrt{x}$ satisfies the hypotheses of the Mean Value Theorem on $[a, b]$. Hence, there exists c in (a, b) such that

$$f'(c) = \frac{1}{2\sqrt{c}} = \frac{f(b) - f(a)}{b - a} = \frac{\sqrt{b} - \sqrt{a}}{b - a}.$$

So $\sqrt{b} - \sqrt{a} = (b - a)\frac{1}{2\sqrt{c}} < \frac{b - a}{2\sqrt{a}}$.

Section 3.3 Increasing and Decreasing Functions and the First Derivative Test

1. (a) Increasing: $(0, 6)$ and $(8, 9)$. Largest: $(0, 6)$

(b) Decreasing: $(6, 8)$ and $(9, 10)$. Largest: $(6, 8)$

2. (a) Increasing: $(4, 5), (6, 7)$. Largest: $(4, 5), (6, 7)$

(b) Decreasing: $(-3, 1), (1, 4), (5, 6)$. Largest: $(-3, 1)$

3. $f(x) = x^2 - 6x + 8$

From the graph, f is decreasing on $(-\infty, 3)$ and increasing on $(3, \infty)$.

Analytically, $f'(x) = 2x - 6$.

Critical number: $x = 3$

Test Intervals:	$-\infty < x < 3$	$3 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

4. $y = -(x + 1)^2$

From the graph, f is increasing on $(-\infty, -1)$ and decreasing on $(-1, \infty)$.

Analytically, $y' = -2(x + 1)$.

Critical number: $x = -1$

Test Intervals:	$-\infty < x < -1$	$-1 < x < \infty$
Sign of y':	$y' > 0$	$y' < 0$
Conclusion:	Increasing	Decreasing

5. $y = \frac{x^3}{4} - 3x$

From the graph, y is increasing on $(-\infty, -2)$ and $(2, \infty)$, and decreasing on $(-2, 2)$.

Analytically, $y' = \frac{3x^2}{4} - 3 = \frac{3}{4}(x^2 - 4) = \frac{3}{4}(x - 2)(x + 2)$

Critical numbers: $x = \pm 2$

Test Intervals:	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Sign of y':	$y' > 0$	$y' < 0$	$y' > 0$
Conclusion:	Increasing	Decreasing	Increasing

6. $f(x) = x^4 - 2x^2$

From the graph, f is decreasing on $(-\infty, -1)$ and $(0, 1)$, and increasing on $(-1, 0)$ and $(1, \infty)$.

Analytically, $f'(x) = 4x^3 - 4x = 4x(x - 1)(x + 1)$.

Critical numbers: $x = 0, \pm 1$.

Test Intervals:	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Sign of f':	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing	Decreasing	Increasing

7. $f(x) = \dfrac{1}{(x + 1)^2}$

From the graph, f is increasing on $(-\infty, -1)$ and decreasing on $(-1, \infty)$.

Analytically, $f'(x) = \dfrac{-2}{(x + 1)^3}$.

No critical numbers. Discontinuity: $x = -1$

Test Intervals:	$-\infty < x < -1$	$-1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

8. $y = \dfrac{x^2}{2x - 1}$

From the graph, y is increasing on $(-\infty, 0)$ and $(1, \infty)$, and decreasing on $(0, 1/2)$ and $(1/2, 1)$.

Analytically, $y' = \dfrac{(2x - 1)2x - x^2(2)}{(2x - 1)^2} = \dfrac{2x^2 - 2x}{(2x - 1)^2} = \dfrac{2x(x - 1)}{(2x - 1)^2}$

Critical numbers: $x = 0, 1$

Discontinuity: $x = 1/2$

Test Intervals:	$-\infty < x < 0$	$0 < x < 1/2$	$1/2 < x < 1$	$1 < x < \infty$
Sign of y':	$y' > 0$	$y' < 0$	$y' < 0$	$y' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

9. $g(x) = x^2 - 2x - 8$

$g'(x) = 2x - 2$

Critical number: $x = 1$

Test Intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $g'(x)$:	$g' < 0$	$g' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(1, \infty)$

Decreasing on: $(-\infty, 1)$

10. $h(x) = 27x - x^3$

$h'(x) = 27 - 3x^2 = 3(3 - x)(3 + x)$

$h'(x) = 0$

Critical numbers: $x = \pm 3$

Test Intervals:	$-\infty < x < -3$	$-3 < x < 3$	$3 < x < \infty$
Sign of $h'(x)$:	$h' < 0$	$h' > 0$	$h' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $(-3, 3)$

Decreasing on: $(-\infty, -3), (3, \infty)$

11. $y = x\sqrt{16 - x^2}$ Domain: $[-4, 4]$

$$y' = \frac{-2(x^2 - 8)}{\sqrt{16 - x^2}} = \frac{-2}{\sqrt{16 - x^2}}(x - 2\sqrt{2})(x + 2\sqrt{2})$$

Critical numbers: $x = \pm 2\sqrt{2}$

Test intervals:	$-4 < x < -2\sqrt{2}$	$-2\sqrt{2} < x < 2\sqrt{2}$	$2\sqrt{2} < x < 4$
Sign of y':	$y' < 0$	$y' > 0$	$y' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $(-2\sqrt{2}, 2\sqrt{2})$

Decreasing on: $(-4, -2\sqrt{2}), (2\sqrt{2}, 4)$

12. $y = x + \frac{4}{x}$

$$y' = \frac{(x - 2)(x + 2)}{x^2}$$

Critical numbers: $x = \pm 2$ Discontinuity: 0

Test intervals:	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of y':	$y' > 0$	$y' < 0$	$y' < 0$	$y' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on: $(-\infty, -2), (2, \infty)$

Decreasing on: $(-2, 0), (0, 2)$

13. $f(x) = \sin x - 1, \quad 0 < x < 2\pi$

$f'(x) = \cos x$

Critical numbers: $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

Test intervals:	$0 < x < \dfrac{\pi}{2}$	$\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$	$\dfrac{3\pi}{2} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{2}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

14. $h(x) = \cos \dfrac{x}{2}, \quad 0 < x < 2\pi$

$h'(x) = -\dfrac{1}{2} \sin \dfrac{x}{2}$

Critical numbers: none

Test interval:	$0 < x < 2\pi$
Sign of $h'(x)$:	$h' < 0$
Conclusion:	Decreasing

Decreasing on $0 < x < 2\pi$

15. $y = x - 2\cos x, \quad 0 < x < 2\pi$

$y' = 1 + 2\sin x$

$y' = 0: \sin x = -\dfrac{1}{2}$

Critical numbers: $x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

Test intervals:	$0 < x < \dfrac{7\pi}{6}$	$\dfrac{7\pi}{6} < x < \dfrac{11\pi}{6}$	$\dfrac{11\pi}{6} < x < 2\pi$
Sign of y':	$y' > 0$	$y' < 0$	$y' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \dfrac{7\pi}{6}\right), \left(\dfrac{11\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\dfrac{7\pi}{6}, \dfrac{11\pi}{6}\right)$

16. $f(x) = \cos^2 x - \cos x, \qquad 0 < x < 2\pi$

$f'(x) = -2\cos x \sin x + \sin x = \sin x(1 - 2\cos x)$

$\sin x = 0 \Rightarrow x = \pi$

$1 - 2\cos x = 0 \Rightarrow \cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$

Critical numbers: $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

Test intervals:	$0 < x < \frac{\pi}{3}$	$\frac{\pi}{3} < x < \pi$	$\pi < x < \frac{5\pi}{3}$	$\frac{5\pi}{3} < x < 2\pi$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing	Decreasing	Increasing

Increasing on: $\left(\frac{\pi}{3}, \pi\right), \left(\frac{5\pi}{3}, 2\pi\right)$

Decreasing on: $\left(0, \frac{\pi}{3}\right), \left(\pi, \frac{5\pi}{3}\right)$

17. (a) $f(x) = x^2 - 4x$

$f'(x) = 2x - 4$

Critical number: $x = 2$

(b)

Test intervals:	$-\infty < x < 2$	$2 < x < \infty$
Sign of f':	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Decreasing on: $(-\infty, 2)$

Increasing on: $(2, \infty)$

(c) Relative minimum: $(2, -4)$

18. (a) $f(x) = x^2 + 6x + 10$

$f'(x) = 2x + 6$

Critical number: $x = -3$

(b)

Test intervals:	$-\infty < x < -3$	$-3 < x < \infty$
Sign of f':	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Decreasing on: $(-\infty, -3)$

Increasing on: $(-3, \infty)$

(c) Relative minimum: $(-3, 1)$

19. (a) $f(x) = -2x^2 + 4x + 3$

$f'(x) = -4x + 4 = 0$

Critical number: $x = 1$

(b)

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, 1)$

Decreasing on: $(1, \infty)$

(c) Relative maximum: $(1, 5)$

20. (a) $f(x) = -(x^2 + 8x + 12)$

$f'(x) = -2x - 8 = 0$

Critical number: $x = -4$

(b)

Test intervals:	$-\infty < x < -4$	$-4 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, -4)$

Decreasing on: $(-4, \infty)$

(c) Relative maximum: $(-4, 4)$

21. (a) $f(x) = 2x^3 + 3x^2 - 12x$

$f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1) = 0$

Critical numbers: $x = -2, 1$

(b)

Test intervals:	$-\infty < x < -2$	$-2 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -2), (1, \infty)$

Decreasing on: $(-2, 1)$

(c) Relative maximum: $(-2, 20)$

Relative minimum: $(1, -7)$

22. (a) $f(x) = x^3 - 6x^2 + 15$

$f'(x) = 3x^2 - 12x = 3x(x - 4)$

Critical numbers: $x = 0, 4$

(b)

Test intervals:	$-\infty < x < 0$	$0 < x < 4$	$4 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, 0), (4, \infty)$

Decreasing on: $(0, 4)$

(c) Relative maximum: $(0, 15)$

Relative minimum: $(4, -17)$

23. (a) $f(x) = (x - 1)^2(x + 3) = x^3 + x^2 - 5x + 3$

$f'(x) = 3x^2 + 2x - 5 = (x - 1)(3x + 5)$

Critical numbers: $x = 1, -\frac{5}{3}$

(b)

Test intervals:	$-\infty < x < -\frac{5}{3}$	$-5/3 < x < 1$	$1 < x < \infty$
Sign of f':	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $\left(-\infty, -\frac{5}{3}\right)$ and $(1, \infty)$

Decreasing on: $\left(-\frac{5}{3}, 1\right)$

(c) Relative maximum: $\left(-\frac{5}{3}, \frac{256}{27}\right)$

Relative minimum: $(1, 0)$

24. (a) $f(x) = (x + 2)^2(x - 1)$

$f'(x) = 3x(x + 2)$

Critical numbers: $x = -2, 0$

(b)

Test intervals:	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -2), (0, \infty)$

Decreasing on: $(-2, 0)$

(c) Relative maximum: $(-2, 0)$

Relative minimum: $(0, -4)$

25. (a) $f(x) = \dfrac{x^5 - 5x}{5}$

$f'(x) = x^4 - 1$

Critical numbers: $x = -1, 1$

(b)

Test intervals:	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -1), (1, \infty)$

Decreasing on: $(-1, 1)$

(c) Relative maximum: $\left(-1, \dfrac{4}{5}\right)$

Relative minimum: $\left(1, -\dfrac{4}{5}\right)$

26. (a) $f(x) = x^4 - 32x + 4$

$f'(x) = 4x^3 - 32 = 4(x^3 - 8)$

Critical number: $x = 2$

(b)

Test intervals:	$-\infty < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(2, \infty)$

Decreasing on: $(-\infty, 2)$

(c) Relative minimum: $(2, -44)$

27. (a) $f(x) = x^{1/3} + 1$

$f'(x) = \dfrac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$

Critical number: $x = 0$

(b)

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, \infty)$

(c) No relative extrema

28. (a) $f(x) = x^{2/3} - 4$

$f'(x) = \dfrac{2}{3}x^{-1/3} = \dfrac{2}{3x^{1/3}}$

Critical number: $x = 0$

(b)

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(0, \infty)$

Decreasing on: $(-\infty, 0)$

(c) Relative minimum: $(0, -4)$

29. (a) $f(x) = (x + 2)^{2/3}$

$f'(x) = \dfrac{2}{3}(x + 2)^{-1/3} = \dfrac{2}{3(x + 2)^{1/3}}$

Critical number: $x = -2$

(b)

Test intervals:	$-\infty < x < -2$	$-2 < x < \infty$
Sign of f':	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Decreasing on: $(-\infty, -2)$

Increasing on: $(-2, \infty)$

(c) Relative minimum: $(-2, 0)$

30. (a) $f(x) = (x-3)^{1/3}$

$$f'(x) = \frac{1}{3}(x-3)^{-2/3} = \frac{1}{3(x-3)^{2/3}}$$

Critical number: $x = 3$

(b)

Test intervals:	$-\infty < x < 3$	$3 < x < \infty$
Sign of f':	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, \infty)$

(c) No relative extrema

31. (a) $f(x) = 5 - |x - 5|$

$$f'(x) = -\frac{x-5}{|x-5|} = \begin{cases} 1, & x < 5 \\ -1, & x > 5 \end{cases}$$

Critical number: $x = 5$

(b)

Test intervals:	$-\infty < x < 5$	$5 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, 5)$

Decreasing on: $(5, \infty)$

(c) Relative maximum: $(5, 5)$

32. (a) $f(x) = |x + 3| - 1$

$$f'(x) = \frac{x+3}{|x+3|} = \begin{cases} 1, & x > -3 \\ -1, & x < -3 \end{cases}$$

