10. $y = \frac{3}{2}x^{2/3} + 4$

$y' = x^{-1/3}, \quad 1 \le x \le 27$

$$s = \int_1^{27} \sqrt{1 + \left(\frac{1}{x^{1/3}}\right)^2}\, dx$$

$$= \int_1^{27} \sqrt{\frac{x^{2/3} + 1}{x^{2/3}}}\, dx$$

$$= \frac{3}{2}\int_1^{27} \sqrt{x^{2/3} + 1}\left(\frac{2}{3x^{1/3}}\right) dx$$

$$= \left[\frac{3}{2} \cdot \frac{2}{3}\left(x^{2/3} + 1\right)^{3/2}\right]_1^{27}$$

$$= 10^{3/2} - 2^{3/2} \approx 28.794$$

11. $y = \ln(\sin x), \quad \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

$$y' = \frac{1}{\sin x}\cos x = \cot x$$

$$1 + (y')^2 = 1 + \cot^2 x = \csc^2 x$$

$$s = \int_{\pi/4}^{3\pi/4} \csc x\, dx$$

$$= \left[\ln\left|\csc x - \cot x\right|\right]_{\pi/4}^{3\pi/4}$$

$$= \ln\left(\sqrt{2} + 1\right) - \ln\left(\sqrt{2} - 1\right) \approx 1.763$$

12. $y = \ln(\cos x), \quad 0 \le x \le \frac{\pi}{3}$

$$y' = \frac{-\sin x}{\cos x} = -\tan x$$

$$1 + (y')^2 = 1 + \tan^2 x = \sec^2 x$$

$$s = \int_0^{\pi/3} \sqrt{\sec^2 x}\, dx$$

$$= \int_0^{\pi/3} \sec x\, dx$$

$$= \ln\left|\sec x + \tan x\right|\Big]_0^{\pi/3}$$

$$= \ln\left(2 + \sqrt{3}\right) \approx 1.3170$$

13. $y = \frac{1}{2}\left(e^x + e^{-x}\right)$

$$y' = \frac{1}{2}\left(e^x - e^{-x}\right), \quad [0, 2]$$

$$1 + (y')^2 = \left[\frac{1}{2}\left(e^x + e^{-x}\right)\right]^2, \quad [0, 2]$$

$$s = \int_0^2 \sqrt{\left[\frac{1}{2}\left(e^x + e^{-x}\right)\right]^2}\, dx$$

$$= \frac{1}{2}\int_0^2 \left(e^x + e^{-x}\right) dx$$

$$= \frac{1}{2}\left[e^x - e^{-x}\right]_0^2 = \frac{1}{2}\left(e^2 - \frac{1}{e^2}\right) \approx 3.627$$

14. $y = \ln\left(\frac{e^x + 1}{e^x - 1}\right) = \ln\left(e^x + 1\right) - \ln\left(e^x - 1\right)$

$$\frac{dy}{dx} = \frac{e^x}{e^x + 1} - \frac{e^x}{e^x - 1} = \frac{-2e^x}{e^{2x} - 1} = \frac{2e^x}{1 - e^{2x}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{4e^{2x}}{1 - 2e^{2x} + e^{4x}}$$

$$= \frac{1 + 2e^{2x} + e^{4x}}{\left(1 - e^{2x}\right)^2} = \left(\frac{1 + e^{2x}}{1 - e^{2x}}\right)^2$$

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{\ln 2}^{\ln 3} \frac{1 + e^{2x}}{e^{2x} - 1}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{e^x + e^{-x}}{e^x - e^{-x}}\, dx = \int_{\ln 2}^{\ln 3} \coth x\, dx$$

$$= \ln\left(\sinh(x)\right)\Big]_{\ln 2}^{\ln 3} = \ln\left(\frac{4}{3}\right) - \ln\left(\frac{3}{4}\right)$$

$$= \ln\left(\frac{4/3}{3/4}\right) = \ln\frac{16}{9} - 2\ln\left(\frac{4}{3}\right) \approx 0.57536$$

15. $x = \frac{1}{3}\left(y^2 + 2\right)^{3/2}, \quad 0 \le y \le 4$

$$\frac{dx}{dy} = y\left(y^2 + 2\right)^{1/2}$$

$$s = \int_0^4 \sqrt{1 + y^2\left(y^2 + 2\right)}\, dy$$

$$= \int_0^4 \sqrt{y^4 + 2y^2 + 1}\, dy$$

$$= \int_0^4 \left(y^2 + 1\right) dy$$

$$= \left[\frac{y^3}{3} + y\right]_0^4 = \frac{64}{3} + 4 = \frac{76}{3}$$

16. $x = \frac{1}{3}\sqrt{y}(y - 3), \quad 1 \le y \le 4$

$$x = \frac{1}{3}\left(y^{3/2} - 3y^{1/2}\right)$$

$$\frac{dx}{dy} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{4}y + \frac{1}{4}y^{-1} - \frac{1}{2}$$

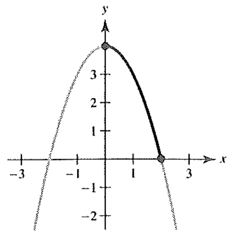
$$= \frac{1}{4}\left(y + 2 + y^{-1}\right) = \frac{1}{4}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)^2$$

$$s = \int_1^4 \frac{1}{2}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right) dy$$

$$= \left[\frac{1}{2}\left(\frac{2}{3}y^{3/2} + 2y^{1/2}\right)\right]_1^4$$

$$= \frac{1}{2}\left(\frac{16}{3} + 4\right) - \frac{1}{2}\left(\frac{2}{3} + 2\right) = \frac{10}{3}$$

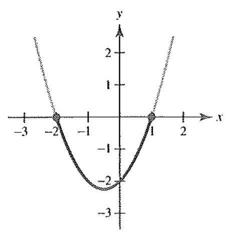
17. (a) $y = 4 - x^2, \quad 0 \le x \le 2$



(b)
$$y' = -2x$$
$$1 + (y')^2 = 1 + 4x^2$$
$$L = \int_0^2 \sqrt{1 + 4x^2}\, dx$$

(c) $L \approx 4.647$

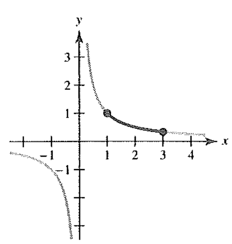
18. (a) $y = x^2 + x - 2, \quad -2 \le x \le 1$



(b)
$$y' = 2x + 1$$
$$1 + (y')^2 = 1 + 4x^2 + 4x + 1$$
$$L = \int_{-2}^1 \sqrt{2 + 4x + 4x^2}\, dx$$

(c) $L \approx 5.653$

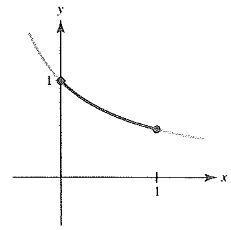
19. (a) $y = \dfrac{1}{x}, \quad 1 \le x \le 3$



(b)
$$y' = -\frac{1}{x^2}$$
$$1 + (y')^2 = 1 + \frac{1}{x^4}$$
$$L = \int_1^3 \sqrt{1 + \frac{1}{x^4}}\, dx$$

(c) $L \approx 2.147$

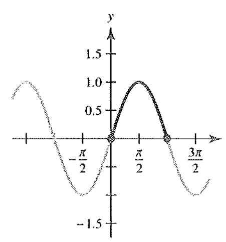
20. (a) $y = \dfrac{1}{1 + x}, \quad 0 \le x \le 1$



(b)
$$y' = -\frac{1}{(1 + x)^2}$$
$$1 + (y')^2 = 1 + \frac{1}{(1 + x)^4}$$
$$L = \int_0^1 \sqrt{1 + \frac{1}{(1 + x)^4}}\, dx$$

(c) $L \approx 1.132$

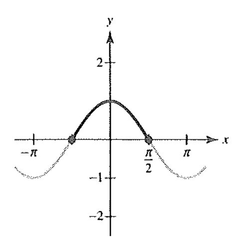
21. (a) $y = \sin x, \quad 0 \le x \le \pi$



(b)
$$y' = \cos x$$
$$1 + (y')^2 = 1 + \cos^2 x$$
$$L = \int_0^\pi \sqrt{1 + \cos^2 x}\, dx$$

(c) $L \approx 3.820$

22. (a) $y = \cos x, \quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$



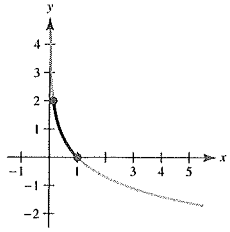
(b)
$$y' = -\sin x$$
$$1 + (y')^2 = 1 + \sin^2 x$$
$$L = \int_{-\pi/2}^{\pi/2} \sqrt{1 + \sin^2 x}\, dx$$

(c) 3.820

23. (a) $x = e^{-y}, \quad 0 \le y \le 2$

$y = -\ln x$

$1 \ge x \ge e^{-2} \approx 0.135$



(b) $y' = -\dfrac{1}{x}$

$1 + (y')^2 = 1 + \dfrac{1}{x^2}$

$L = \displaystyle\int_{e^{-2}}^{1} \sqrt{1 + \frac{1}{x^2}}\, dx$

(c) $L \approx 2.221$

Alternatively, you can do all the computations with respect to y.