Critical number: $x = -3$

(b)

Test intervals:	$-\infty < x < -3$	$-3 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(-3, \infty)$

Decreasing on: $(-\infty, -3)$

(c) Relative minimum: $(-3, -1)$

33. (a) $f(x) = 2x + \dfrac{1}{x}$

$$f'(x) = 2 - \frac{1}{x^2} = \frac{2x^2 - 1}{x^2}$$

Critical numbers: $x = \pm\dfrac{\sqrt{2}}{2}$

Discontinuity: $x = 0$

(b)

Test intervals:	$-\infty < x < -\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2} < x < 0$	$0 < x < \frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2} < x < \infty$
Sign of f':	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on: $\left(-\infty, -\dfrac{\sqrt{2}}{2}\right)$ and $\left(\dfrac{\sqrt{2}}{2}, \infty\right)$

Decreasing on: $\left(-\dfrac{\sqrt{2}}{2}, 0\right)$ and $\left(0, \dfrac{\sqrt{2}}{2}\right)$

(c) Relative maximum: $\left(-\dfrac{\sqrt{2}}{2}, -2\sqrt{2}\right)$

Relative minimum: $\left(\dfrac{\sqrt{2}}{2}, 2\sqrt{2}\right)$

34. (a) $f(x) = \dfrac{x}{x + 3}$

$f'(x) = \dfrac{(x + 3) - x}{(x + 3)^2} = \dfrac{3}{(x + 3)^2}$

No critical numbers

Discontinuity: $x = -3$

(b)

Test intervals:	$-\infty < x < -3$	$-3 < x < \infty$
Sign of f':	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, -3)$ and $(-3, \infty)$

(c) No relative extrema

35. (a) $f(x) = \dfrac{x^2}{x^2 - 9}$

$f'(x) = \dfrac{(x^2 - 9)(2x) - (x^2)(2x)}{(x^2 - 9)^2} = \dfrac{-18x}{(x^2 - 9)^2}$

Critical number: $x = 0$

Discontinuities: $x = -3, 3$

(b)

Test intervals:	$-\infty < x < -3$	$-3 < x < 0$	$0 < x < 3$	$3 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$	$f' < 0$	$f' < 0$
Conclusion:	Increasing	Increasing	Decreasing	Decreasing

Increasing on: $(-\infty, -3), (-3, 0)$

Decreasing on: $(0, 3), (3, \infty)$

(c) Relative maximum: $(0, 0)$

36. (a) $f(x) = \dfrac{x + 4}{x^2}$

$f'(x) = \dfrac{x^2 - (x + 4)(2x)}{x^4} = \dfrac{-x^2 - 8x}{x^4} = \dfrac{-(x + 8)}{x^3}$

Critical number: $x = -8$

Discontinuity: $x = 0$

(b)

Test intervals:	$-\infty < x < -8$	$-8 < x < 0$	$0 < x < \infty$
Sign of f':	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $(-8, 0)$

Decreasing on: $(-\infty, -8)$ and $(0, \infty)$

(c) Relative minimum: $\left(-8, -\dfrac{1}{16}\right)$

37. (a) $f(x) = \dfrac{x^2 - 2x + 1}{x + 1}$

$$f'(x) = \frac{(x+1)(2x-2) - (x^2 - 2x + 1)(1)}{(x+1)^2} = \frac{x^2 + 2x - 3}{(x+1)^2} = \frac{(x+3)(x-1)}{(x+1)^2}$$

Critical numbers: $x = -3, 1$

Discontinuity: $x = -1$

(b)

Test intervals:	$-\infty < x < -3$	$-3 < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on: $(-\infty, -3), (1, \infty)$

Decreasing on: $(-3, -1), (-1, 1)$

(c) Relative maximum: $(-3, -8)$

Relative minimum: $(1, 0)$

38. (a) $f(x) = \dfrac{x^2 - 3x - 4}{x - 2}$

$$f'(x) = \frac{(x-2)(2x-3) - (x^2 - 3x - 4)(1)}{(x-2)^2} = \frac{x^2 - 4x + 10}{(x-2)^2}$$

Discontinuity: $x = 2$

(b)

Test intervals:	$-\infty < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, 2), (2, \infty)$

(c) No relative extrema

39. (a) $f(x) = \begin{cases} 4 - x^2, & x \le 0 \\ -2x, & x > 0 \end{cases}$

$$f'(x) = \begin{cases} -2x, & x < 0 \\ -2, & x > 0 \end{cases}$$

Critical number: $x = 0$

(b)

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of f':	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, 0)$

Decreasing on: $(0, \infty)$

(c) No relative extrema. (Note: $(0, 4)$ is an absolute maximum)

40. (a) $f(x) = \begin{cases} 2x + 1, & x \le -1 \\ x^2 - 2, & x > -1 \end{cases}$

$f'(x) = \begin{cases} 2, & x < -1 \\ 2x, & x > -1 \end{cases}$

Critical numbers: $x = -1, 0$

(b)

Test intervals:	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < \infty$
Sign of f':	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -1)$ and $(0, \infty)$

Decreasing on: $(-1, 0)$

(c) Relative maximum: $(-1, -1)$

Relative minimum: $(0, -2)$

41. (a) $f(x) = \begin{cases} 3x + 1, & x \le 1 \\ 5 - x^2, & x > 1 \end{cases}$

$f'(x) = \begin{cases} 3, & x < 1 \\ -2x, & x > 1 \end{cases}$

Critical number: $x = 1$

(b)

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of f':	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, 1)$

Decreasing on: $(1, \infty)$

(c) Relative maximum: $(1, 4)$

42. (a) $f(x) = \begin{cases} -x^3 + 1, & x \le 0 \\ -x^2 + 2x, & x > 0 \end{cases}$

$f'(x) = \begin{cases} -3x^2, & x < 0 \\ -2x + 2, & x > 0 \end{cases}$

Critical numbers: $x = 0, 1$

(b)

Test intervals:	$-\infty < x < 0$	$0 < x < 1$	$1 < x < \infty$
Sign of f':	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $(0, 1)$

Decreasing on: $(-\infty, 0)$ and $(1, \infty)$

(c) Relative maximum: $(1, 1)$

Note: $(0, 1)$ is not a relative minimum

43. (a) $f(x) = \dfrac{x}{2} + \cos x, 0 < x < 2\pi$

$f'(x) = \dfrac{1}{2} - \sin x = 0$

Critical numbers: $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

Test intervals:	$0 < x < \dfrac{\pi}{6}$	$\dfrac{\pi}{6} < x < \dfrac{5\pi}{6}$	$\dfrac{5\pi}{6} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

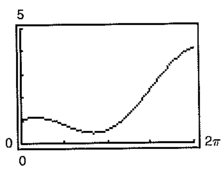
Increasing on: $\left(0, \dfrac{\pi}{6}\right), \left(\dfrac{5\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{6}, \dfrac{5\pi}{6}\right)$

(b) Relative maximum: $\left(\dfrac{\pi}{6}, \dfrac{\pi + 6\sqrt{3}}{12}\right)$

Relative minimum: $\left(\dfrac{5\pi}{6}, \dfrac{5\pi - 6\sqrt{3}}{12}\right)$

(c)



44. (a) $f(x) = \sin x \cos x + 5 = \dfrac{1}{2}\sin 2x + 5, 0 < x < 2\pi$

$f'(x) = \cos 2x$

Critical numbers: $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

Test intervals:	$0 < x < \dfrac{\pi}{4}$	$\dfrac{\pi}{4} < x < \dfrac{3\pi}{4}$	$\dfrac{3\pi}{4} < x < \dfrac{5\pi}{4}$	$\dfrac{5\pi}{4} < x < \dfrac{7\pi}{4}$	$\dfrac{7\pi}{4} < x < 2\pi$
Sign of f':	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing	Decreasing	Increasing

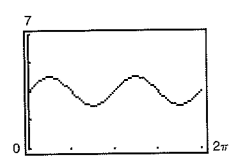
Increasing on: $\left(0, \dfrac{\pi}{4}\right), \left(\dfrac{3\pi}{4}, \dfrac{5\pi}{4}\right), \left(\dfrac{7\pi}{4}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right), \left(\dfrac{5\pi}{4}, \dfrac{7\pi}{4}\right)$

(b) Relative maxima: $\left(\dfrac{\pi}{4}, \dfrac{11}{2}\right), \left(\dfrac{5\pi}{4}, \dfrac{11}{2}\right)$

Relative minima: $\left(\dfrac{3\pi}{4}, \dfrac{9}{2}\right), \left(\dfrac{7\pi}{4}, \dfrac{9}{2}\right)$

(c)



45. (a) $f(x) = \sin x + \cos x, \quad 0 < x < 2\pi$

$f'(x) = \cos x - \sin x = 0 \Rightarrow \sin x = \cos x$

Critical numbers: $x = \frac{\pi}{4}, \frac{5\pi}{4}$

Test intervals:	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

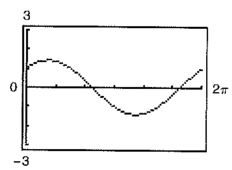
Increasing on: $\left(0, \frac{\pi}{4}\right), \left(\frac{5\pi}{4}, 2\pi\right)$

Decreasing on: $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

(b) Relative maximum: $\left(\frac{\pi}{4}, \sqrt{2}\right)$

Relative minimum: $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$

(c)



46. (a) $f(x) = x + 2\sin x, \quad 0 < x < 2\pi$

$f'(x) = 1 + 2\cos x = 0 \Rightarrow \cos x = -\frac{1}{2}$

Critical numbers: $\frac{2\pi}{3}, \frac{4\pi}{3}$

Test intervals:	$0 < x < \frac{2\pi}{3}$	$\frac{2\pi}{3} < x < \frac{4\pi}{3}$	$\frac{4\pi}{3} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

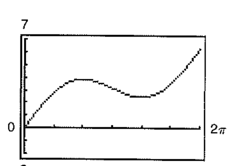
Increasing on: $\left(0, \frac{2\pi}{3}\right), \left(\frac{4\pi}{3}, 2\pi\right)$

Decreasing on: $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$

(b) Relative maximum: $\left(\frac{2\pi}{3}, \frac{2\pi}{3} + \sqrt{3}\right) \approx \left(\frac{2\pi}{3}, 3.826\right)$

Relative minimum: $\left(\frac{4\pi}{3}, \frac{4\pi}{3} - \sqrt{3}\right) \approx \left(\frac{4\pi}{3}, 2.457\right)$

(c)



47. (a) $f(x) = \cos^2(2x), \qquad 0 < x < 2\pi$

$f'(x) = -4\cos 2x \sin 2x = 0 \Rightarrow \cos 2x = 0 \text{ or } \sin 2x = 0$

Critical numbers: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

Test intervals:	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \pi$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing	Decreasing	Increasing

Test intervals:	$\pi < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < \frac{7\pi}{4}$	$\frac{7\pi}{4} < x < 2\pi$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing	Decreasing	Increasing

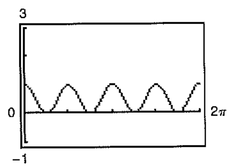
Increasing on: $\left(\frac{\pi}{4}, \frac{\pi}{2}\right), \left(\frac{3\pi}{4}, \pi\right), \left(\frac{5\pi}{4}, \frac{3\pi}{2}\right), \left(\frac{7\pi}{4}, 2\pi\right)$

Decreasing on: $\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{2}, \frac{3\pi}{4}\right), \left(\pi, \frac{5\pi}{4}\right), \left(\frac{3\pi}{2}, \frac{7\pi}{4}\right)$

(b) Relative maxima: $\left(\frac{\pi}{2}, 1\right), (\pi, 1), \left(\frac{3\pi}{2}, 1\right)$

Relative minima: $\left(\frac{\pi}{4}, 0\right), \left(\frac{3\pi}{4}, 0\right), \left(\frac{5\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$

(c)



48. (a) $f(x) = \sqrt{3}\sin x + \cos x$

$f'(x) = \sqrt{3}\cos x - \sin x = 0 \Rightarrow \tan x = \sqrt{3}$

Critical numbers: $x = \frac{\pi}{3}, \frac{4\pi}{3}$

Test intervals:	$0 < x < \frac{\pi}{3}$	$\frac{\pi}{3} < x < \frac{4\pi}{3}$	$\frac{4\pi}{3} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

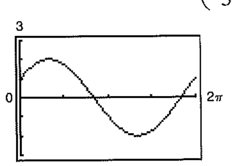
Increasing on: $\left(0, \frac{\pi}{3}\right), \left(\frac{4\pi}{3}, 2\pi\right)$

Decreasing on: $\left(\frac{\pi}{3}, \frac{4\pi}{3}\right)$

(b) Relative maximum: $\left(\frac{\pi}{3}, 2\right)$

Relative minimum: $\left(\frac{4\pi}{3}, -2\right)$

(c)



49. (a) $f(x) = \sin^2 x + \sin x, \quad 0 < x < 2\pi$

$f'(x) = 2\sin x \cos x + \cos x = \cos x(2\sin x + 1) = 0$

Critical numbers: $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{7\pi}{6}$	$\frac{7\pi}{6} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < \frac{11\pi}{6}$	$\frac{11\pi}{6} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing	Decreasing	Increasing

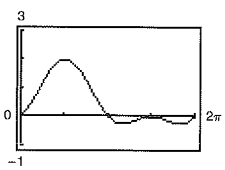
Increasing on: $\left(0, \frac{\pi}{2}\right), \left(\frac{7\pi}{6}, \frac{3\pi}{2}\right), \left(\frac{11\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\frac{\pi}{2}, \frac{7\pi}{6}\right), \left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)$

(b) Relative minima: $\left(\frac{7\pi}{6}, -\frac{1}{4}\right), \left(\frac{11\pi}{6}, -\frac{1}{4}\right)$

Relative maxima: $\left(\frac{\pi}{2}, 2\right), \left(\frac{3\pi}{2}, 0\right)$

(c)



50. (a) $f(x) = \frac{\sin x}{1 + \cos^2 x}, 0 < x < 2\pi$

$f'(x) = \frac{\cos x(2 + \sin^2 x)}{(1 + \cos^2 x)^2} = 0$

Critical numbers: $x = \frac{\pi}{2}, \frac{3\pi}{2}$

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

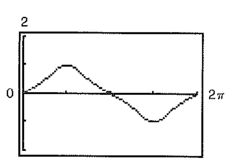
Increasing on: $\left(0, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, 2\pi\right)$

Decreasing on: $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

(b) Relative maximum: $\left(\frac{\pi}{2}, 1\right)$

Relative minimum: $\left(\frac{3\pi}{2}, -1\right)$

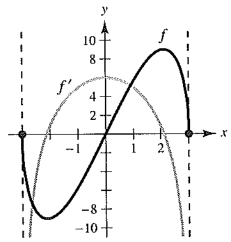
(c)



51. $f(x) = 2x\sqrt{9 - x^2}, [-3, 3]$

(a) $f'(x) = \dfrac{2(9 - 2x^2)}{\sqrt{9 - x^2}}$

(b)



(c) $\dfrac{2(9 - 2x^2)}{\sqrt{9 - x^2}} = 0$

Critical numbers: $x = \pm\dfrac{3}{\sqrt{2}} = \pm\dfrac{3\sqrt{2}}{2}$

(d) Intervals:

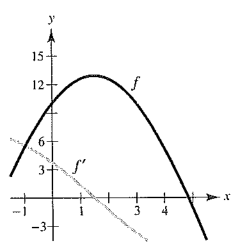
$\left(-3, -\frac{3\sqrt{2}}{2}\right)$	$\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$	$\left(\frac{3\sqrt{2}}{2}, 3\right)$
$f'(x) < 0$	$f'(x) > 0$	$f'(x) < 0$
Decreasing	Increasing	Decreasing

f is increasing when f' is positive and decreasing when f' is negative.

52. $f(x) = 10\left(5 - \sqrt{x^2 - 3x + 16}\right), [0, 5]$

(a) $f'(x) = -\dfrac{5(2x - 3)}{\sqrt{x^2 - 3x + 16}}$

(b)



(c) $-\dfrac{5(2x - 3)}{\sqrt{x^2 - 3x + 16}} = 0$

Critical number: $x = \dfrac{3}{2}$

(d) Intervals:

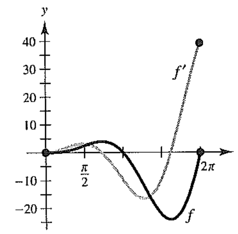
$\left(0, \frac{3}{2}\right)$	$\left(\frac{3}{2}, 5\right)$
$f'(x) > 0$	$f'(x) < 0$
Increasing	Decreasing

f is increasing when f' is positive and decreasing when f' is negative.

53. $f(t) = t^2 \sin t, [0, 2\pi]$

(a) $f'(t) = t^2 \cos t + 2t \sin t = t(t \cos t + 2 \sin t)$

(b)



(c) $t(t \cos t + 2 \sin t) = 0$

$t = 0$ or $t = -2 \tan t$

$t \cot t = -2$

$t \approx 2.2889, 5.0870$ (graphing utility)

Critical numbers: $t = 2.2889, 5.0870$

(d) Intervals:

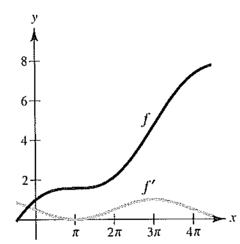
$(0, 2.2889)$	$(2.2889, 5.0870)$	$(5.0870, 2\pi)$
$f'(t) > 0$	$f'(t) < 0$	$f'(t) > 0$
Increasing	Decreasing	Increasing

f is increasing when f' is positive and decreasing when f' is negative.

54. $f(x) = \dfrac{x}{2} + \cos\dfrac{x}{2}, [0, 4\pi]$

(a) $f'(x) = \dfrac{1}{2} - \dfrac{1}{2}\sin\dfrac{x}{2}$

(b)



(c) $\dfrac{1}{2} - \dfrac{1}{2}\sin\dfrac{x}{2} = 0$

$\sin\dfrac{x}{2} = 1$

$\dfrac{x}{2} = \dfrac{\pi}{2}$

Critical number: $x = \pi$

(d) Intervals:

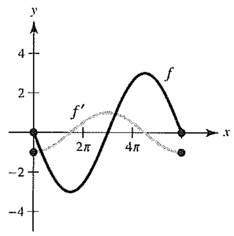
$(0, \pi)$	$(\pi, 4\pi)$
$f'(x) > 0$	$f'(x) > 0$
Increasing	Increasing

f is increasing when f' is positive.

55. (a) $f(x) = -3\sin\dfrac{x}{3}, [0, 6\pi]$

$f'(x) = -\cos\dfrac{x}{3}$

(b)



(c) Critical numbers: $x = \dfrac{3\pi}{2}, \dfrac{9\pi}{2}$

(d) Intervals:

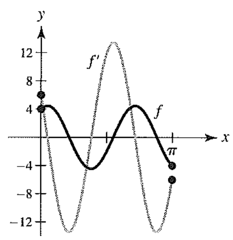
$\left(0, \dfrac{3\pi}{2}\right)$	$\left(\dfrac{3\pi}{2}, \dfrac{9\pi}{2}\right)$	$\left(\dfrac{9\pi}{2}, 6\pi\right)$
$f' < 0$	$f' > 0$	$f' < 0$
Decreasing	Increasing	Decreasing

f is increasing when f' is positive and decreasing when f' is negative.

56. (a) $f(x) = 2\sin 3x + 4\cos 3x, [0, \pi]$

$f'(x) = 6\cos 3x - 12\sin 3x$

(b)



(c) $f'(x) = 0 \Rightarrow \tan 3x = \dfrac{1}{2}$

Critical numbers: $x \approx 0.1545, 1.2017, 2.2489$

(d) Intervals:

$(0, 0.1545)$	$(0.1545, 1.2017)$	$(1.2017, 2.2489)$	$(2.2489, \pi)$
$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$
Increasing	Decreasing	Increasing	Decreasing

f is increasing when f' is positive and decreasing when f' is negative.

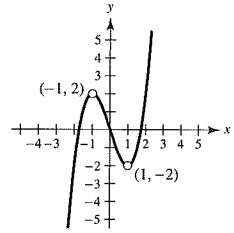
57. $f(x) = \dfrac{x^5 - 4x^3 + 3x}{x^2 - 1} = \dfrac{(x^2 - 1)(x^3 - 3x)}{x^2 - 1} = x^3 - 3x, x \neq \pm 1$

$f(x) = g(x) = x^3 - 3x$ for all $x \neq \pm 1$.

$f'(x) = 3x^2 - 3 = 3(x^2 - 1), x \neq \pm 1 \Rightarrow f'(x) \neq 0$

f symmetric about origin

zeros of f: $(0, 0), (\pm\sqrt{3}, 0)$



$g(x)$ is continuous on $(-\infty, \infty)$ and $f(x)$ has holes at $(-1, 2)$ and $(1, -2)$.

58. $f(t) = \cos^2 t - \sin^2 t = 1 - 2\sin^2 t = g(t)$

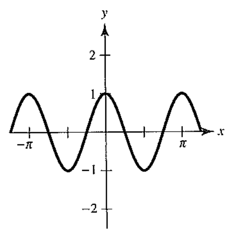
$f'(t) = -4 \sin t \cos t = -2 \sin 2t$

f symmetric with respect to y-axis

zeros of f: $\pm\dfrac{\pi}{4}$

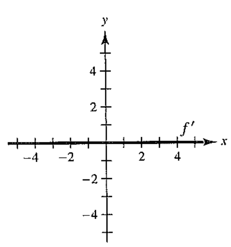
Relative maximum: $(0, 1)$

Relative minimum: $\left(-\dfrac{\pi}{2}, -1\right), \left(\dfrac{\pi}{2}, -1\right)$

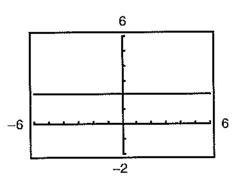


The graphs of $f(x)$ and $g(x)$ are the same.

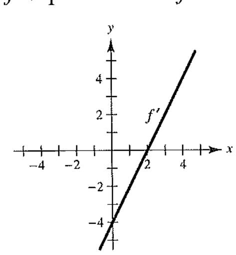
59. $f(x) = c$ is constant $\Rightarrow f'(x) = 0$.



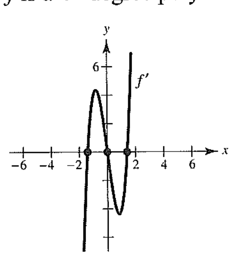
60. $f(x)$ is a line of slope $\approx 2 \Rightarrow f'(x) = 2$.



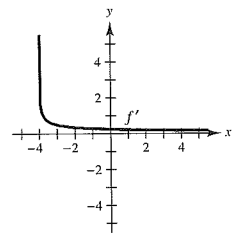
61. f is quadratic $\Rightarrow f'$ is a line.



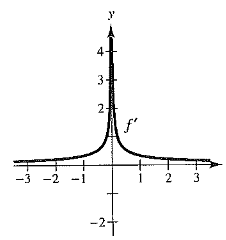
62. f is a 4$^{\text{th}}$ degree polynomial $\Rightarrow f'$ is a cubic polynomial.



63. f has positive, but decreasing slope



64. f has positive slope



65. (a) f increasing on $(2, \infty)$ because $f' > 0$ on $(2, \infty)$

f decreasing on $(-\infty, 2)$ because $f' < 0$ on $(-\infty, 2)$

(b) f has a relative minimum at $x = 2$.

66. (a) f increasing on $(-\infty, 0)$ and $(1, \infty)$ because $f' > 0$

f decreasing on $(0, 1)$ because $f' < 0$

(b) f has a relative maximum at $x = 0$, and a relative minimum at $x = 1$.

67. (a) f increasing on $(-\infty, -1)$ and $(0, 1)$ because $f' > 0$

f decreasing on $(-1, 0)$ and $(1, \infty)$ because $f' < 0$

(b) f has a relative maximum at $x = -1$ and $x = 1$.

f has a relative minimum at $x = 0$.

68. (a) f is increasing on $(-1, 0)$ and $(0, \infty)$ because $f' > 0$

f is decreasing on $(-\infty, -1)$ because $f' < 0$.

(b) f has a relative minimum at $x = -1$.

69. (a) $f' = 0$ at $x = -1, 1, 2$

Critical numbers: $x = -1, 1, 2$

(b) $x = 1$ is a relative maximum because f' changes from positive to negative.

$x = 2$ is a relative minimum because f' changes from negative to positive. $x = -1$ is not a relative extremum.

70. (a) $f' = 0$ at $x = -3, 1$ and 5.

Critical numbers: $x = -3, 1, 5$

(b) $x = -3$ is a relative minimum because f' changes from negative to positive.

$x = 5$ is a relative maximum because f' changes from positive to negative.

$x = 1$ is not a relative extremum.

In Exercises 71–76, $f'(x) > 0$ on $(-\infty, -4)$, $f'(x) < 0$ on $(-4, 6)$ and $f'(x) > 0$ on $(6, \infty)$.

71. $g(x) = f(x) + 5$

$g'(x) = f'(x)$

$g'(0) = f'(0) < 0$

72. $g(x) = 3f(x) - 3$

$g'(x) = 3f'(x)$

$g'(-5) = 3f'(-5) > 0$

73. $g(x) = -f(x)$

$g'(x) = -f'(x)$

$g'(-6) = -f'(-6) < 0$

74. $g(x) = -f(x)$

$g'(x) = -f'(x)$

$g'(0) = -f'(0) > 0$

75. $g(x) = f(x - 10)$

$g'(x) = f'(x - 10)$

$g'(0) = f'(-10) > 0$

76. $g(x) = f(x - 10)$

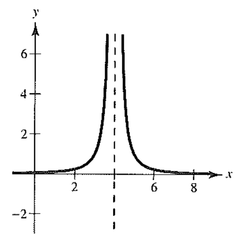
$g'(x) = f'(x - 10)$

$g'(8) = f'(-2) < 0$

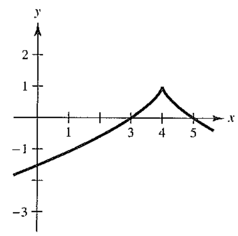
77. $f'(x)\begin{cases} > 0, & x < 4 \Rightarrow f \text{ is increasing on } (-\infty, 4). \\ \text{undefined}, & x = 4 \\ < 0, & x > 4 \Rightarrow f \text{ is decreasing on } (4\ \infty). \end{cases}$

Two possibilities for $f(x)$ are given below.

(a)



(b)



78. Critical number: $x = 5$

$f'(4) = -2.5 \Rightarrow f$ is decreasing at $x = 4$.