(a) $x = e^{-y}, \quad 0 \le y \le 2$

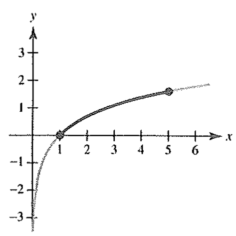
(b) $\dfrac{dx}{dy} = -e^{-y}$

$1 + \left(\dfrac{dx}{dy}\right)^2 = 1 + e^{-2y}$

$L = \displaystyle\int_0^2 \sqrt{1 + e^{-2y}}\, dy$

(c) $L \approx 2.221$

24. (a) $y = \ln x, \quad 1 \le x \le 5$



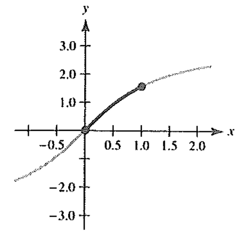
(b) $y' = \dfrac{1}{x}$

$1 + (y')^2 = 1 + \dfrac{1}{x^2}$

$L = \displaystyle\int_1^5 \sqrt{1 + \frac{1}{x^2}}\, dx$

(c) $L \approx 4.367$

25. (a) $y = 2 \arctan x, \quad 0 \le x \le 1$



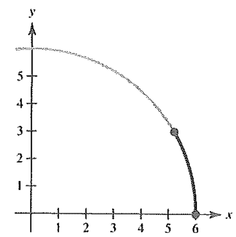
(b) $y' = \dfrac{2}{1 + x^2}$

$L = \displaystyle\int_0^1 \sqrt{1 + \frac{4}{\left(1 + x^2\right)^2}}\, dx$

(c) $L \approx 1.871$

26. (a) $x = \sqrt{36 - y^2}, \quad 0 \le y \le 3$

$y = \sqrt{36 - x^2}, \quad 3\sqrt{3} \le x \le 6$



(b) $\dfrac{dx}{dy} = \dfrac{1}{2}\left(36 - y^2\right)^{-1/2}(-2y)$

$= \dfrac{-y}{\sqrt{36 - y^2}}$

$L = \displaystyle\int_0^3 \sqrt{1 + \frac{y^2}{36 - y^2}}\, dy$

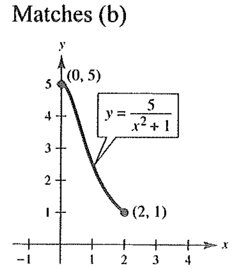
$= \displaystyle\int_0^3 \frac{6}{\sqrt{36 - y^2}}\, dy$

(c) $L \approx 3.142 \quad (\pi)$

27. $\displaystyle\int_0^2 \sqrt{1 + \left[\frac{d}{dx}\left(\frac{5}{x^2 + 1}\right)\right]^2}\, dx$

$s \approx 5$

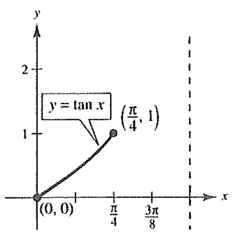
Matches (b)



28. $\int_0^{\pi/4} \sqrt{1 + \left[\frac{d}{dx}(\tan x)\right]^2}\,dx$

$s \approx 1$

Matches (e)



29. $y = x^3, \quad [0, 4]$

(a) $d = \sqrt{(4-0)^2 + (64-0)^2} \approx 64.125$

(b) $d = \sqrt{(1-0)^2 + (1-0)^2} + \sqrt{(2-1)^2 + (8-1)^2} + \sqrt{(3-2)^2 + (27-8)^2} + \sqrt{(4-3)^2 + (64-27)^2}$

≈ 64.525

(c) $s = \int_0^4 \sqrt{1 + (3x^2)^2}\,dx = \int_0^4 \sqrt{1 + 9x^4}\,dx \approx 64.666$ (Simpson's Rule, $n = 10$)

(d) 64.672

30. $f(x) = (x^2 - 4)^2, \quad [0, 4]$

(a) $d = \sqrt{(4-0)^2 + (144-16)^2} \approx 128.062$

(b) $d = \sqrt{(1-0)^2 + (9-16)^2} + \sqrt{(2-1)^2 + (0-9)^2} + \sqrt{(3-2)^2 + (25-0)^2} + \sqrt{(4-3)^2 + (144-25)^2}$

≈ 160.151

(c) $s = \int_0^4 \sqrt{1 + \left[4x(x^2-4)\right]^2}\,dx \approx 159.087$

(d) 160.287

31.

$$y = 20\cosh\frac{x}{20}, \quad -20 \le x \le 20$$

$$y' = \sinh\frac{x}{20}$$

$$1 + (y')^2 = 1 + \sinh^2\frac{x}{20} = \cosh^2\frac{x}{20}$$

$$L = \int_{-20}^{20} \cosh\frac{x}{20}\,dx = 2\int_0^{20} \cosh\frac{x}{20}\,dx = \left[2(20)\sinh\frac{x}{20}\right]_0^{20} = 40\sinh(1) \approx 47.008 \text{ m}$$

32.

$$y = 31 - 10\left(e^{x/20} + e^{-x/20}\right)$$

$$y' = -\frac{1}{2}\left(e^{x/20} - e^{-x/20}\right)$$

$$1 + (y')^2 = 1 + \frac{1}{4}\left(e^{x/10} - 2 + e^{-x/10}\right) = \left[\frac{1}{2}\left(e^{x/20} + e^{-x/20}\right)\right]^2$$

$$s = \int_{-20}^{20} \sqrt{\left[\frac{1}{2}\left(e^{x/20} + e^{-x/20}\right)\right]^2}\,dx = \frac{1}{2}\int_{-20}^{20} \left(e^{x/20} + e^{-x/20}\right)dx = \left[10\left(e^{x/20} - e^{-x/20}\right)\right]_{-20}^{20} = 20\left(e - \frac{1}{e}\right) \approx 47 \text{ ft}$$

So, there are $100(47) = 4700$ square feet of roofing on the barn.

33. $y = 693.8597 - 68.7672 \cosh 0.0100333x$

$y' = -0.6899619478 \sinh 0.0100333x$

$$s = \int_{-299.2239}^{299.2239} \sqrt{1 + (-0.6899619478 \sinh 0.0100333x)^2}\, dx \approx 1480$$

(Use Simpson's Rule with $n = 100$ or a graphing utility.)

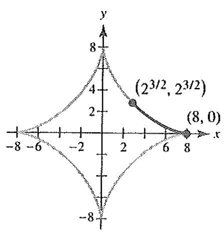
34. $x^{2/3} + y^{2/3} = 4$

$y^{2/3} = 4 - x^{2/3}$

$y = \left(4 - x^{2/3}\right)^{3/2}$

$$y' = \frac{3}{2}\left(4 - x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = \frac{-\left(4 - x^{2/3}\right)^{1/2}}{x^{1/3}}$$

In order to avoid division by 0, compute the arc length for $2^{3/2} \le x \le 8$, and multiply the answer by 8, as indicated in the figure.



$$1 + (y')^2 = 1 + \frac{4 - x^{2/3}}{x^{2/3}}, \quad 2^{3/2} \le x \le 8$$

$$= \frac{4}{x^{2/3}}$$

$$s = 8\int_{2^{3/2}}^{8} \sqrt{\frac{4}{x^{2/3}}}\, dx$$

$$= 16\int_{2^{3/2}}^{8} x^{-1/3}\, dx$$

$$= 16\left[\frac{3}{2}x^{2/3}\right]_{2^{3/2}}^{8}$$

$$= 24(4 - 2) = 48$$

35. $y = \sqrt{9 - x^2}$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$1 + (y')^2 = \frac{9}{9 - x^2}$$

$$s = \int_0^2 \sqrt{\frac{9}{9 - x^2}}\, dx = \int_0^2 \frac{3}{\sqrt{9 - x^2}}\, dx$$

$$= \left[3 \arcsin \frac{x}{3}\right]_0^2 = 3\left(\arcsin \frac{2}{3} - \arcsin 0\right)$$

$$= 3 \arcsin \frac{2}{3} \approx 2.1892$$

36. $y = \sqrt{25 - x^2}$

$$y' = \frac{-x}{\sqrt{25 - x^2}}$$

$$1 + (y')^2 = \frac{25}{25 - x^2}$$

$$s = \int_{-3}^{4} \sqrt{\frac{25}{25 - x^2}}\, dx = \int_{-3}^{4} \frac{5}{\sqrt{25 - x^2}}\, dx$$

$$= \left[5 \arcsin \frac{x}{5}\right]_{-3}^{4} = 5\left[\arcsin \frac{4}{5} - \arcsin\left(-\frac{3}{5}\right)\right]$$

$$\approx 7.8540$$

$$\frac{1}{4}\left[2\pi(5)\right] \approx 7.8540 = s$$

37. $y = \dfrac{x^3}{3}$

$y' = x^2, \quad [0, 3]$

$$S = 2\pi \int_0^3 \frac{x^3}{3}\sqrt{1 + x^4}\, dx$$

$$= \frac{\pi}{6}\int_0^3 \left(1 + x^4\right)^{1/2}\left(4x^3\right) dx$$

$$= \left[\frac{\pi}{9}\left(1 + x^4\right)^{3/2}\right]_0^3$$

$$= \frac{\pi}{9}\left(82\sqrt{82} - 1\right) \approx 258.85$$

38. $y = 2\sqrt{x}$

$y' = \dfrac{1}{\sqrt{x}}, \quad [4, 9]$

$$S = 2\pi \int_4^9 2\sqrt{x}\sqrt{1 + \frac{1}{x}}\, dx$$

$$= 4\pi \int_4^9 \sqrt{x + 1}\, dx$$

$$= \left[\frac{8}{3}\pi(x + 1)^{3/2}\right]_4^9$$

$$= \frac{8\pi}{3}\left(10^{3/2} - 5^{3/2}\right) \approx 171.258$$

39.
$$y = \frac{x^3}{6} + \frac{1}{2x}$$
$$y' = \frac{x^2}{2} - \frac{1}{2x^2}$$
$$1 + (y')^2 = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2, \quad [1, 2]$$
$$S = 2\pi\int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx$$
$$= 2\pi\int_1^2 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3}\right) dx$$
$$= 2\pi\left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2}\right]_1^2 = \frac{47\pi}{16}$$