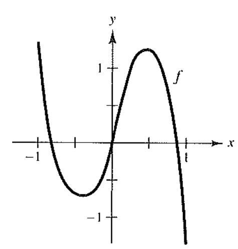
$f'(6) = 3 \Rightarrow f$ is increasing at $x = 6$.

$(5, f(5))$ is a relative minimum.

In Exercises 79 and 80, answers will vary.

Sample answers:

79. (a)



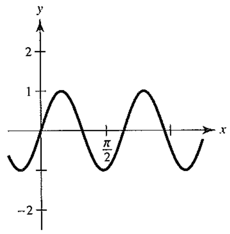
(b) The critical numbers are in intervals $(-0.50, -0.25)$ and $(0.25, 0.50)$ because the sign of f' changes in these intervals.

f is decreasing on approximately $(-1, -0.40)$, $(0.48, 1)$, and increasing on $(-0.40, 0.48)$.

(c) Relative minimum when $x \approx -0.40$: $(-0.40, 0.75)$

Relative maximum when $x \approx 0.48$: $(0.48, 1.25)$

80. (a)



(b) The critical numbers are in the intervals $\left(0, \frac{\pi}{6}\right), \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$, and $\left(\frac{3\pi}{4}, \frac{5\pi}{6}\right)$ because the sign of f' changes in these intervals.

f is increasing on approximately $\left(0, \frac{\pi}{7}\right)$ and $\left(\frac{3\pi}{7}, \frac{6\pi}{7}\right)$ and decreasing on $\left(\frac{\pi}{7}, \frac{3\pi}{7}\right)$ and $\left(\frac{6\pi}{7}, \pi\right)$.

(c) Relative minima when $x \approx \frac{3\pi}{7}, \pi$

Relative maxima when $x \approx \frac{\pi}{7}, \frac{6\pi}{7}$

81. $s(t) = 4.9(\sin\theta)t^2$

(a) $s'(t) = 4.9(\sin\theta)(2t) = 9.8(\sin\theta)t$

speed $= |s'(t)| = |9.8(\sin\theta)t|$

(b)

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π
$\lvert s'(t)\rvert$	0	$4.9\sqrt{2}t$	$4.9\sqrt{3}t$	$9.8t$	$4.9\sqrt{3}t$	$4.9\sqrt{2}t$	0

The speed is maximum for $\theta = \frac{\pi}{2}$.

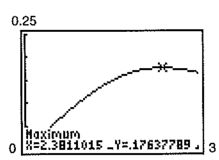
82. $C = \frac{3t}{27 + t^3}, t \geq 0$

(a)

t	0	0.5	1	1.5	2	2.5	3
$C(t)$	0	0.055	0.107	0.148	0.171	0.176	0.167

The concentration seems greatest near $t = 2.5$ hours.

(b)



The concentration is greatest when $t \approx 2.38$ hours.

(c) $$C' = \frac{(27 + t^3)(3) - (3t)(3t^2)}{(27 + t^3)^2}$$

$$= \frac{3(27 - 2t^3)}{(27 + t^3)^2}$$

$C' = 0$ when $t = 3/\sqrt[3]{2} \approx 2.38$ hours.

By the First Derivative Test, this is a maximum.

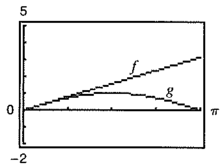
83. $f(x) = x, g(x) = \sin x, 0 < x < \pi$

(a)

x	0.5	1	1.5	2	2.5	3
$f(x)$	0.5	1	1.5	2	2.5	3
$g(x)$	0.479	0.841	0.997	0.909	0.598	0.141

$f(x)$ seems greater than $g(x)$ on $(0, \pi)$.

(b)



$x > \sin x$ on $(0, \pi)$ so, $f(x) > g(x)$.

(c) Let $h(x) = f(x) - g(x) = x - \sin x$

$$h'(x) = 1 - \cos x > 0 \text{ on } (0, \pi).$$

Therefore, $h(x)$ is increasing on $(0, \pi)$. Because $h(0) = 0$ and $h'(x) > 0$ on $(0, \pi)$,

$$h(x) > 0$$
$$x - \sin x > 0$$
$$x > \sin x$$
$$f(x) > g(x) \text{ on } (0, \pi)$$

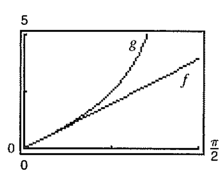
84. $f(x) = x, g(x) = \tan x$

(a)

x	0.25	0.5	0.75	1.0	1.25	1.5
$f(x)$	0.25	0.5	0.75	1.0	1.25	1.5
$g(x)$	0.2553	0.5463	0.9316	1.5574	3.0096	14.1014

$g(x)$ seems greater than $f(x)$ on $\left(0, \frac{\pi}{2}\right)$.

(b)



$\tan x > x$ on $\left(0, \frac{\pi}{2}\right)$ so, $g(x) > f(x)$

(c) Let $h(x) = f(x) - g(x) = \tan x - x$

$$h'(x) = \sec^2 x - 1 > 0 \text{ on } \left(0, \frac{\pi}{2}\right).$$

Because $h(0) = 0$ and $h'(x) > 0$ on $\left(0, \frac{\pi}{2}\right)$,

$$h(x) > 0$$
$$\tan x - x > 0$$
$$\tan x > x$$
$$g(x) > f(x) \text{ on } \left(0, \frac{\pi}{2}\right).$$

85. $v = k(R - r)r^2 = k(Rr^2 - r^3)$

$v' = k(2Rr - 3r^2)$

$= kr(2R - 3r) = 0$

$r = 0 \text{ or } \frac{2}{3}R$

Maximum when $r = \frac{2}{3}R$.

86. $P = \dfrac{vR_1R_2}{(R_1 + R_2)^2}$, v and R_1 are constant

$$\frac{dP}{dR_2} = \frac{(R_1 + R_2)^2(vR_1) - vR_1R_2[2(R_1 + R_2)(1)]}{(R_1 + R_2)^4}$$

$$= \frac{vR_1(R_1 - R_2)}{(R_1 + R_2)^3} = 0 \Rightarrow R_2 = R_1$$

Maximum when $R_1 = R_2$.

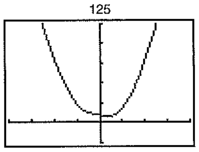
87. $R = \sqrt{0.001T^4 - 4T + 100}$

(a) $R' = \dfrac{0.004T^3 - 4}{2\sqrt{0.001T^4 - 4T + 100}} = 0$

Critical number: $T = 10°$

Minimum resistance: $R \approx 8.3666$ ohms

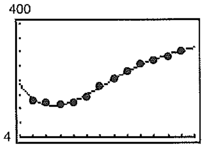
(b)



The minimum resistance is approximately $R \approx 8.37$ ohms at $T = 10°$.

88. (a) $M = 0.03723t^4 - 1.9931t^3 + 37.986t^2 - 282.74t + 825.7$

(b)



(c) Using a graphing utility, the minimum is (6.5, 111.9) which compares well with the minimum (7, 115.6).

89. (a) $s(t) = 6t - t^2, t \geq 0$

$v(t) = 6 - 2t$

(b) $v(t) = 0$ when $t = 3$.

Moving in positive direction for $0 \leq t < 3$ because $v(t) > 0$ on $0 \leq t < 3$.

(c) Moving in negative direction when $t > 3$.

(d) The particle changes direction at $t = 3$.

90. (a) $s(t) = t^2 - 7t + 10, t \geq 0$

$v(t) = 2t - 7$

(b) $v(t) = 0$ when $t = \frac{7}{2}$

Particle moving in positive direction for $t > \frac{7}{2}$ because $v'(t) > 0$ on $\left(\frac{7}{2}, \infty\right)$.

(c) Particle moving in negative direction on $\left[0, \frac{7}{2}\right)$.

(d) The particle changes direction at $t = \frac{7}{2}$.

91. (a) $s(t) = t^3 - 5t^2 + 4t, t \geq 0$

$v(t) = 3t^2 - 10t + 4$

(b) $v(t) = 0$ for $t = \dfrac{10 \pm \sqrt{100 - 48}}{6} = \dfrac{5 \pm \sqrt{13}}{3}$

Particle is moving in a positive direction on $\left[0, \dfrac{5 - \sqrt{13}}{3}\right) \approx [0, 0.4648)$ and $\left(\dfrac{5 + \sqrt{13}}{3}, \infty\right) \approx (2.8685, \infty)$ because $v > 0$ on these intervals.

(c) Particle is moving in a negative direction on $\left(\dfrac{5 - \sqrt{13}}{3}, \dfrac{5 + \sqrt{13}}{3}\right) \approx (0.4648, 2.8685)$

(d) The particle changes direction at $t = \dfrac{5 \pm \sqrt{13}}{3}$.

92. (a) $s(t) = t^3 - 20t^2 + 128t - 280$

$v(t) = 3t^2 - 40t + 128$

(b) $v(t) = (3t - 16)(t - 8)$

$v(t) = 0$ when $t = \frac{16}{3}, 8$

$v(t) > 0$ for $\left[0, \frac{16}{3}\right)$ and $(8, \infty)$

(c) $v(t) < 0$ for $\left(\frac{16}{3}, 8\right)$

(d) The particle changes direction at $t = \frac{16}{3}$ and 8.

93. Answers will vary.

94. Answers will vary.

95. (a) Use a cubic polynomial

$$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

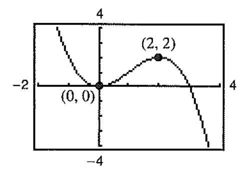
(b) $f'(x) = 3a_3x^2 + 2a_2x + a_1.$

$$\begin{aligned}
f(0) = 0:&\quad a_3(0)^3 + a_2(0)^2 + a_1(0) + a_0 = 0 \Rightarrow &\quad a_0 &= 0\\
f'(0) = 0:&\quad 3a_3(0)^2 + 2a_2(0) + a_1 = 0 \Rightarrow &\quad a_1 &= 0\\
f(2) = 2:&\quad a_3(2)^3 + a_2(2)^2 + a_1(2) + a_0 = 2 \Rightarrow &\quad 8a_3 + 4a_2 &= 2\\
f'(2) = 0:&\quad 3a_3(2)^2 + 2a_2(2) + a_1 = 0 \Rightarrow &\quad 12a_3 + 4a_2 &= 0
\end{aligned}$$

(c) The solution is $a_0 = a_1 = 0, a_2 = \frac{3}{2}, a_3 = -\frac{1}{2}$:

$$f(x) = -\frac{1}{2}x^3 + \frac{3}{2}x^2.$$

(d)



96. (a) Use a cubic polynomial

$$f(x) = 3a_3x^3 + a_2x^2 + a_1x + a_0$$

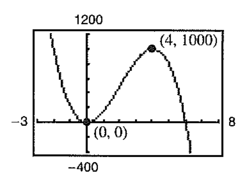
(b) $f'(x) = 3a_3x^2 + 2a_2x + a_1$

$$\begin{aligned}
f(0) = 0:&\quad a_3(0)^3 + a_2(0)^2 + a_1(0) + a_0 = 0 \Rightarrow &\quad a_0 &= 0\\
f'(0) = 0:&\quad 3a_3(0)^2 + 2a_2(0) + a_1 = 0 \Rightarrow &\quad a_1 &= 0\\
f(4) = 1000:&\quad a_3(4)^3 + a_2(4)^2 + a_1(4) + a_0 = 1000 \Rightarrow &\quad 64a_3 + 16a_2 &= 100\\
f'(4) = 0:&\quad 3a_3(4)^2 + 2a_2(4) + a_1 = 0 \Rightarrow &\quad 48a_3 + 8a_2 &= 0
\end{aligned}$$

(c) The solution is $a_0 = a_1 = 0, a_2 = \frac{375}{2}, a_3 = -\frac{125}{4}$

$$f(x) = -\frac{125}{4}x^3 + \frac{375}{2}x^2.$$

(d)



97. (a) Use a fourth degree polynomial

$$f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

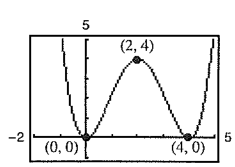
(b) $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

$$\begin{aligned}
f(0) = 0:&\quad a_4(0)^4 + a_3(0)^3 + a_2(0)^2 + a_1(0) + a_0 = 0 \Rightarrow &\quad a_0 &= 0\\
f'(0) = 0:&\quad 4a_4(0)^3 + 3a_3(0)^2 + 2a_2(0) + a_1 = 0 \Rightarrow &\quad a_1 &= 0\\
f(4) = 0:&\quad a_4(4)^4 + a_3(4)^3 + a_2(4)^2 + a_1(4) + a_0 = 0 \Rightarrow &\quad 256a_4 + 64a_3 + 16a_2 &= 0\\
f'(4) = 0:&\quad 4a_4(4)^3 + 3a_3(4)^2 + 2a_2(4) + a_1 = 0 \Rightarrow &\quad 256a_4 + 48a_3 + 8a_2 &= 0\\
f(2) = 4:&\quad a_4(2)^4 + a_3(2)^3 + a_2(2)^2 + a_1(2) + a_0 = 4 \Rightarrow &\quad 16a_4 + 8a_3 + 4a_2 &= 4\\
f'(2) = 0:&\quad 4a_4(2)^3 + 3a_3(2)^2 + 2a_2(2) + a_1 = 0 \Rightarrow &\quad 32a_4 + 12a_3 + 4a_2 &= 0
\end{aligned}$$

(c) The solution is $a_0 = a_1 = 0, a_2 = 4, a_3 = -2, \quad a_4 = \frac{1}{4}.$

$$f(x) = \frac{1}{4}x^4 - 2x^3 + 4x^2$$

(d)