40.
$$y = \frac{x}{2}$$
$$y' = \frac{1}{2}$$
$$1 + (y')^2 = \frac{5}{4}, \quad [0, 6]$$
$$S = 2\pi\int_0^6 \frac{x}{2}\sqrt{\frac{5}{4}}\, dx$$
$$= \left[\frac{2\pi\sqrt{5}}{8}x^2\right]_0^6 = 9\sqrt{5}\pi$$

41.
$$y = \sqrt{4 - x^2}$$
$$y' = \frac{1}{2}(4 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{4 - x^2}}, \ -1 \le x \le 1$$
$$1 + (y')^2 = 1 + \frac{x^2}{4 - x^2} = \frac{4}{4 - x^2}$$
$$S = 2\pi\int_{-1}^1 \sqrt{4 - x^2}\cdot\sqrt{\frac{4}{4 - x^2}}\, dx$$
$$= 4\pi\int_{-1}^1 dx = 4\pi[x]_{-1}^1 = 8\pi$$

42. $y = \sqrt{9 - x^2}, \quad -2 \le x \le 2$
$$y' = \frac{1}{2}(9 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{9 - x^2}}$$
$$1 + (y')^2 + 1 + \frac{x^2}{(9 - x^2)} = \frac{9}{9 - x^2}$$
$$S = 2\pi\int_{-2}^2 \sqrt{9 - x^2}\,\frac{3}{\sqrt{9 - x^2}}\, dx = 2\pi\int_{-2}^2 3\, dx$$
$$= 2\pi[3x]_{-2}^2 = 24\pi$$

43. $y = \sqrt[3]{x} + 2$
$$y' = \frac{1}{3x^{2/3}}, \quad [1, 8]$$
$$S = 2\pi\int_1^8 x\sqrt{1 + \frac{1}{9x^{4/3}}}\, dx$$
$$= \frac{2\pi}{3}\int_1^8 x^{1/3}\sqrt{9x^{4/3} + 1}\, dx$$
$$= \frac{\pi}{18}\int_1^8 (9x^{4/3} + 1)^{1/2}(12x^{1/3})\, dx$$
$$= \left[\frac{\pi}{27}(9x^{4/3} + 1)^{3/2}\right]_1^8$$
$$= \frac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right) \approx 199.48$$

44. $y = 9 - x^2, \quad [0, 3]$
$$y' = -2x$$
$$S = 2\pi\int_0^3 x\sqrt{1 + 4x^2}\, dx$$
$$= \frac{\pi}{4}\int_0^3 (1 + 4x^2)^{1/2}(8x)\, dx$$
$$= \left[\frac{\pi}{6}(1 + 4x^2)^{3/2}\right]_0^3 = \frac{\pi}{6}(37^{3/2} - 1) \approx 117.319$$

45.
$$y = 1 - \frac{x^2}{4}$$
$$y' = -\frac{x}{2}, \quad 0 \le x \le 2$$
$$1 + (y')^2 = 1 + \frac{x^2}{4} = \frac{4 + x^2}{4}$$
$$S = 2\pi\int_0^2 x\sqrt{\frac{4 + x^2}{4}}\, dx$$
$$= \pi\int_0^2 x\sqrt{4 + x^2}\, dx$$
$$= \frac{1}{2}\pi\int_0^2 (4 + x^2)^{1/2}(2x)\, dx$$
$$= \frac{1}{2}\pi\left[\frac{2}{3}(4 + x^2)^{3/2}\right]_0^2$$
$$= \frac{\pi}{3}(8^{3/2} - 4^{3/2})$$
$$= \frac{\pi}{3}(16\sqrt{2} - 8) \approx 15.318$$

46. $y = 2x + 5$
$$y' = 2, \quad 1 \le x \le 4$$
$$V = \int_1^4 2\pi x\sqrt{1 + 4}\, dx$$
$$= 2\pi\sqrt{5}\left[\frac{x^2}{2}\right]_1^4$$
$$= 2\pi\sqrt{5}\left(8 - \frac{1}{2}\right) = 15\sqrt{5}\pi$$

47. $y = \sin x$

$y' = \cos x, \quad [0, \pi]$

$$S = 2\pi \int_0^{\pi} \sin x\sqrt{1 + \cos^2 x}\, dx \approx 14.4236$$

48.
$$y = \ln x$$
$$y' = \frac{1}{x}$$
$$1 + (y')^2 = \frac{x^2 + 1}{x^2}, \quad [1, e]$$
$$S = 2\pi \int_1^e x\sqrt{\frac{x^2+1}{x^2}}\, dx = 2\pi \int_1^e \sqrt{x^2 + 1}\, dx$$
$$\approx 22.943$$

49. A rectifiable curve is one that has a finite arc length.

50. The precalculus formula is the distance formula between two points. The representative element is

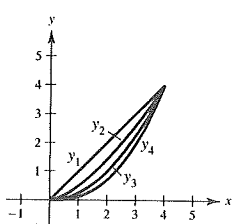
$$\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\, \Delta xi.$$

51. The precalculus formula is the surface area formula for the lateral surface of the frustum of a right circular cone. The formula is $S = 2\pi rL$, where $r = \frac{1}{2}(r_1 + r_2)$, which is the average radius of the frustum, and L is the length of a line segment on the frustum. The representative element is

$$2\pi f(d_i)\sqrt{\Delta x_i^2 + \Delta y_i^2} = 2\pi f(d_i)\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\, \Delta x_i.$$

52. The surface of revolution given by f_1 will be larger. $r(x)$ is larger for f_1.

53. (a)



(b) y_1, y_2, y_3, y_4

(c) $y_1' = 1, \quad s_1 = \int_0^4 \sqrt{2}\, dx \approx 5.657$

$$y_2' = \frac{3}{4}x^{1/2}, \quad s_2 = \int_0^4 \sqrt{1 + \frac{9x}{16}}\, dx \approx 5.759$$

$$y_3' = \frac{1}{2}x, \quad s_3 = \int_0^4 \sqrt{1 + \frac{x^2}{4}}\, dx \approx 5.916$$

$$y_4' = \frac{5}{16}x^{3/2}, \quad s_4 = \int_0^4 \sqrt{1 + \frac{25}{256}x^3}\, dx \approx 6.063$$

54. Let $y = \ln x, 1 \le x \le e, y' = \frac{1}{x}$ and

$$s_1 = \int_1^e \sqrt{1 + \frac{1}{x^2}}\, dx.$$

Equivalently, $x = e^y, 0 \le y \le 1, \frac{dx}{dy} = e^y$, and

$$s_2 = \int_0^1 \sqrt{1 + e^{2y}}\, dy = \int_0^1 \sqrt{1 + e^{2x}}\, dx.$$

Numerically, both integrals yield $s = 2.0035$.

55.
$$y = \frac{3x}{4}, \quad y' = \frac{3}{4}$$
$$1 + (y')^2 = 1 + \frac{9}{16} = 25/16$$
$$S = 2\pi \int_0^4 x\sqrt{\frac{25}{16}}\, dx = \frac{5\pi}{2}\left[\frac{x^2}{2}\right]_0^4 = 20\pi$$

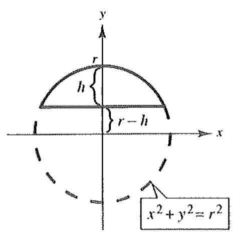
56.
$$y = \frac{hx}{r}$$
$$y' = \frac{h}{r}$$
$$1 + (y')^2 = \frac{r^2 + h^2}{r^2}$$
$$S = 2\pi \int_0^r x\sqrt{\frac{r^2 + h^2}{r^2}}\, dx$$
$$= \left[\frac{2\pi\sqrt{r^2 + h^2}}{r}\left(\frac{x^2}{2}\right)\right]_0^r = \pi r\sqrt{r^2 + h^2}$$

57.
$$y = \sqrt{9 - x^2}$$
$$y' = \frac{-x}{\sqrt{9 - x^2}}$$
$$\sqrt{1 + (y')^2} = \frac{3}{\sqrt{9 - x^2}}$$
$$S = 2\pi \int_0^2 \frac{3x}{\sqrt{9 - x^2}}\, dx$$
$$= -3\pi \int_0^2 \frac{-2x}{\sqrt{9 - x^2}}\, dx$$
$$= \left[-6\pi\sqrt{9 - x^2}\right]_0^2$$
$$= 6\pi\left(3 - \sqrt{5}\right) \approx 14.40$$

See figure in Exercise 58.

58. From Exercise 57 you have:

$$S = 2\pi\int_0^a \frac{rx}{\sqrt{r^2 - x^2}}\,dx$$
$$= -r\pi\int_0^a \frac{-2x\,dx}{\sqrt{r^2 - x^2}}$$
$$= \left[-2r\pi\sqrt{r^2 - x^2}\right]_0^a$$
$$= 2r^2\pi - 2r\pi\sqrt{r^2 - a^2}$$
$$= 2r\pi\left(r - \sqrt{r^2 - a^2}\right)$$
$= 2\pi rh$ (where h is the height of the zone)



59. (a) Approximate the volume by summing six disks of thickness 3 and circumference C_i equal to the average of the given circumferences:

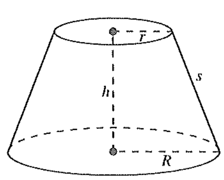
$$V \approx \sum_{i=1}^{6}\pi r_i^2(3) = \sum_{i=1}^{6}\pi\left(\frac{C_i}{2\pi}\right)^2(3) = \frac{3}{4\pi}\sum_{i=1}^{6}C_i^2$$
$$= \frac{3}{4\pi}\left[\left(\frac{50+65.5}{2}\right)^2 + \left(\frac{65.5+70}{2}\right)^2 + \left(\frac{70+66}{2}\right)^2 + \left(\frac{66+58}{2}\right)^2 + \left(\frac{58+51}{2}\right)^2 + \left(\frac{51+48}{2}\right)^2\right]$$
$$= \frac{3}{4\pi}\left[57.75^2 + 67.75^2 + 68^2 + 62^2 + 54.5^2 + 49.5^2\right] = \frac{3}{4\pi}(21813.625) = 5207.62 \text{ in.}^3$$

(b) The lateral surface area of a frustum of a right circular cone is $\pi s(R + r)$. For the first frustum:

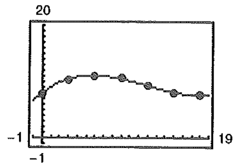
$$S_1 \approx \pi\left[3^2 + \left(\frac{65.5 - 50}{2\pi}\right)^2\right]^{1/2}\left[\frac{50}{2\pi} + \frac{65.5}{2\pi}\right]$$
$$= \left(\frac{50+65.5}{2}\right)\left[9 + \left(\frac{65.5-50}{2\pi}\right)^2\right]^{1/2}.$$

Adding the six frustums together:

$$S \approx \left(\frac{50+65.5}{2}\right)\left[9 + \left(\frac{15.5}{2\pi}\right)^2\right]^{1/2} + \left(\frac{65.5+70}{2}\right)\left[9 + \left(\frac{4.5}{2\pi}\right)^2\right]^{1/2}$$
$$+\left(\frac{70+66}{2}\right)\left[9 + \left(\frac{4}{2\pi}\right)^2\right]^{1/2} + \left(\frac{66+58}{2}\right)\left[9 + \left(\frac{8}{2\pi}\right)^2\right]^{1/2}$$
$$+\left(\frac{58+51}{2}\right)\left[9 + \left(\frac{7}{2\pi}\right)^2\right]^{1/2} + \left(\frac{51+48}{2}\right)\left[9 + \left(\frac{3}{2\pi}\right)^2\right]^{1/2}$$
$$\approx 224.30 + 208.96 + 208.54 + 202.06 + 174.41 + 150.37 = 1168.64$$



(c) $r = 0.00401y^3 - 0.1416y^2 + 1.232y + 7.943$



(d) $V = \int_0^{18} \pi r^2 dy \approx 5275.9 \text{ in.}^3$

$S = \int_0^{18} 2\pi r(y)\sqrt{1 + r'(y)^2}\, dy \approx 1179.5 \text{ in.}^2$

60. (a) $y = f(x) = 0.0000001953x^4 - 0.0001804x^3 + 0.0496x^2 - 4.8323x + 536.9270$

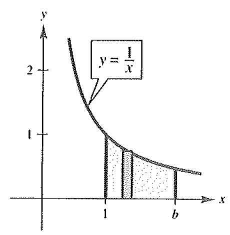
(b) $\text{Area} = \int_0^{400} f(x)\, dx \approx 131{,}734.5 \text{ ft}^2 \approx 3.0 \text{ acres} \quad (1 \text{ acre} = 43{,}560 \text{ ft}^2)$

(Answers will vary.)

(c) $L = \int_0^{400} \sqrt{1 + f'(x)^2}\, dx \approx 794.9 \text{ ft}$

(Answers will vary.)

61. (a) $V = \pi \int_1^b \frac{1}{x^2}\, dx = \left[-\frac{\pi}{x}\right]_1^b = \pi\left(1 - \frac{1}{b}\right)$



(b) $S = 2\pi \int_1^b \frac{1}{x}\sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\, dx$

$= 2\pi \int_1^b \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx$

$= 2\pi \int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\, dx$

(c) $\lim_{b\to\infty} V = \lim_{b\to\infty} \pi\left(1 - \frac{1}{b}\right) = \pi$

(d) Because

$\frac{\sqrt{x^4 + 1}}{x^3} > \frac{\sqrt{x^4}}{x^3} = \frac{1}{x} > 0$ on $[1, b]$,

you have

$\int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\, dx > \int_1^b \frac{1}{x}\, dx = \left[\ln x\right]_1^b = \ln b$

and $\lim_{b\to\infty} \ln b \to \infty$. So,

$\lim_{b\to\infty} 2\pi \int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\, dx = \infty.$

62. (a) Area of circle with radius L: $A = \pi L^2$

Area of sector with central angle θ (in radians):

$S = \frac{\theta}{2\pi} A = \frac{\theta}{2\pi}(\pi L^2) = \frac{1}{2} L^2 \theta$

(b) Let s be the arc length of the sector, which is the circumference of the base of the cone. Here, $s = L\theta = 2\pi r$, and you have

$S = \frac{1}{2} L^2 \theta = \frac{1}{2} L^2 \left(\frac{s}{L}\right) = \frac{1}{2} Ls = \frac{1}{2} L(2\pi r) = \pi r L.$

(c) The lateral surface area of the frustum is the difference of the large cone and the small one.

$S = \pi r_2(L + L_1) - \pi r_1 L_1$

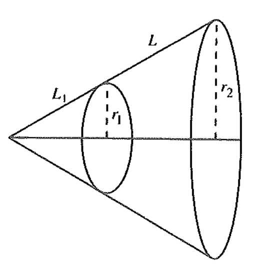
$= \pi r_2 L + \pi L_1(r_2 - r_1)$

By similar triangles,

$\frac{L + L_1}{r_2} = \frac{L_1}{r_1} \Rightarrow L r_1 = L_1(r_2 - r_1)$. So,

$S = \pi r_2 L + \pi L_1(r_2 - r_1) = \pi r_2 L + \pi L r_1$

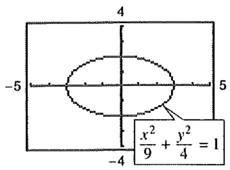
$= \pi L(r_1 + r_2).$



63. (a) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

Ellipse: $y_1 = 2\sqrt{1 - \dfrac{x^2}{9}}$

$y_2 = -2\sqrt{1 - \dfrac{x^2}{9}}$



(b) $y = 2\sqrt{1 - \dfrac{x^2}{9}}, \quad 0 \le x \le 3$

$$y' = 2\left(\frac{1}{2}\right)\left(1 - \frac{x^2}{9}\right)^{-1/2}\left(\frac{-2x}{9}\right)$$

$$= \frac{-2x}{9\sqrt{1 - (x^2/9)}} = \frac{-2x}{3\sqrt{9 - x^2}}$$

$$L = \int_0^3 \sqrt{1 + \frac{4x^2}{81 - 9x^2}}\, dx$$

(c) You cannot evaluate this definite integral, because the integrand is not defined at $x = 3$. Simpson's Rule will not work for the same reason. Also, the integrand does not have an elementary antiderivative.

64. Essay

65. $y = \dfrac{1}{3}\left(x^{3/2} - 3x^{1/2} + 2\right)$

When $x = 0$, $y = \dfrac{2}{3}$. So, the fleeing object has traveled $\dfrac{2}{3}$ unit when it is caught.

$$y' = \frac{1}{3}\left(\frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}\right) = \left(\frac{1}{2}\right)\frac{x - 1}{x^{1/2}}$$

$$1 + (y')^2 = 1 + \frac{(x - 1)^2}{4x} = \frac{(x + 1)^2}{4x}$$

$$s = \int_0^1 \frac{x + 1}{2x^{1/2}}\, dx = \frac{1}{2}\int_0^1 \left(x^{1/2} + x^{-1/2}\right) dx$$

$$= \frac{1}{2}\left[\frac{2}{3}x^{3/2} + 2x^{1/2}\right]_0^1 = \frac{4}{3} = 2\left(\frac{2}{3}\right)$$

The pursuer has traveled twice the distance that the fleeing object has traveled when it is caught.