98. (a) Use a fourth degree polynomial

$$f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

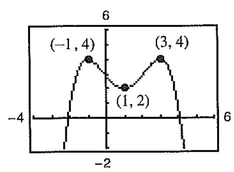
(b) $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

$$
\begin{aligned}
f(1) = 2&: & a_4(1)^4 + a_3(1)^3 + a_2(1)^2 + a_1(1) + a_0 = 2 &\Rightarrow & a_4 + a_3 + a_2 + a_1 + a_0 &= 2\\
f'(1) = 0&: & 4a_4(1)^3 + 3a_3(1)^2 + 2a_2(1) + a_1 = 0 &\Rightarrow & 4a_4 + 3a_3 + 2a_2 + a_1 &= 0\\
f(-1) = 4&: & a_4(-1)^4 + a_3(-1)^3 + a_2(-1)^2 + a_1(-1) + a_0 = 4 &\Rightarrow & a_4 - a_3 + a_2 - a_1 + a_0 &= 4\\
f'(-1) = 0&: & 4a_4(-1)^3 + 3a_3(-1)^2 + 2a_2(-1) + a_1 = 0 &\Rightarrow & -4a_4 + 3a_3 - 2a_2 + a_1 &= 0\\
f(3) = 4&: & a_4(3)^4 + a_3(3)^3 + a_2(3)^2 + a_1(3) + a_0 = 4 &\Rightarrow & 81a_4 + 27a_3 + 9a_2 + a_1 + a_0 &= 4\\
f'(3) = 0&: & 4a_4(3)^3 + 3a_3(3)^2 + 2a_2(3) + a_1 = 0 &\Rightarrow & 108a_4 + 27a_3 + 6a_2 + a_1 &= 0
\end{aligned}
$$

(c) The solution is $a_0 = \frac{23}{8}, a_1 = -\frac{3}{2}, a_2 = \frac{1}{4}, a_3 = \frac{1}{2}, a_4 = -\frac{1}{8}$

$$f(x) = -\frac{1}{8}x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2 - \frac{3}{2}x + \frac{23}{8}.$$

(d)



99. True.

Let $h(x) = f(x) + g(x)$ where f and g are increasing. Then $h'(x) = f'(x) + g'(x) > 0$ because $f'(x) > 0$ and $g'(x) > 0$.

100. False.

Let $h(x) = f(x)g(x)$ where $f(x) = g(x) = x$. Then $h(x) = x^2$ is decreasing on $(-\infty, 0)$.

101. False.

Let $f(x) = x^3$, then $f'(x) = 3x^2$ and f only has one critical number. Or, let $f(x) = x^3 + 3x + 1$, then $f'(x) = 3(x^2 + 1)$ has no critical numbers.

102. True.

If $f(x)$ is an nth-degree polynomial, then the degree of $f'(x)$ is $n - 1$.

103. False. For example, $f(x) = x^3$ does not have a relative extrema at the critical number $x = 0$.

104. Assume that $f'(x) < 0$ for all x in the interval (a, b) and let $x_1 < x_2$ be any two points in the interval. By the Mean Value Theorem, you know there exists a number c such that $x_1 < c < x_2$, and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Because $f'(c) < 0$ and $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) < 0$, which implies that $f(x_2) < f(x_1)$. So, f is decreasing on the interval.

105. Suppose $f'(x)$ changes from positive to negative at c. Then there exists a and b in I such that $f'(x) > 0$ for all x in (a, c) and $f'(x) < 0$ for all x in (c, b). By Theorem 3.5, f is increasing on (a, c) and decreasing on (c, b). Therefore, $f(c)$ is a maximum of f on (a, b) and so, a relative maximum of f.

106. Let x_1 and x_2 be two real numbers, $x_1 < x_2$. Then $x_1^3 < x_2^3 \Rightarrow f(x_1) < f(x_2)$. So f is increasing on $(-\infty, \infty)$.

107. Let x_1 and x_2 be two positive real numbers, $0 < x_1 < x_2$. Then

$$\frac{1}{x_1} > \frac{1}{x_2}$$

$$f(x_1) > f(x_2)$$

So, f is decreasing on $(0, \infty)$.

108. First observe that

$$\tan x + \cot x + \sec x + \csc x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} + \frac{1}{\cos x} + \frac{1}{\sin x}$$
$$= \frac{\sin^2 x + \cos^2 x + \sin x + \cos x}{\sin x \cos x}$$
$$= \frac{1 + \sin x + \cos x}{\sin x \cos x}\left(\frac{\sin x + \cos x - 1}{\sin x + \cos x - 1}\right)$$
$$= \frac{(\sin x + \cos x)^2 - 1}{\sin x \cos x(\sin x + \cos x - 1)}$$
$$= \frac{2 \sin x \cos x}{\sin x \cos x(\sin x + \cos x - 1)}$$
$$= \frac{2}{\sin x + \cos x - 1}$$

Let $t = \sin x + \cos x - 1$. The expression inside the absolute value sign is

$$f(t) = \sin x + \cos x + \frac{2}{\sin x + \cos x - 1}$$
$$= (\sin x + \cos x - 1) + 1 + \frac{2}{\sin x + \cos x - 1}$$
$$= t + 1 + \frac{2}{t}$$

Because $\sin\left(x + \frac{\pi}{4}\right) = \sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}$

$$= \frac{\sqrt{2}}{2}(\sin x + \cos x),$$

$\sin x + \cos x \in \left[-\sqrt{2}, \sqrt{2}\right]$ and

$t = \sin x + \cos x - 1 \in \left[-1 - \sqrt{2}, -1 + \sqrt{2}\right].$

$$f'(t) = 1 - \frac{2}{t^2} = \frac{t^2 - 2}{t^2} = \frac{(t + \sqrt{2})(t - \sqrt{2})}{t^2}$$

$$f(-1 + \sqrt{2}) = -1 + \sqrt{2} + 1 + \frac{2}{-1 + \sqrt{2}} = \sqrt{2} + \frac{2}{\sqrt{2} - 1}$$
$$= \frac{4 - \sqrt{2}}{\sqrt{2} - 1}\left(\frac{\sqrt{2} + 1}{\sqrt{2} + 1}\right) = \frac{4\sqrt{2} - 2 + 4 - \sqrt{2}}{1} = 2 + 3\sqrt{2}$$

For $t > 0$, f is decreasing and $f(t) > f(-1 + \sqrt{2}) = 2 + 3\sqrt{2}$

For $t < 0$, f is increasing on $(-\sqrt{2} - 1, -\sqrt{2})$, then decreasing on $(-\sqrt{2}, 0)$. So $f(t) < f(-\sqrt{2}) = 1 - 2\sqrt{2}$.

Finally, $|f(t)| \geq 2\sqrt{2} - 1$.

(You can verify this easily with a graphing utility.)

Section 3.4 Concavity and the Second Derivative Test

1. The graph of f is increasing and concave upwards:
$f' > 0, f'' > 0$

2. The graph of f is increasing and concave downwards:
$f' > 0, f'' < 0$

3. The graph of f is decreasing and concave downward:
$f' < 0, f'' < 0$

4. The graph of f is decreasing and concave upward:
$f' < 0, f'' > 0$

5. $y = x^2 - x - 2$

$y' = 2x - 1$

$y'' = 2$

Concave upward: $(-\infty, \infty)$

6. $y = -x^3 + 3x^2 - 2$

$y' = -3x^2 + 6x$

$y'' = -6x + 6$

Concave upward: $(-\infty, 1)$

Concave downward: $(1, \infty)$

7. $g(x) = 3x^2 - x^3$

$g'(x) = 6x - 3x^2$

$g''(x) = 6 - 6x$

Concave upward: $(-\infty, 1)$

Concave downward: $(1, \infty)$

8. $h(x) = x^5 - 5x + 2$

$h'(x) = 5x^4 - 5$

$h''(x) = 20x^3$

Concave upward: $(0, \infty)$

Concave downward: $(-\infty, 0)$

9. $f(x) = -x^3 + 6x^2 - 9x - 1$

$f'(x) = -3x^2 + 12x - 9$

$f''(x) = -6x + 12 = -6(x - 2)$

Concave upward: $(-\infty, 2)$

Concave downward: $(2, \infty)$

10. $f(x) = x^5 + 5x^4 - 40x^2$

$f'(x) = 5x^4 + 20x^3 - 80x$

$f''(x) = 20x^3 + 60x^2 - 80$

$= 20(x^3 + 3x^2 - 4)$

$= 20(x - 1)(x + 2)^2$

Test Interval:	$-\infty < x < -2$	$-2 < x < 1$	$1 < x < \infty$
Sign of f'':	$f'' < 0$	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave downward	Concave upward

Concave upward: $(1, \infty)$

Concave downward: $(-\infty, 1)$

11. $f(x) = \dfrac{24}{x^2 + 12}$

$f' = \dfrac{-48x}{(x^2 + 12)^2}$

$f'' = \dfrac{-144(4 - x^2)}{(x^2 + 12)^3}$

Concave upward: $(-\infty, -2), (2, \infty)$

Concave downward: $(-2, 2)$

12. $f(x) = \dfrac{x^2}{x^2 + 1}$

$f'(x) = \dfrac{2x}{(x^2 + 1)^2}$

$f''(x) = \dfrac{-2(3x^2 - 1)}{(x^2 + 1)^3} = \dfrac{2(1 + \sqrt{3}x)(1 - \sqrt{3}x)}{(x^2 + 1)^3}$

Concave upward: $\left(-\dfrac{\sqrt{3}}{3}, \dfrac{\sqrt{3}}{3}\right)$

Concave downward: $\left(-\infty, -\dfrac{\sqrt{3}}{3}\right), \left(\dfrac{\sqrt{3}}{3}, \infty\right)$

13. $f(x) = \dfrac{x^2+1}{x^2-1}$

$f' = \dfrac{-4x}{(x^2-1)^2}$

$f'' = \dfrac{4(3x^2+1)}{(x^2-1)^3}$

Concave upward: $(-\infty, -1), (1, \infty)$

Concave downward: $(-1, 1)$

14. $y = \frac{1}{270}(-3x^5 + 40x^3 + 135x)$

$y' = \frac{1}{270}(-15x^4 + 120x^2 + 135)$

$y'' = -\frac{2}{9}x(x-2)(x+2)$

Concave upward: $(-\infty, -2), (0, 2)$

Concave downward: $(-2, 0), (2, \infty)$

15. $g(x) = \dfrac{x^2+4}{4-x^2}$

$g'(x) = \dfrac{16x}{(4-x^2)^2}$

$g''(x) = \dfrac{16(3x^2+4)}{(4-x^2)^3} = \dfrac{16(3x^2+4)}{(2-x)^3(2+x)^3}$

Concave upward: $(-2, 2)$

Concave downward: $(-\infty, -2), (2, \infty)$

16. $h(x) = \dfrac{x^2-1}{2x-1}$

$h'(x) = \dfrac{2(x^2-x+1)}{(2x-1)^2}$

$h''(x) = \dfrac{-6}{(2x-1)^3}$

Concave upward: $\left(-\infty, \dfrac{1}{2}\right)$

Concave downward: $\left(\dfrac{1}{2}, \infty\right)$

17. $y = 2x - \tan x, \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$y' = 2 - \sec^2 x$

$y'' = -2\sec^2 x \tan x$

Concave upward: $\left(-\dfrac{\pi}{2}, 0\right)$

Concave downward: $\left(0, \dfrac{\pi}{2}\right)$

18. $y = x + 2\csc x, \quad (-\pi, \pi)$

$y' = 1 - 2\csc x \cot x$

$y'' = -2\csc x(-\csc^2 x) - 2\cot x(-\csc x \cot x)$

$= 2(\csc^3 x + \csc x \cot^2 x)$

Concave upward: $(0, \pi)$

Concave downward: $(-\pi, 0)$

19. $f(x) = \frac{1}{2}x^4 + 2x^3$

$f'(x) = 2x^3 + 6x^2$

$f''(x) = 6x^2 + 12x = 6x(x+2)$

$f''(x) = 0$ when $x = 0, -2$

Concave upward: $(-\infty, -2), (0, \infty)$

Concave downward: $(-2, 0)$

Points of inflection: $(-2, -8)$ and $(0, 0)$

20. $f(x) = -x^4 + 24x^2$

$f'(x) = -4x^3 + 48x$

$f''(x) = -12x^2 + 48 = 12(4-x^2) = 12(2+x)(2-x)$

$f''(x) = 0 \quad \text{for} \quad x = -2, 2$

Concave upward: $(-2, 2)$

Concave downward: $(-\infty, -2), (2, \infty)$

Points of inflection: $(-2, 80), (2, 80)$

21. $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12$

$f''(x) = 6(x-2) = 0$ when $x = 2$.

Concave upward: $(2, \infty)$

Concave downward: $(-\infty, 2)$

Point of inflection: $(2, 8)$

22. $f(x) = 2x^3 - 3x^2 - 12x + 5$

$f'(x) = 6x^2 - 6x - 12$

$f''(x) = 12x - 6$

$f''(x) = 12x - 6 = 0$ when $x = \frac{1}{2}$.

Test interval:	$-\infty < x < \frac{1}{2}$	$\frac{1}{2} < x < \infty$
Sign of $f''(x)$:	$f''(x) < 0$	$f''(x) > 0$
Conclusion:	Concave downward	Concave upward

Point of inflection: $\left(\frac{1}{2}, -\frac{13}{2}\right)$

23. $f(x) = \frac{1}{4}x^4 - 2x^2$

$f'(x) = x^3 - 4x$

$f''(x) = 3x^2 - 4$

$f''(x) = 3x^2 - 4 = 0$ when $x = \pm\frac{2}{\sqrt{3}}$.

Test interval:	$-\infty < x < -\frac{2}{\sqrt{3}}$	$-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}} < x < \infty$
Sign of $f''(x)$:	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

Points of inflection: $\left(\pm\frac{2}{\sqrt{3}}, -\frac{20}{9}\right)$

24. $f(x) = 2x^4 - 8x + 3$

$f'(x) = 8x^3 - 8$

$f''(x) = 24x^2 = 0$ when $x = 0$.

However, $(0, 3)$ is not a point of inflection because $f''(x) \geq 0$ for all x.

Concave upward: $(-\infty, \infty)$

25. $f(x) = x(x-4)^3$

$f'(x) = x\left[3(x-4)^2\right] + (x-4)^3 = (x-4)^2(4x-4)$

$f''(x) = 4(x-1)\left[2(x-4)\right] + 4(x-4)^2 = 4(x-4)\left[2(x-1) + (x-4)\right] = 4(x-4)(3x-6) = 12(x-4)(x-2)$

$f''(x) = 12(x-4)(x-2) = 0$ when $x = 2, 4$.

Test interval:	$-\infty < x < 2$	$2 < x < 4$	$4 < x < \infty$
Sign of $f''(x)$:	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

Points of inflection: $(2, -16), (4, 0)$

26. $f(x) = (x-2)^3(x-1)$

$f'(x) = (x-2)^2(4x-5)$

$f''(x) = 6(x-2)(2x-3)$

$f''(x) = 0$ when $x = \frac{3}{2}, 2$

Concave upward: $\left(-\infty, \frac{3}{2}\right)$ and $(2, \infty)$

Concave downward: $\left(\frac{3}{2}, 2\right)$

Points of inflection: $\left(\frac{3}{2}, -\frac{1}{16}\right), (2, 0)$

27. $f(x) = x\sqrt{x+3}$, Domain: $[-3, \infty)$

$f'(x) = x\left(\frac{1}{2}\right)(x+3)^{-1/2} + \sqrt{x+3} = \frac{3(x+2)}{2\sqrt{x+3}}$

$f''(x) = \frac{6\sqrt{x+3} - 3(x+2)(x+3)^{-1/2}}{4(x+3)}$

$= \frac{3(x+4)}{4(x+3)^{3/2}}$

$f''(x) > 0$ on the entire domain of f (except for $x = -3$, for which $f''(x)$ is undefined). There are no points of inflection.

Concave upward: $(-3, \infty)$

28. $f(x) = x\sqrt{9-x}$ Domain: $x \le 9$

$f'(x) = \dfrac{3(6-x)}{2\sqrt{9-x}}$

$f''(x) = \dfrac{3(x-12)}{4(9-x)^{3/2}}$

Concave downward: $(-\infty, 9)$

No point of inflection

29. $f(x) = \dfrac{4}{x^2+1}$

$f'(x) = \dfrac{-8x}{(x^2+1)^2}$

$f''(x) = \dfrac{8(3x^2-1)}{(x^2+1)^3}$

$f''(x) = 0$ for $x = \pm\dfrac{\sqrt{3}}{3}$

Concave upward: $\left(-\infty, -\dfrac{\sqrt{3}}{3}\right), \left(\dfrac{\sqrt{3}}{3}, \infty\right)$

Concave downward: $\left(-\dfrac{\sqrt{3}}{3}, \dfrac{\sqrt{3}}{3}\right)$

Points of inflection: $\left(-\dfrac{\sqrt{3}}{3}, 3\right)$ and $\left(\dfrac{\sqrt{3}}{3}, 3\right)$

30. $f(x) = \dfrac{x+1}{\sqrt{x}}$, Domain: $x > 0$

$f'(x) = \dfrac{x-1}{2x^{3/2}}$

$f''(x) = \dfrac{3-x}{4x^{5/2}}$

Point of inflection: $\left(3, \dfrac{4}{\sqrt{3}}\right) = \left(3, \dfrac{4\sqrt{3}}{3}\right)$

Test intervals:	$0 < x < 3$	$3 < x < \infty$
Sign of $f''(x)$:	$f'' > 0$	$f'' < 0$
Conclusion:	Concave upward	Concave downward

31. $f(x) = \sin\dfrac{x}{2}, 0 \le x \le 4\pi$

$f'(x) = \dfrac{1}{2}\cos\left(\dfrac{x}{2}\right)$

$f''(x) = -\dfrac{1}{4}\sin\left(\dfrac{x}{2}\right)$

$f''(x) = 0$ when $x = 0, 2\pi, 4\pi$.

Point of inflection: $(2\pi, 0)$

Test interval:	$0 < x < 2\pi$	$2\pi < x < 4\pi$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward

32. $f(x) = 2\csc\frac{3x}{2}, 0 < x < 2\pi$

$f'(x) = -3\csc\frac{3x}{2}\cot\frac{3x}{2}$

$f''(x) = \frac{9}{2}\left(\csc^3\frac{3x}{2} + \csc\frac{3x}{2}\cot^2\frac{3x}{2}\right) \neq 0$ for any x in the domain of f.

Concave upward: $\left(0, \frac{2\pi}{3}\right), \left(\frac{4\pi}{3}, 2\pi\right)$

Concave downward: $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$

No point of inflection

33. $f(x) = \sec\left(x - \frac{\pi}{2}\right), 0 < x < 4\pi$

$f'(x) = \sec\left(x - \frac{\pi}{2}\right)\tan\left(x - \frac{\pi}{2}\right)$

$f''(x) = \sec^3\left(x - \frac{\pi}{2}\right) + \sec\left(x - \frac{\pi}{2}\right)\tan^2\left(x - \frac{\pi}{2}\right) \neq 0$ for any x in the domain of f.

Concave upward: $(0, \pi), (2\pi, 3\pi)$

Concave downward: $(\pi, 2\pi), (3\pi, 4\pi)$

No point of inflection

34. $f(x) = \sin x + \cos x, 0 \le x \le 2\pi$

$f'(x) = \cos x - \sin x$

$f''(x) = \sin x - \cos x$

$f''(x) = 0$ when $x = \frac{3\pi}{4}, \frac{7\pi}{4}$.

Test interval:	$0 < x < \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \frac{7\pi}{4}$	$\frac{7\pi}{4} < x < 2\pi$
Sign of $f''(x)$:	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

Points of inflection: $\left(\frac{3\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$

35. $f(x) = 2\sin x + \sin 2x, 0 \le x \le 2\pi$

$f'(x) = 2\cos x + 2\cos 2x$

$f''(x) = -2\sin x - 4\sin 2x = -2\sin x(1 + 4\cos x)$

$f''(x) = 0$ when $x = 0, 1.823, \pi, 4.460$.

Test interval:	$0 < x < 1.823$	$1.823 < x < \pi$	$\pi < x < 4.460$	$4.460 < x < 2\pi$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward	Concave downward	Concave upward

Points of inflection: $(1.823, 1.452), (\pi, 0), (4.46, -1.452)$

36. $f(x) = x + 2\cos x, [0, 2\pi]$

$f'(x) = 1 - 2\sin x$

$f''(x) = -2\cos x$

$f''(x) = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$	$f'' < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

Points of inflection: $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

37. $f(x) = (x - 5)^2$

$f'(x) = 2(x - 5)$

$f''(x) = 2$

Critical number: $x = 5$

$f''(5) > 0$

Therefore, $(5, 0)$ is a relative minimum.

38. $f(x) = -(x - 5)^2$

$f'(x) = -2(x - 5)$

$f''(x) = -2$

Critical number: $x = 5$

$f''(5) < 0$

Therefore, $(5, 0)$ is a relative maximum.

39. $f(x) = 6x - x^2$

$f'(x) = 6 - 2x$

$f''(x) = -2$

Critical number: $x = 3$

$f''(3) < 0$

Therefore, $(3, 9)$ is a relative maximum.

40. $f(x) = x^2 + 3x - 8$

$f'(x) = 2x + 3$

$f''(x) = 2$

Critical number: $x = -\frac{3}{2}$

$f''\left(-\frac{3}{2}\right) > 0$

Therefore, $\left(-\frac{3}{2}, -\frac{41}{4}\right)$ is a relative minimum.

41. $f(x) = x^3 - 3x^2 + 3$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

$f''(x) = 6x - 6 = 6(x - 1)$

Critical numbers: $x = 0, x = 2$

$f''(0) = -6 < 0$

Therefore, $(0, 3)$ is a relative maximum.

$f''(2) = 6 > 0$

Therefore, $(2, -1)$ is a relative minimum.

42. $f(x) = x^3 - 5x^2 + 7x$

$f'(x) = 3x^2 - 10x + 7 = (3x - 7)(x - 1)$

$f''(x) = 6x - 10$

Critical numbers: $x = \frac{7}{3}, 1$

$f''\left(\frac{7}{3}\right) = 4 > 0$

Therefore, $\left(\frac{7}{3}, \frac{49}{27}\right)$ is a relative minimum.

$f''(1) = -4 < 0$

Therefore, $(1, 3)$ is a relative maximum.

43. $f(x) = x^4 - 4x^3 + 2$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$

$f''(x) = 12x^2 - 24x = 12x(x - 2)$

Critical numbers: $x = 0, x = 3$

However, $f''(0) = 0$, so you must use the First Derivative Test. $f'(x) < 0$ on the intervals $(-\infty, 0)$ and $(0, 3)$; so, $(0, 2)$ is not an extremum. $f''(3) > 0$ so $(3, -25)$ is a relative minimum.

44. $f(x) = -x^4 + 4x^3 + 8x^2$

$f'(x) = -4x^3 + 12x^2 + 16x = -4x(x-4)(x+1)$

$f''(x) = -12x^2 + 24x + 16 = -4(3x^2 - 6x - 4)$

Critical numbers: $x = -1, 0, 4$

$f''(-1) = -20$

Therefore $(-1, 3)$ is a relative maximum.

$f''(0) = 16$

Therefore, $(0, 0)$ is a relative minimum.

$f''(4) = -80$

Therefore, $(4, 128)$ is a relative maximum.

45. $g(x) = x^2(6-x)^3$

$g'(x) = x(x-6)^2(12-5x)$

$g''(x) = 4(6-x)(5x^2 - 24x + 18)$

Critical numbers: $x = 0, \frac{12}{5}, 6$

$g''(0) = 432 > 0$

Therefore, $(0, 0)$ is a relative minimum.

$g''\left(\frac{12}{5}\right) = -155.52 < 0$

Therefore, $\left(\frac{12}{5}, 268.7\right)$ is a relative maximum.

$g''(6) = 0$

Test fails. By the First Derivative Test, $(6, 0)$ is not an extremum.

46. $g(x) = -\dfrac{1}{8}(x+2)^2(x-4)^2$

$g'(x) = \dfrac{-(x-4)(x-1)(x+2)}{2}$

$g''(x) = 3 + 3x - \dfrac{3}{2}x^2$

Critical numbers: $x = -2, 1, 4$

$g''(-2) = -9 < 0$

$(-2, 0)$ is a relative maximum.

$g''(1) = \dfrac{9}{2} > 0$

$(1, -10.125)$ is a relative minimum.

$g''(4) = -9 < 0$

$(4, 0)$ is a relative maximum.

47. $f(x) = x^{2/3} - 3$

$f'(x) = \dfrac{2}{3x^{1/3}}$

$f''(x) = -\dfrac{2}{9x^{4/3}}$

Critical number: $x = 0$

However, $f''(0)$ is undefined, so you must use the First Derivative Test. Because $f'(x) < 0$ on $(-\infty, 0)$ and $f'(x) > 0$ on $(0, \infty)$, $(0, -3)$ is a relative minimum.

48. $f(x) = \sqrt{x^2+1}$

$f'(x) = \dfrac{x}{\sqrt{x^2+1}}$

$f''(x) = \dfrac{1}{(x^2+1)^{3/2}}$

Critical number: $x = 0$

$f''(0) = 1 > 0$

Therefore, $(0, 1)$ is a relative minimum.

49. $f(x) = x + \dfrac{4}{x}$

$f'(x) = 1 - \dfrac{4}{x^2} = \dfrac{x^2-4}{x^2}$

$f''(x) = \dfrac{8}{x^3}$

Critical numbers: $x = \pm 2$

$f''(-2) < 0$

Therefore, $(-2, -4)$ is a relative maximum.

$f''(2) > 0$

Therefore, $(2, 4)$ is a relative minimum.

50. $f(x) = \dfrac{x}{x-1}$

$f'(x) = \dfrac{-1}{(x-1)^2}$

There are no critical numbers and $x = 1$ is not in the domain. There are no relative extrema.

51. $f(x) = \cos x - x, 0 \le x \le 4\pi$

$f'(x) = -\sin x - 1 \le 0$

Therefore, f is non-increasing and there are no relative extrema.

52. $f(x) = 2\sin x + \cos 2x, 0 \le x \le 2\pi$

$f'(x) = 2\cos x - 2\sin 2x = 2\cos x - 4\sin x\cos x$

$= 2\cos x(1 - 2\sin x) = 0$ when $x = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}, \dfrac{3\pi}{2}$.