66.

$$y = \frac{1}{3}x^{1/2} - x^{3/2}$$

$$y' = \frac{1}{6}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{6}\left(x^{-1/2} - 9x^{1/2}\right)$$

$$1 + (y')^2 = 1 + \frac{1}{36}\left(x^{-1} - 18 + 81x\right) = \frac{1}{36}\left(x^{-1/2} + 9x^{1/2}\right)^2$$

$$S = 2\pi\int_0^{1/3}\left(\frac{1}{3}x^{1/2} - x^{3/2}\right)\sqrt{\frac{1}{36}\left(x^{-1/2} + 9x^{1/2}\right)^2}\, dx = \frac{2\pi}{6}\int_0^{1/3}\left(\frac{1}{3}x^{1/2} - x^{3/2}\right)\left(x^{-1/2} + 9x^{1/2}\right) dx$$

$$= \frac{\pi}{3}\int_0^{1/3}\left(\frac{1}{3} + 2x - 9x^2\right) dx = \frac{\pi}{3}\left[\frac{1}{3}x + x^2 - 3x^3\right]_0^{1/3} = \frac{\pi}{27}\text{ ft}^2 \approx 0.1164\text{ ft}^2 \approx 16.8\text{ in.}^2$$

Amount of glass needed: $V = \dfrac{\pi}{27}\left(\dfrac{0.015}{12}\right) \approx 0.00015\text{ ft}^3 \approx 0.25\text{ in.}^3$

67. $x^{2/3} + y^{2/3} = 4$

$$y^{2/3} = 4 - x^{2/3}$$

$$y = \left(4 - x^{2/3}\right)^{3/2}, \quad 0 \le x \le 8$$

$$y' = \frac{3}{2}\left(4 - x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = \frac{-\left(4 - x^{2/3}\right)^{1/2}}{x^{1/3}}$$

$$1 + (y')^2 = 1 + \frac{4 - x^{2/3}}{x^{2/3}} = \frac{4}{x^{2/3}}$$

$$S = 2\pi\int_0^8 \left(4 - x^{2/3}\right)^{3/2}\sqrt{\frac{4}{x^{2/3}}}\, dx = 4\pi\int_0^8 \frac{\left(4 - x^{2/3}\right)^{3/2}}{x^{1/3}}\, dx = \left[-\frac{12\pi}{5}\left(4 - x^{2/3}\right)^{5/2}\right]_0^8 = \frac{384\pi}{5}$$

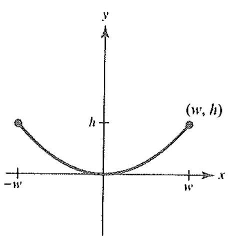
[Surface area of portion above the x-axis]

68.
$$y^2 = \frac{1}{12}x(4-x)^2, \quad 0 \le x \le 4$$
$$y = \frac{(4-x)\sqrt{x}}{\sqrt{12}}$$
$$y' = \frac{(4-3x)\sqrt{3}}{12\sqrt{x}}$$
$$1 + (y')^2 = 1 + \frac{(4-3x)^2}{48x}$$
$$= \frac{48x + 16 - 24x + 9x^2}{48x} = \frac{(4+3x)^2}{48x}, \quad x \ne 0$$
$$S = 2\pi \int_0^4 \frac{(4-x)\sqrt{x}}{\sqrt{12}} \cdot \frac{(4+3x)}{\sqrt{48x}}\,dx$$
$$= 2\pi \int_0^4 \frac{(4-x)(4+3x)}{24}\,dx$$
$$= \frac{\pi}{12}\int_0^4 (16 + 8x - 3x^2)\,dx = \frac{\pi}{12}\Big[16x + 4x^2 - x^3\Big]_0^4 = \frac{\pi}{12}(64 + 64 - 64) = \frac{16\pi}{3}$$

69. $y = kx^2,\ y' = 2kx$

$$1 + (y')^2 = 1 + 4k^2x^2$$
$$h = kw^2 \Rightarrow k = \frac{h}{w^2} \Rightarrow 1 + (y') = 1 + \frac{4h^2}{w^4}x^2$$

By symmetry, $C = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2}\,dx.$



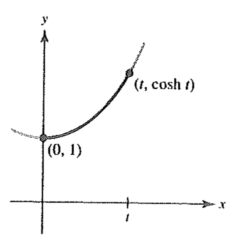
70. $C = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2}\,dx$

$$= 2\int_0^{700} \sqrt{1 + \frac{4(155)^2x^2}{700^4}}\,dx = 1444.5 \text{ m}$$

71.
$$y = f(x) = \cosh x$$
$$y' = \sinh x$$
$$1 + (y')^2 = 1 + \sinh^2 x = \cosh^2 x$$
$$\text{Area} = \int_0^t \cosh x\,dx = \big[\sinh x\big]_0^t = \sinh t$$
$$\text{Arc length} = \int_0^t \sqrt{1 + (y')^2}\,dx$$
$$= \int_0^t \cosh x\,dx = \sinh x\Big]_0^t$$
$$= \sinh t.$$

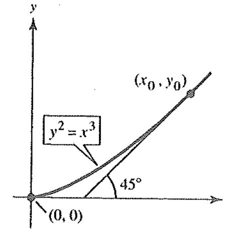
Another curve with this property is $g(x) = 1$.

$$\text{Area} = \int_0^t dx = t$$
$$\text{Arc length} = t$$



72. Let (x_0, y_0) be the point on the graph of $y^2 = x^3$ where the tangent line makes an angle of 45° with the x-axis.

$$y = x^{3/2}$$
$$y' = \tfrac{3}{2}x^{1/2} = 1$$
$$x_0 = \tfrac{4}{9}$$
$$L = \int_0^{4/9} \sqrt{1 + \tfrac{9}{4}x}\,dx$$
$$= \tfrac{8}{27}\left(2\sqrt{2} - 1\right)$$



Section 7.5 Work

1. $W = Fd = (100)(20) = 2000$ ft-lb

2. $W = Fd = (3500)(4) = 14{,}000$ ft-lb

3. $W = Fd = (112)(8) = 896$ joules (Newton-meters)

4. $W = Fd = \left[9(2000)\right]\left[\frac{1}{2}(5280)\right] = 47{,}520{,}000$ ft-lb

5. $F(x) = kx$

$5 = k(3)$

$k = \dfrac{5}{3}$

$F(x) = \dfrac{5}{3}x$

$$W = \int_0^7 F(x)\,dx = \int_0^7 \frac{5}{3}x\,dx = \left[\frac{5}{6}x^2\right]_0^7 = \frac{245}{6} \text{ in.-lb}$$

≈ 40.833 in.-lb ≈ 3.403 ft-lb

6. From Exercise 5, $F(x) = \dfrac{5}{3}x$.

$$W = \int_{15-10}^{15-6} F(x)\,dx = \int_5^9 \frac{5}{3}x\,dx = \left.\frac{5x^2}{6}\right]_5^9 = \frac{140}{3} \text{ in-lb}$$

7. $F(x) = kx$

$250 = k(30) \Rightarrow k = \dfrac{25}{3}$

$$W = \int_{20}^{50} F(x)\,dx$$

$$= \int_{20}^{50} \frac{25}{3}x\,dx = \left.\frac{25x^2}{6}\right]_{20}^{50}$$

$= 8750$ n-cm

$= 87.5$ joules or Nm

8. $F(x) = kx$

$800 = k(70) \Rightarrow k = \dfrac{80}{7}$

$$W = \int_0^{70} F(x)\,dx$$

$$= \int_0^{70} \frac{80}{7}x\,dx = \left.\frac{40x^2}{7}\right]_0^{70}$$

$= 28{,}000$ n-cm $= 280$ Nm

9. $F(x) = kx$

$20 = k(9)$

$k = \frac{20}{9}$

$$W = \int_0^{12} \tfrac{20}{9}x\,dx = \left[\tfrac{10}{9}x^2\right]_0^{12} = 160 \text{ in.-lb} = \tfrac{40}{3} \text{ ft-lb}$$

10. $F(x) = kx$

$15 = k(1) = k$

$$W = 2\int_0^4 15x\,dx = \left[15x^2\right]_0^4 = 240 \text{ ft-lb}$$

11. $W = 18 = \displaystyle\int_0^{1/3} kx\,dx = \left.\frac{kx^2}{2}\right]_0^{1/3} = \frac{k}{18} \Rightarrow k = 324$

$$W = \int_{1/3}^{7/12} 324x\,dx = \left[162x^2\right]_{1/3}^{7/12} = 37.125 \text{ ft-lb}$$

[**Note:** 4 inches $= \dfrac{1}{3}$ foot]