$f''(x) = -2\sin x - 4\cos 2x$

$f''\left(\dfrac{\pi}{6}\right) < 0$

$f''\left(\dfrac{\pi}{2}\right) > 0$

$f''\left(\dfrac{5\pi}{6}\right) < 0$

$f''\left(\dfrac{3\pi}{2}\right) > 0$

Relative maxima: $\left(\dfrac{\pi}{6}, \dfrac{3}{2}\right), \left(\dfrac{5\pi}{6}, \dfrac{3}{2}\right)$

Relative minima: $\left(\dfrac{\pi}{2}, 1\right), \left(\dfrac{3\pi}{2}, -3\right)$

53. $f(x) = 0.2x^2(x - 3)^3, [-1, 4]$

(a) $f'(x) = 0.2x(5x - 6)(x - 3)^2$

$f''(x) = (x - 3)(4x^2 - 9.6x + 3.6)$

$= 0.4(x - 3)(10x^2 - 24x + 9)$

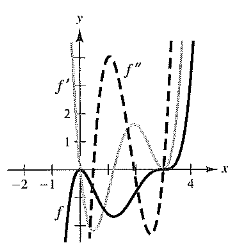
(b) $f''(0) < 0 \Rightarrow (0, 0)$ is a relative maximum.

$f''\left(\frac{6}{5}\right) > 0 \Rightarrow (1.2, -1.6796)$ is a relative minimum.

Points of inflection:

$(3, 0), (0.4652, -0.7048), (1.9348, -0.9049)$

(c)



f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

54. $f(x) = x^2\sqrt{6 - x^2}, \left[-\sqrt{6}, \sqrt{6}\right]$

(a) $f'(x) = \dfrac{3x(4 - x^2)}{\sqrt{6 - x^2}}$

$f'(x) = 0$ when $x = 0, x = \pm 2$.

$f''(x) = \dfrac{6(x^4 - 9x^2 + 12)}{(6 - x^2)^{3/2}}$

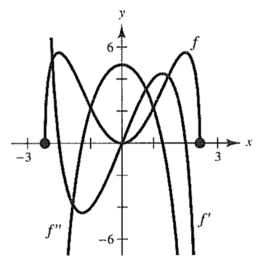
$f''(x) = 0$ when $x = \pm\sqrt{\dfrac{9 - \sqrt{33}}{2}}$.

(b) $f''(0) > 0 \Rightarrow (0, 0)$ is a relative minimum.

$f''(\pm 2) < 0 \Rightarrow \left(\pm 2, 4\sqrt{2}\right)$ are relative maxima.

Points of inflection: $(\pm 1.2758, 3.4035)$

(c)



The graph of f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

55. $f(x) = \sin x - \dfrac{1}{3}\sin 3x + \dfrac{1}{5}\sin 5x, \quad [0, \pi]$

(a) $f'(x) = \cos x - \cos 3x + \cos 5x$

$f'(x) = 0$ when $x = \dfrac{\pi}{6}, x = \dfrac{\pi}{2}, x = \dfrac{5\pi}{6}$.

$f''(x) = -\sin x + 3\sin 3x - 5\sin 5x$

$f''(x) = 0$ when $x = \dfrac{\pi}{6}, x = \dfrac{5\pi}{6}$,

$x \approx 1.1731, x \approx 1.9685$

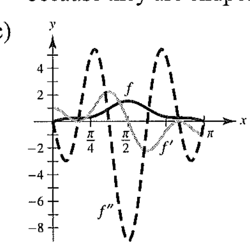
(b) $f''\left(\dfrac{\pi}{2}\right) < 0 \Rightarrow \left(\dfrac{\pi}{2}, 1.53333\right)$ is a relative maximum.

Points of inflection: $\left(\dfrac{\pi}{6}, 0.2667\right), (1.1731, 0.9638),$

$(1.9685, 0.9637), \left(\dfrac{5\pi}{6}, 0.2667\right)$

Note: $(0, 0)$ and $(\pi, 0)$ are not points of inflection because they are endpoints.

(c)



The graph of f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

56. $f(x) = \sqrt{2x}\sin x, [0, 2\pi]$

(a) $f'(x) = \sqrt{2x}\cos x + \dfrac{\sin x}{\sqrt{2x}}$

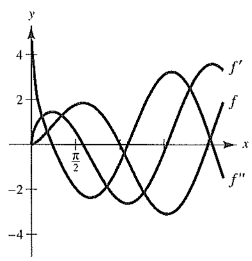
Critical numbers: $x \approx 1.84, 4.82$

$$f''(x) = -\sqrt{2x}\sin x + \frac{\cos x}{\sqrt{2x}} + \frac{\cos x}{\sqrt{2x}} - \frac{\sin x}{2x\sqrt{2x}}$$
$$= \frac{2\cos x}{\sqrt{2x}} - \frac{(4x^2+1)\sin x}{2x\sqrt{2x}}$$
$$= \frac{4x\cos x - (4x^2+1)\sin x}{2x\sqrt{2x}}$$

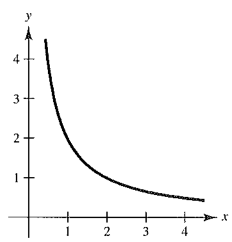
(b) Relative maximum: $(1.84, 1.85)$

Relative minimum: $(4.82, -3.09)$

Points of inflection: $(0.75, 0.83), (3.42, -0.72)$

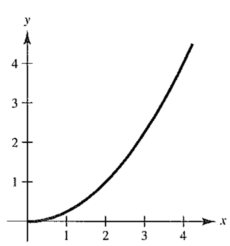
(c) 

f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

57. (a) 

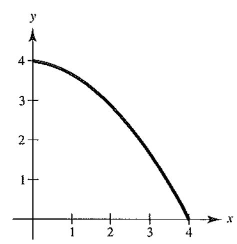
$f' < 0$ means f decreasing

f' increasing means concave upward

(b) 

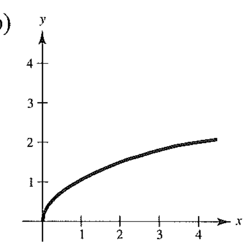
$f' > 0$ means f increasing

f' increasing means concave upward

58. (a) 

$f' < 0$ means f decreasing

f' decreasing means concave downward

(b) 

$f' > 0$ means f increasing

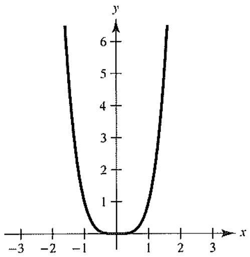
f' decreasing means concave downward

59. Answers will vary. *Sample answer*:

Let $f(x) = x^4$.

$f''(x) = 12x^2$

$f''(0) = 0$, but $(0, 0)$ is not a point of inflection.



60. (a) The rate of change of sales is increasing.

$S'' > 0$

(b) The rate of change of sales is decreasing.

$S' > 0, S'' < 0$

(c) The rate of change of sales is constant.

$S' = C, S'' = 0$

(d) Sales are steady.

$S = C, S' = 0, S'' = 0$

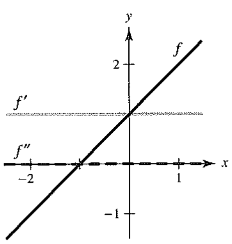
(e) Sales are declining, but at a lower rate.

$S' < 0, S'' > 0$

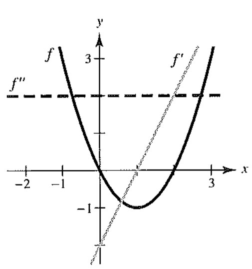
(f) Sales have bottomed out and have started to rise.

$S' > 0$

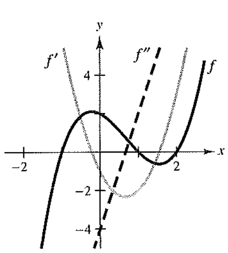
61.



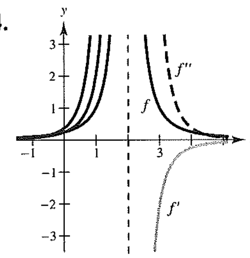
62.



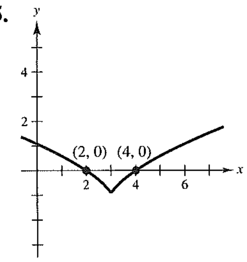
63.



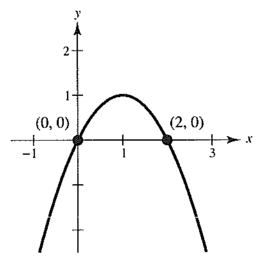
64.



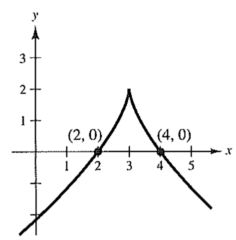
65.



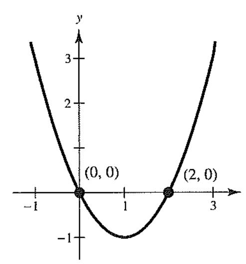
66.



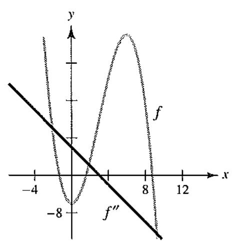
67.



68.



69.



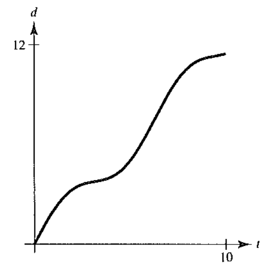
f'' is linear.

f' is quadratic.

f is cubic.

f concave upward on $(-\infty, 3)$, downward on $(3, \infty)$.

70. (a)



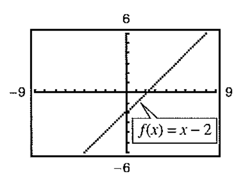
(b) Because the depth d is always increasing, there are no relative extrema. $f'(x) > 0$

(c) The rate of change of d is decreasing until you reach the widest point of the jug, then the rate increases until you reach the narrowest part of the jug's neck, then the rate decreases until you reach the top of the jug.

71. (a) $n = 1$:

$$f(x) = x - 2$$
$$f'(x) = 1$$
$$f''(x) = 0$$

No point of inflection

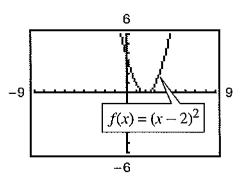


$n = 2$:

$$f(x) = (x-2)^2$$
$$f'(x) = 2(x-2)$$
$$f''(x) = 2$$

No point of inflection

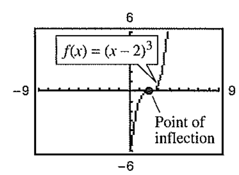
Relative minimum: $(2, 0)$



$n = 3$:

$$f(x) = (x-2)^3$$
$$f'(x) = 3(x-2)^2$$
$$f''(x) = 6(x-2)$$

Point of inflection: $(2, 0)$

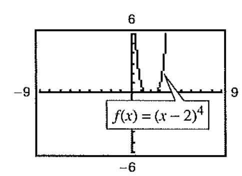


$n = 4$:

$$f(x) = (x-2)^4$$
$$f'(x) = 4(x-2)^3$$
$$f''(x) = 12(x-2)^2$$

No point of inflection

Relative minimum: $(2, 0)$



Conclusion: If $n \geq 3$ and n is odd, then $(2, 0)$ is point of inflection. If $n \geq 2$ and n is even, then $(2, 0)$ is a relative minimum.

(b) Let $f(x) = (x-2)^n$, $f'(x) = n(x-2)^{n-1}$, $f''(x) = n(n-1)(x-2)^{n-2}$.

For $n \geq 3$ and odd, $n - 2$ is also odd and the concavity changes at $x = 2$.

For $n \geq 4$ and even, $n - 2$ is also even and the concavity does not change at $x = 2$.

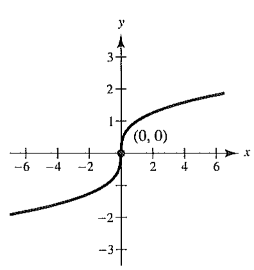
So, $x = 2$ is point of inflection if and only if $n \geq 3$ is odd.

72. (a) $f(x) = \sqrt[3]{x}$

$$f'(x) = \frac{1}{3}x^{-2/3}$$
$$f''(x) = -\frac{2}{9}x^{-5/3}$$

Point of inflection: $(0, 0)$

(b) $f''(x)$ does not exist at $x = 0$.