12. $W = 7.5 = \displaystyle\int_0^{1/6} kx\,dx = \left.\frac{kx^2}{2}\right]_0^{1/6} = \frac{k}{72} \Rightarrow k = 540$

$$W = \int_{1/6}^{5/24} 540x\,dx = \left[270x^2\right]_{1/6}^{5/24} = 4.21875 \text{ ft-lb}$$

13. Assume that Earth has a radius of 4000 miles.

$F(x) = \dfrac{k}{x^2}$

$5 = \dfrac{k}{(4000)^2}$

$k = 80{,}000{,}000$

$F(x) = \dfrac{80{,}000{,}000}{x^2}$

(a) $W = \displaystyle\int_{4000}^{4100} \frac{80{,}000{,}000}{x^2}\,dx = \left[\frac{-80{,}000{,}000}{x}\right]_{4000}^{4100}$

≈ 487.8 mi-tons $\approx 5.15 \times 10^9$ ft-lb

(b) $W = \displaystyle\int_{4000}^{4300} \frac{80{,}000{,}000}{x^2}\,dx$

≈ 1395.3 mi-ton $\approx 1.47 \times 10^{10}$ ft-ton

14. $W = \displaystyle\int_{4000}^{h} \frac{80{,}000{,}000}{x^2}\,dx$

$$= \left[-\frac{80{,}000{,}000}{x}\right]_{4000}^{h}$$

$$= \frac{-80{,}000{,}000}{h} + 20{,}000$$

$\displaystyle\lim_{h\to\infty} W = 20{,}000$ mi-ton $\approx 2.1 \times 10^{11}$ ft-lb

15. Assume that Earth has a radius of 4000 miles.

$$F(x) = \frac{k}{x^2}$$

$$10 = \frac{k}{(4000)^2}$$

$$k = 160{,}000{,}000$$

$$F(x) = \frac{160{,}000{,}000}{x^2}$$

(a) $W = \int_{4000}^{15{,}000} \frac{160{,}000{,}000}{x^2}\,dx = \left[-\frac{160{,}000{,}000}{x}\right]_{4000}^{15{,}000} \approx -10{,}666.667 + 40{,}000$

$= 29{,}333.333$ mi-ton

$\approx 2.93 \times 10^4$ mi-ton

$\approx 3.10 \times 10^{11}$ ft-lb

(b) $W = \int_{4000}^{26{,}000} \frac{160{,}000{,}000}{x^2}\,dx = \left[-\frac{160{,}000{,}000}{x}\right]_{4000}^{26{,}000} \approx -6{,}153.846 + 40{,}000$

$= 33{,}846.154$ mi-ton

$\approx 3.38 \times 10^4$ mi-ton

$\approx 3.57 \times 10^{11}$ ft-lb

16. Weight on surface of moon: $\frac{1}{6}(12) = 2$ tons

Weight varies inversely as the square of distance from the center of the moon. Therefore:

$$F(x) = \frac{k}{x^2}$$

$$2 = \frac{k}{(1100)^2}$$

$$k = 2.42 \times 10^6$$

$$W = \int_{1100}^{1150} \frac{2.42 \times 10^6}{x^2}\,dx = \left[\frac{-2.42 \times 10^6}{x}\right]_{1100}^{1150} = 2.42 \times 10^6\left(\frac{1}{1100} - \frac{1}{1150}\right)$$

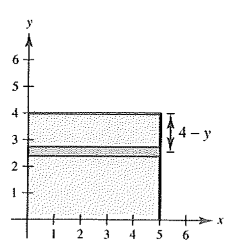
≈ 95.652 mi-ton $\approx 1.01 \times 10^9$ ft-lb

17. Weight of each layer: $62.4(20)\,\Delta y$

Distance: $4 - y$

(a) $W = \int_2^4 62.4(20)(4 - y)\,dy = \left[4992y - 624y^2\right]_2^4 = 2496$ ft-lb

(b) $W = \int_0^4 62.4(20)(4 - y)\,dy = \left[4992y - 624y^2\right]_0^4 = 9984$ ft-lb



18. The bottom half had to be pumped a greater distance than the top half.

19. Volume of disk: $\pi(2)^2\ \Delta y = 4\pi\ \Delta y$

Weight of disk of water: $9800(4\pi)\ \Delta y$

Distance the disk of water is moved: $5 - y$

$$W = \int_0^4 (5 - y)(9800)4\pi\ dy = 39{,}200\pi \int_0^4 (5 - y)\ dy$$

$$= 39{,}200\pi\left[5y - \frac{y^2}{2}\right]_0^4$$

$$= 39{,}200\pi(12) = 470{,}400\pi \text{ newton–meters}$$

20. Volume of disk: $4\pi\ \Delta y$

Weight of disk: $9800(4\pi)\ \Delta y$

Distance the disk of water is moved: y

$$W = \int_{10}^{12} y(9800)(4\pi)\ dy = 39{,}200\pi\left[\frac{y^2}{2}\right]_{10}^{12}$$

$$= 39{,}200\pi(22)$$

$$= 862{,}400\pi \text{ newton–meters}$$

21. Volume of disk: $\pi\left(\frac{2}{3}y\right)^2\ \Delta y$

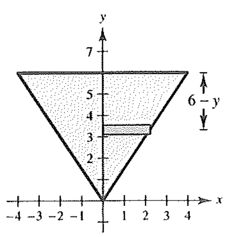
Weight of disk: $62.4\pi\left(\frac{2}{3}y\right)^2\ \Delta y$

Distance: $6 - y$

$$W = \frac{4(62.4)\pi}{9}\int_0^6 (6 - y)y^2\ dy$$

$$= \frac{4}{9}(62.4)\pi\left[2y^3 - \frac{1}{4}y^4\right]_0^6$$

$$= 2995.2\pi \text{ ft-lb}$$



22. Volume of disk: $\pi\left(\frac{2}{3}y\right)^2\ \Delta y$

Weight of disk: $62.4\pi\left(\frac{2}{3}y\right)^2\ \Delta y$

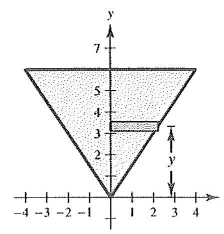
Distance: y

(a) $W = \frac{4}{9}(62.4)\pi\int_0^2 y^3 dy$

$$= \left[\tfrac{4}{9}(62.4)\pi\left(\tfrac{1}{4}y^4\right)\right]_0^2 \approx 110.9\pi \text{ ft} \cdot \text{lb}$$

(b) $W = \frac{4}{9}(62.4)\pi\int_4^6 y^3 dy$

$$= \left[\tfrac{4}{9}(62.4)\pi\left(\tfrac{1}{4}y^4\right)\right]_4^6 \approx 7210.7\pi \text{ ft-lb}$$



23. Volume of disk: $\pi\left(\sqrt{36 - y^2}\right)^2\ \Delta y$

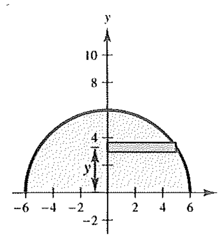
Weight of disk: $62.4\pi\left(36 - y^2\right)\ \Delta y$

Distance: y

$$W = 62.4\pi\int_0^6 y\left(36 - y^2\right)\ dy$$

$$= 62.4\pi\int_0^6 \left(36y - y^3\right)\ dy = 62.4\pi\left[18y^2 - \tfrac{1}{4}y^4\right]_0^6$$

$$= 20{,}217.6\pi \text{ ft-lb}$$

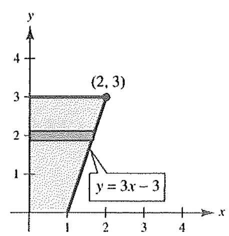


24. Volume of each layer: $\frac{y+3}{3}(3)\,\Delta y = (y+3)\,\Delta y$

Weight of each layer: $53.1(y+3)\,\Delta y$

Distance: $6 - y$

$$W = \int_0^3 53.1(6-y)(y+3)\,dy$$
$$= 53.1\int_0^3 \left(18 + 3y - y^2\right) dy$$
$$= 53.1\left[18y + \frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3$$
$$= 53.1\left(\frac{117}{2}\right)$$
$$= 3106.35 \text{ ft-lb}$$



25. Volume of layer: $V = lwh = 4(2)\sqrt{(9/4) - y^2}\,\Delta y$

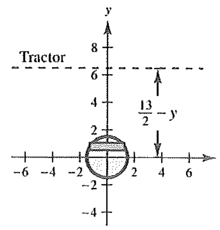
Weight of layer: $W = 42(8)\sqrt{(9/4) - y^2}\,\Delta y$

Distance: $\frac{13}{2} - y$

$$W = \int_{-1.5}^{1.5} 42(8)\sqrt{\tfrac{9}{4} - y^2}\left(\tfrac{13}{2} - y\right) dy$$
$$= 336\left[\frac{13}{2}\int_{-1.5}^{1.5}\sqrt{\tfrac{9}{4} - y^2}\,dy - \int_{-1.5}^{1.5}\sqrt{\tfrac{9}{4} - y^2}\,y\,dy\right]$$

The second integral is zero because the integrand is odd and the limits of integration are symmetric to the origin. The first integral represents the area of a semicircle of radius $\frac{3}{2}$. So, the work is

$$W = 336\left(\tfrac{13}{2}\right)\pi\left(\tfrac{3}{2}\right)^2\left(\tfrac{1}{2}\right) = 2457\pi \text{ ft-lb.}$$



26. Volume of layer: $V = 12(2)\sqrt{(25/4) - y^2}\,\Delta y$

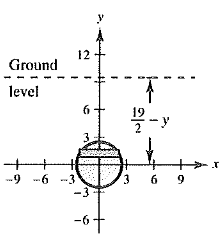
Weight of layer: $W = 42(24)\sqrt{(25/4) - y^2}\,\Delta y$

Distance: $\frac{19}{2} - y$

$$W = \int_{-2.5}^{2.5} 42(24)\sqrt{\tfrac{25}{4} - y^2}\left(\tfrac{19}{2} - y\right) dy = 1008\left[\frac{19}{2}\int_{-2.5}^{2.5}\sqrt{\tfrac{25}{4} - y^2}\,dy + \int_{-2.5}^{2.5}\sqrt{\tfrac{25}{4} - y^2}(-y)\,dy\right]$$

The second integral is zero because the integrand is odd and the limits of integration are symmetric to the origin. The first integral represents the area of a semicircle of radius $\frac{5}{2}$. So, the work is

$$W = 1008\left(\tfrac{19}{2}\right)\pi\left(\tfrac{5}{2}\right)^2\left(\tfrac{1}{2}\right) = 29{,}925\pi \text{ ft-lb} \approx 94{,}012.16 \text{ ft-lb.}$$



27. Weight of section of chain: $3\,\Delta y$

Distance: $20 - y$. $\Delta W = (\text{force increment})(\text{distance}) = (3\,\Delta y)(20 - y)$

$$W = \int_0^{20} (20 - y)3\,dy = 3\left[20y - \frac{y^2}{2}\right]_0^{20} = 3\left[400 - \frac{400}{2}\right] = 600 \text{ ft-lb}$$

28. The lower $\frac{2}{3}(20)$ feet of fence are raised with a constant force

$$W_1 = 3\left(\frac{2}{3}(20)\right)\left(\frac{20}{3}\right) = \frac{800}{3} \text{ ft-lb}$$

The top $\frac{1}{3}(20)$ feet are raised with a variable force.

Weight of section: $3\,\Delta y$

Distance: $\frac{1}{3}(20) - y$

$$W_2 = \int_0^{20/3} 3\left(\frac{20}{3} - y\right) dy = 3\left[\frac{20}{3}y - \frac{y^2}{2}\right]_0^{20/3}$$
$$= \frac{200}{3} \text{ ft-lb}$$

$$W = W_1 + W_2 = \frac{800}{3} + \frac{200}{3} = \frac{1000}{3} \text{ ft-lb}$$

29. The lower 10 feet of fence are raised 10 feet with a constant force.

$W_1 = 3(10)(10) = 300$ ft-lb

The top 10 feet are raised with a variable force.

Weight of section: $3\,\Delta y$

Distance: $10 - y$

$$W_2 = \int_0^{10} 3(10 - y)\,dy = 3\left[10y - \frac{y^2}{2}\right]_0^{10} = 150 \text{ ft-lb}$$

$W = W_1 + W_2 = 300 + 150 = 450$ ft-lb

30. From Exercise 27, the work required to lift the chain is 600 ft-lb.

The work required to lift the 500-pound load is $500(20) = 10{,}000$ ft-lb.

The total is $600 + 10{,}000 = 10{,}600$ ft-lb.

31. Weight of section of chain: $3\,\Delta y$

Distance: $15 - 2y$

$$W = 3\int_0^{7.5}(15 - 2y)\,dy = \left[-\tfrac{3}{4}(15 - 2y)^2\right]_0^{7.5}$$
$$= \tfrac{3}{4}(15)^2 = 168.75 \text{ ft-lb}$$

32. $W = 3\int_0^{6}(12 - 2y)\,dy = \left[-\tfrac{3}{4}(12 - 2y)^2\right]_0^{6}$

$= \tfrac{3}{4}(12)^2 = 108$ ft-lb

33. If an object is moved a distance D in the direction of an applied constant force F, then the work W done by the force is defined as force times distance, $W = FD$.

34. $W = \int_a^b F(x)\,dx$ is the work done by a force F moving an object along a straight line from $x = a$ to $x = b$.

35. (a) requires more work. In part (b) no work is done because the books are not moved:
W = force × distance

36. Because the work equals the area under the force function, you have $(c) < (d) < (a) < (b)$.

37. (a) $W = \int_0^9 6\,dx = 54$ ft-lb

(b) $W = \int_0^7 20\,dx + \int_7^9(-10x + 90)\,dx = 140 + 20$
$= 160$ ft-lb

(c) $W = \int_0^9 \frac{1}{27}x^2\,dx = \left.\frac{x^3}{81}\right]_0^9 = 9$ ft-lb

(d) $W = \int_0^9 \sqrt{x}\,dx = \left.\frac{2}{3}x^{3/2}\right]_0^9 = \frac{2}{3}(27) = 18$ ft-lb

38. (a) Work to pull up the ball:
$W_1 = 50(15) = 750$ ft-lb

Work to wind up the top 15 feet of cable:
Weight of section: $2\,\Delta y$

Distance: $15 - y$

$$W_2 = \int_0^{15} 2(15 - y)\,dy = \left[30y - y^2\right]_0^{15} = 225 \text{ ft-lb}$$

Work to lift bottom 25 feet of cable
$W_3 = 2(25)(15) = 750$ ft-lb

$W = W_1 + W_2 + W_3 = 750 + 225 + 750 = 1725$ ft-lb

(b) Work to pull up the ball: $W_1 = 50(40) = 2000$ ft-lb

Work to wind up the cable.

$$W_2 = \int_0^{40} 2(40 - y)\,dy = \left[80y - y^2\right]_0^{40} = 1600 \text{ ft-lb}$$

$W = W_1 + W_2 = 2000 + 1600 = 3600$ ft-lb

39.
$$p = \frac{k}{V}$$
$$1000 = \frac{k}{2}$$
$$k = 2000$$
$$W = \int_2^3 \frac{2000}{V}\,dV$$
$$= \left[2000 \ln|V|\right]_2^3 = 2000 \ln\left(\frac{3}{2}\right) \approx 810.93 \text{ ft-lb}$$

40. $p = \dfrac{k}{V}$

$2500 = \dfrac{k}{1} \Rightarrow k = 2500$

$W = \int_1^3 \dfrac{2500}{V}\,dV = \left[2500 \ln V\right]_1^3 = 2500 \ln 3 \approx 2746.53$ ft-lb

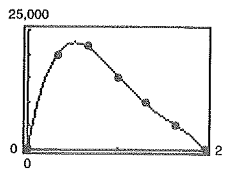
41. $F(x) = \dfrac{k}{(2-x)^2}$

$W = \int_{-2}^{1} \dfrac{k}{(2-x)^2}\,dx = \left[\dfrac{k}{2-x}\right]_{-2}^{1} = k\left(1 - \dfrac{1}{4}\right) = \dfrac{3k}{4}$ (units of work)

42. (a) $W = FD = (8000\pi)(2) = 16{,}000\pi$ ft · lbs

(b) $W \approx \frac{2-0}{3(6)}\left[0 + 4(20{,}000) + 2(22{,}000) + 4(15{,}000) + 2(10{,}000) + 4(5000) + 0\right] \approx 24{,}88.889$ ft-lb

(c) $F(x) = -16{,}261.36x^4 + 85{,}295.45x^3 - 157{,}738.64x^2 + 104{,}386.36x - 32.4675$



(d) $F(x) = 0$ when $x \approx 0.524$ feet. $F(x)$ is a maximum when $x \approx 0.524$ feet.

(e) $W = \int_0^2 F(x)\,dx \approx 25{,}180.5$ ft-lb

43. $W = \int_0^5 1000\left[1.8 - \ln(x+1)\right]dx \approx 3249.44$ ft-lb

44. $W = \int_0^4 \left(\dfrac{e^{x^2} - 1}{100}\right)dx \approx 11{,}494$ ft-lb

45. $W = \int_0^5 100x\sqrt{125 - x^3}\,dx \approx 10{,}330.3$ ft-lb

46. $W = \int_0^2 1000 \sinh x\,dx \approx 2762.2$ ft-lb

Section 7.6 Moments, Centers of Mass, and Centroids

1. $\bar{x} = \dfrac{7(-5) + 3(0) + 5(3)}{7 + 3 + 5} = \dfrac{-20}{15} = -\dfrac{4}{3}$

2. $\bar{x} = \dfrac{7(-3) + 4(-2) + 3(5) + 8(4)}{7 + 4 + 3 + 8} = \dfrac{9}{11}$

3. $\bar{x} = \dfrac{1(7) + 1(8) + 1(12) + 1(15) + 1(18)}{1 + 1 + 1 + 1 + 1} = 12$

4. $\bar{x} = \dfrac{12(-6) + 1(-4) + 6(-2) + 3(0) + 11(8)}{12 + 1 + 6 + 3 + 11} = 0$

5. (a) Add 4 to each x-value because each point is translated 4 units to the right.