73. $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: $(3, 3)$

Relative minimum: $(5, 1)$

Point of inflection: $(4, 2)$

$f'(x) = 3ax^2 + 2bx + c,\ f''(x) = 6ax + 2b$

$$\left.\begin{aligned} f(3) &= 27a + 9b + 3c + d = 3 \\ f(5) &= 125a + 25b + 5c + d = 1 \end{aligned}\right\} 98a + 16b + 2c = -2 \Rightarrow 49a + 8b + c = -1$$

$f'(3) = 27a + 6b + c = 0,\ f''(4) = 24a + 2b = 0$

$$\begin{array}{lcl} 49a + 8b + c = -1 & \quad & 24a + 2b = 0 \\ \underline{27a + 6b + c = 0} & & \underline{22a + 2b = -1} \\ 22a + 2b = -1 & & 2a = 1 \end{array}$$

$a = \frac{1}{2},\ b = -6,\ c = \frac{45}{2},\ d = -24$

$f(x) = \frac{1}{2}x^3 - 6x^2 + \frac{45}{2}x - 24$

74. $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: $(2, 4)$

Relative minimum: $(4, 2)$

Point of inflection: $(3, 3)$

$f'(x) = 3ax^2 + 2bx + c,\ f''(x) = 6ax + 2b$

$$\left.\begin{aligned} f(2) &= 8a + 4b + 2c + d = 4 \\ f(4) &= 64a + 16b + 4c + d = 2 \end{aligned}\right\} 56a + 12b + 2c = -2 \Rightarrow 28a + 6b + c = -1$$

$f'(2) = 12a + 4b + c = 0,\ f'(4) = 48a + 8b + c = 0,\ f''(3) = 18a + 2b = 0$

$$\begin{array}{lcl} 28a + 6b + c = -1 & \quad & 18a + 2b = 0 \\ 12a + 4b + c = 0 & & 16a + 2b = -1 \\ 16a + 2b = -1 & & 2a = 1 \end{array}$$

$a = \frac{1}{2},\ b = -\frac{9}{2},\ c = 12,\ d = -6$

$f(x) = \frac{1}{2}x^3 - \frac{9}{2}x^2 + 12x - 6$

75. $f(x) = ax^3 + bx^2 + cx + d$

Maximum: $(-4, 1)$

Minimum: $(0, 0)$

(a) $f'(x) = 3ax^2 + 2bx + c, \qquad f''(x) = 6ax + 2b$

$$\begin{aligned} f(0) &= 0 \Rightarrow d = 0 \\ f(-4) &= 1 \Rightarrow -64a + 16b - 4c = 1 \\ f'(-4) &= 0 \Rightarrow 48a - 8b + c = 0 \\ f'(0) &= 0 \Rightarrow c = 0 \end{aligned}$$

Solving this system yields $a = \frac{1}{32}$ and $b = 6a = \frac{3}{16}$.

$f(x) = \frac{1}{32}x^3 + \frac{3}{16}x^2$

(b) The plane would be descending at the greatest rate at the point of inflection.

$f''(x) = 6ax + 2b = \frac{3}{16}x + \frac{3}{8} = 0 \Rightarrow x = -2.$

Two miles from touchdown.

76. (a) line OA: $y = -0.06x$ slope: -0.06

line CB: $y = 0.04x + 50$ slope: 0.04

$f(x) = ax^3 + bx^2 + cx + d$

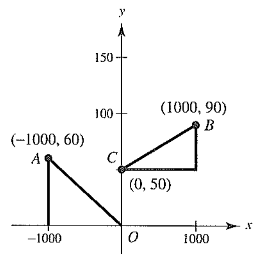
$f'(x) = 3ax^2 + 2bx + c$

$(-1000, 60)$: $60 = (-1000)^3 a + (1000)^2 b - 1000c + d$

$-0.06 = (1000)^2\, 3a - 2000b + c$

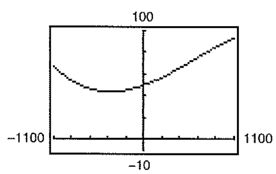
$(1000, 90)$: $90 = (1000)^3 a + (1000)^2 b + 1000c + d$

$0.04 = (1000)^2\, 3a + 2000b + c$

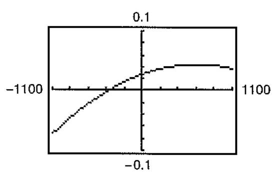


The solution to this system of four equations is $a = -1.25 \times 10^{-8}$, $b = 0.000025$, $c = 0.0275$, and $d = 50$.

(b) $y = -1.25 \times 10^{-8}x^3 + 0.000025x^2 + 0.0275x + 50$



(c)



(d) The steepest part of the road is 6% at the point A.

77. $D = 2x^4 - 5Lx^3 + 3L^2x^2$

$D' = 8x^3 - 15Lx^2 + 6L^2x = x(8x^2 - 15Lx + 6L^2) = 0$

$$x = 0 \text{ or } x = \frac{15L \pm \sqrt{33}L}{16} = \left(\frac{15 \pm \sqrt{33}}{16}\right)L$$

By the Second Derivative Test, the deflection is maximum when

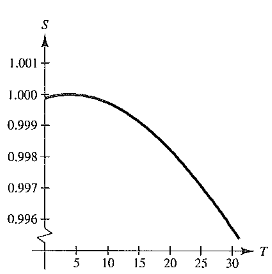
$$x = \left(\frac{15 - \sqrt{33}}{16}\right)L \approx 0.578L.$$

78. $S = \dfrac{5.755T^3}{10^8} - \dfrac{8.521T^2}{10^6} + \dfrac{0.654T}{10^4} + 0.99987,$

$0 < T < 25$

(a) The maximum occurs when $T \approx 4°$ and $S \approx 0.999999$.

(b)



(c) $S(20°) \approx 0.9982$

79. $C = 0.5x^2 + 15x + 5000$

$$\overline{C} = \frac{C}{x} = 0.5x + 15 + \frac{5000}{x}$$

$\overline{C}$ = average cost per unit

$$\frac{d\overline{C}}{dx} = 0.5 - \frac{5000}{x^2} = 0 \text{ when } x = 100$$

By the First Derivative Test, $\overline{C}$ is minimized when $x = 100$ units.

80. $C = 2x + \dfrac{300{,}000}{x}$

$$C' = 2 - \frac{300{,}000}{x^2} = 0 \text{ when } x = 100\sqrt{15} \approx 387$$

By the First Derivative Test, is C minimized when $x \approx 387$ units.

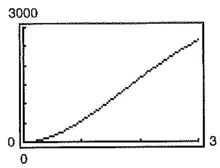
81. $S = \dfrac{5000t^2}{8 + t^2}, 0 \le t \le 3$

(a)

t	0.5	1	1.5	2	2.5	3
S	151.5	555.6	1097.6	1666.7	2193.0	2647.1

Increasing at greatest rate when $1.5 < t < 2$

(b)



Increasing at greatest rate when $t \approx 1.5$.

(c)

$$S = \frac{5000t^2}{8 + t^2}$$

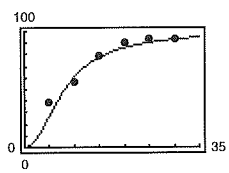
$$S'(t) = \frac{80{,}000t}{\left(8 + t^2\right)^2}$$

$$S''(t) = \frac{80{,}000\left(8 - 3t^2\right)}{\left(8 + t^2\right)^3}$$

$$S''(t) = 0 \text{ for } t = \pm\sqrt{\frac{8}{3}}. \text{ So, } t = \frac{2\sqrt{6}}{3} \approx 1.633 \text{ yrs.}$$

82. $S = \dfrac{100t^2}{65 + t^2}, t > 0$

(a)



(b) $S'(t) = \dfrac{13{,}000t}{\left(65 + t^2\right)^2}$

$$S''(t) = \frac{13{,}000\left(65 - 3t^2\right)}{\left(65 + t^2\right)^3} = 0 \Rightarrow t = 4.65$$

S is concave upwards on $(0, 4.65)$, concave downwards on $(4.65, 30)$.

(c) $S'(t) > 0$ for $t > 0$.

As t increases, the speed increases, but at a slower rate.

83. $f(x) = 2(\sin x + \cos x), \quad f\left(\frac{\pi}{4}\right) = 2\sqrt{2}$

$f'(x) = 2(\cos x - \sin x), \quad f'\left(\frac{\pi}{4}\right) = 0$

$f''(x) = 2(-\sin x - \cos x), \quad f''\left(\frac{\pi}{4}\right) = -2\sqrt{2}$

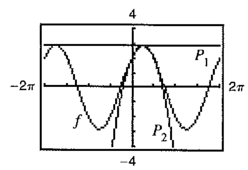
$P_1(x) = 2\sqrt{2} + 0\left(x - \frac{\pi}{4}\right) = 2\sqrt{2}$

$P_1'(x) = 0$

$P_2(x) = 2\sqrt{2} + 0\left(x - \frac{\pi}{4}\right) + \frac{1}{2}\left(-2\sqrt{2}\right)\left(x - \frac{\pi}{4}\right)^2 = 2\sqrt{2} - \sqrt{2}\left(x - \frac{\pi}{4}\right)^2$

$P_2'(x) = -2\sqrt{2}\left(x - \frac{\pi}{4}\right)$

$P_2''(x) = -2\sqrt{2}$



The values of f, P_1, P_2, and their first derivatives are equal at $x = \pi/4$. The values of the second derivatives of f and P_2 are equal at $x = \pi/4$. The approximations worsen as you move away from $x = \pi/4$.

84. $f(x) = 2(\sin x + \cos x), \quad f(0) = 2$

$f'(x) = 2(\cos x - \sin x), \quad f'(0) = 2$

$f''(x) = 2(-\sin x - \cos x), \quad f''(0) = -2$

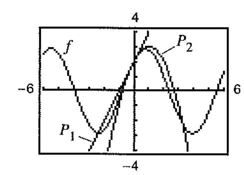
$P_1(x) = 2 + 2(x - 0) = 2(1 + x)$

$P_1'(x) = 2$

$P_2(x) = 2 + 2(x - 0) + \frac{1}{2}(-2)(x - 0)^2 = 2 + 2x - x^2$

$P_2'(x) = 2 - 2x$

$P_2''(x) = -2$



The values of f, P_1, P_2, and their first derivatives are equal at $x = 0$. The values of the second derivatives of f and P_2 are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

85. $f(x) = \sqrt{1 - x}, \quad f(0) = 1$

$f'(x) = -\frac{1}{2\sqrt{1 - x}}, \quad f'(0) = -\frac{1}{2}$

$f''(x) = -\frac{1}{4(1 - x)^{3/2}}, \quad f''(0) = -\frac{1}{4}$

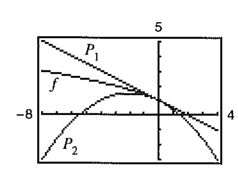
$P_1(x) = 1 + \left(-\frac{1}{2}\right)(x - 0) = 1 - \frac{x}{2}$

$P_1'(x) = -\frac{1}{2}$

$P_2(x) = 1 + \left(-\frac{1}{2}\right)(x - 0) + \frac{1}{2}\left(-\frac{1}{4}\right)(x - 0)^2 = 1 - \frac{x}{2} - \frac{x^2}{8}$

$P_2'(x) = -\frac{1}{2} - \frac{x}{4}$

$P_2''(x) = -\frac{1}{4}$



The values of f, P_1, P_2, and their first derivatives are equal at $x = 0$. The values of the second derivatives of f and P_2 are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

86. $f(x) = \dfrac{\sqrt{x}}{x-1}, \qquad f(2) = \sqrt{2}$

$f'(x) = \dfrac{-(x+1)}{2\sqrt{x}(x-1)^2}, \qquad f'(2) = -\dfrac{3}{2\sqrt{2}} = -\dfrac{3\sqrt{2}}{4}$

$f''(x) = \dfrac{3x^2+6x-1}{4x^{3/2}(x-1)^3}, \qquad f''(2) = \dfrac{23}{8\sqrt{2}} = \dfrac{23\sqrt{2}}{16}$

$P_1(x) = \sqrt{2} + \left(-\dfrac{3\sqrt{2}}{4}\right)(x-2) = -\dfrac{3\sqrt{2}}{4}x + \dfrac{5\sqrt{2}}{2}$

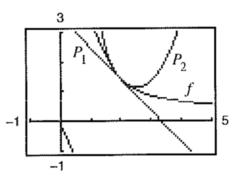
$P_1'(x) = -\dfrac{3\sqrt{2}}{4}$

$P_2(x) = \sqrt{2} + \left(-\dfrac{3\sqrt{2}}{4}\right)(x-2) + \dfrac{1}{2}\left(\dfrac{23\sqrt{2}}{16}\right)(x-2)^2 = \sqrt{2} - \dfrac{3\sqrt{2}}{4}(x-2) + \dfrac{23\sqrt{2}}{32}(x-2)^2$

$P_2'(x) = -\dfrac{3\sqrt{2}}{4} + \dfrac{23\sqrt{2}}{16}(x-2)$

$P_2''(x) = \dfrac{23\sqrt{2}}{16}$

The values of f, P_1, P_2 and their first derivatives are equal at $x = 2$. The values of the second derivatives of f and P_2 are equal at $x = 2$. The approximations worsen as you move away from $x = 2$.



87. $f(x) = x\sin\left(\dfrac{1}{x}\right)$

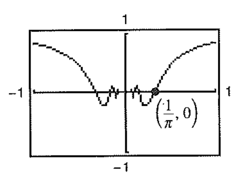
$f'(x) = x\left[-\dfrac{1}{x^2}\cos\left(\dfrac{1}{x}\right)\right] + \sin\left(\dfrac{1}{x}\right) = -\dfrac{1}{x}\cos\left(\dfrac{1}{x}\right) + \sin\left(\dfrac{1}{x}\right)$

$f''(x) = -\dfrac{1}{x}\left[\dfrac{1}{x^2}\sin\left(\dfrac{1}{x}\right)\right] + \dfrac{1}{x^2}\cos\left(\dfrac{1}{x}\right) - \dfrac{1}{x^2}\cos\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^3}\sin\left(\dfrac{1}{x}\right) = 0$

$x = \dfrac{1}{\pi}$

Point of inflection: $\left(\dfrac{1}{\pi}, 0\right)$

When $x > 1/\pi$, $f'' < 0$, so the graph is concave downward.



88. $f(x) = x(x-6)^2 = x^3 - 12x^2 + 36x$

$f'(x) = 3x^2 - 24x + 36 = 3(x-2)(x-6) = 0$

$f''(x) = 6x - 24 = 6(x-4) = 0$

Relative extrema: $(2, 32)$ and $(6, 0)$

Point of inflection $(4, 16)$ is midway between the relative extrema of f.