$\bar{x} = \dfrac{1(7+4) + 1(8+4) + 1(12+4) + 1(15+4) + 1(18+4)}{1 + 1 + 1 + 1 + 1} = \dfrac{80}{5} = 16$

Note: From Exercise 3, $12 + 4 = 16$.

(b) Subtract 2 from each x-value because each point is translated 2 units to the left.

$\bar{x} = \dfrac{12(-6-2) + 1(-4-2) + 6(-2-2) + 3(0-2) + 11(8-2)}{12 + 1 + 6 + 3 + 11} = \dfrac{-66}{33} = -2$

Note: From Exercise 4, $0 - 2 = -2$.

6. The center of mass is translated k units as well.

7. $48x = 72(L - x) = 72(10 - x)$

$48x = 720 - 72x$

$120x = 720$

$x = 6 \text{ ft}$

8. $200x = 600(5 - x)$ (person is on the left)

$200x = 3000 - 600x$

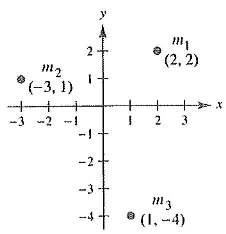
$800x = 3000$

$x = \frac{15}{4} = 3\frac{3}{4} \text{ ft}$

9. $$\bar{x} = \frac{5(2) + 1(-3) + 3(1)}{5 + 1 + 3} = \frac{10}{9}$$

$$\bar{y} = \frac{5(2) + 1(1) + 3(-4)}{5 + 1 + 3} = -\frac{1}{9}$$

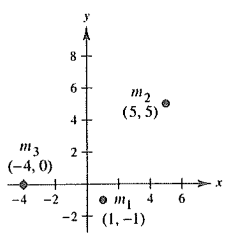
$$(\bar{x}, \bar{y}) = \left(\frac{10}{9}, -\frac{1}{9}\right)$$



10. $$\bar{x} = \frac{10(1) + 2(5) + 5(-4)}{10 + 2 + 5} = 0$$

$$\bar{y} = \frac{10(-1) + 2(5) + 5(0)}{10 + 2 + 5} = 0$$

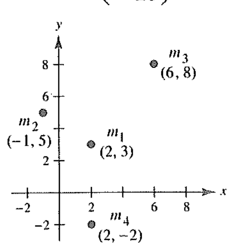
$$(\bar{x}, \bar{y}) = (0, 0)$$



11. $$\bar{x} = \frac{12(2) + 6(-1) + (9/2)(6) + 15(2)}{12 + 6 + (9/2) + 15} = \frac{75}{37.5} = 2$$

$$\bar{y} = \frac{12(3) + 6(5) + (9/2)(8) + 15(-2)}{12 + 6 + (9/2) + 15} = \frac{72}{37.5} = \frac{48}{25}$$

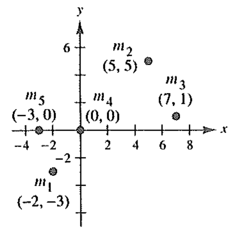
$$(\bar{x}, \bar{y}) = \left(2, \frac{48}{25}\right)$$



12. $$\bar{x} = \frac{3(-2) + 4(5) + 2(7) + 1(0) + 6(-3)}{3 + 4 + 2 + 1 + 6} = \frac{5}{8}$$

$$\bar{y} = \frac{3(-3) + 4(5) + 2(1) + 1(0) + 6(0)}{3 + 4 + 2 + 1 + 6} = \frac{13}{16}$$

$$(\bar{x}, \bar{y}) = \left(\frac{5}{8}, \frac{13}{16}\right)$$



13. $$m = \rho\int_0^2 \frac{x}{2}\,dx = \left[\rho\frac{x^2}{4}\right]_0^2 = \rho$$

$$M_x = \rho\int_0^2 \frac{1}{2}\left(\frac{x}{2}\right)^2 dx$$

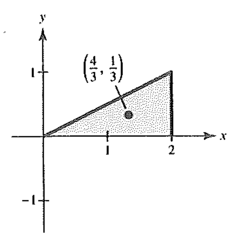
$$= \frac{\rho}{8}\left[\frac{x^3}{3}\right]_0^2 = \frac{\rho}{3}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\rho/3}{\rho} = \frac{1}{3}$$

$$M_y = \rho\int_0^2 x\left(\frac{x}{2}\right) dx = \frac{\rho}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{4}{3}\rho$$

$$\bar{x} = \frac{M_y}{m} = \frac{4/3\rho}{\rho} = \frac{4}{3}$$

$$(\bar{x}, \bar{y}) = \left(\frac{4}{3}, \frac{1}{3}\right)$$



14. $$m = \rho\int_0^3 (-x+3)\,dx = \rho\left[-\frac{x^2}{2}+3x\right]_0^3 = \frac{9}{2}\rho$$

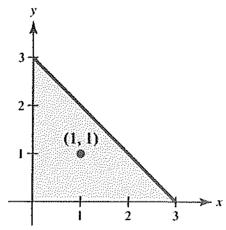
$$M_x = \rho\int_0^3 \frac{1}{2}(-x+3)^2\,dx = \frac{\rho}{2}\int_0^3 \left(x^2-6x+9\right)dx = \frac{\rho}{2}\left[\frac{x^3}{3}-3x^2+9x\right]_0^3 = \frac{\rho}{2}[9-27+27] = \frac{9}{2}\rho$$

$$\bar{y} = \frac{M_x}{m} = \frac{9/2\rho}{9/2\rho} = 1$$

$$M_y = \rho\int_0^3 x(-x+3)\,dx = \rho\left[\frac{-x^3}{3}+\frac{3x^2}{2}\right]_0^3 = \rho\left[-9+\frac{27}{2}\right] = \frac{9}{2}\rho$$

$$\bar{x} = \frac{M_y}{m} = \frac{9/2\rho}{9/2\rho} = 1$$

$$(\bar{x}, \bar{y}) = (1, 1)$$



15. $$m = \rho\int_0^4 \sqrt{x}\,dx = \left[\frac{2\rho}{3}x^{3/2}\right]_0^4 = \frac{16\rho}{3}$$

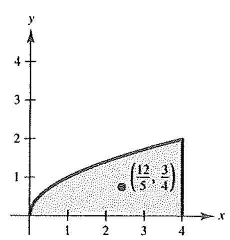
$$M_x = \rho\int_0^4 \frac{\sqrt{x}}{2}\left(\sqrt{x}\right)dx = \left[\rho\frac{x^2}{4}\right]_0^4 = 4\rho$$

$$\bar{y} = \frac{M_x}{m} = 4\rho\left(\frac{3}{16\rho}\right) = \frac{3}{4}$$

$$M_y = \rho\int_0^4 x\sqrt{x}\,dx = \left[\rho\frac{2}{5}x^{5/2}\right]_0^4 = \frac{64\rho}{5}$$

$$\bar{x} = \frac{M_y}{m} = \frac{64\rho}{5}\left(\frac{3}{16\rho}\right) = \frac{12}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{12}{5}, \frac{3}{4}\right)$$



16. $$m = \rho\int_0^3 \frac{1}{3}x^2\,dx$$

$$= \rho\left[\frac{x^3}{9}\right]_0^3 = 3\rho$$

$$M_x = \rho\int_0^3 \frac{1}{2}\left(\frac{1}{3}x^2\right)^2 dx = \frac{\rho}{18}\int_0^3 x^4\,dx$$

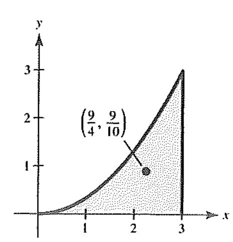
$$= \frac{\rho}{18}\left[\frac{x^5}{5}\right]_0^3 = \frac{27}{10}\rho$$

$$\bar{y} = \frac{M_x}{m} = \frac{27/10\rho}{3\rho} = \frac{9}{10}$$

$$M_y = \rho\int_0^3 x\left(\frac{1}{3}x^2\right)dx = \frac{\rho}{3}\left[\frac{x^4}{4}\right]_0^3 = \frac{27}{4}\rho$$

$$\bar{x} = \frac{M_y}{m} = \frac{27/4\rho}{3\rho} = \frac{9}{4}$$

$$(\bar{x}, \bar{y}) = \left(\frac{9}{4}, \frac{9}{10}\right)$$



17. $m = \rho\int_0^1 \left(x^2 - x^3\right) dx = \rho\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\rho}{12}$

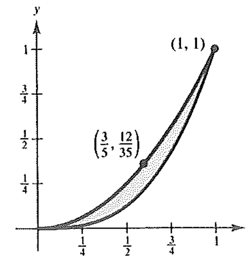
$M_x = \rho\int_0^1 \frac{\left(x^2 + x^3\right)}{2}\left(x^2 - x^3\right) dx = \frac{\rho}{2}\int_0^1\left(x^4 - x^6\right) dx = \frac{\rho}{2}\left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1 = \frac{\rho}{35}$

$\bar{y} = \frac{M_x}{m} = \frac{\rho}{35}\left(\frac{12}{\rho}\right) = \frac{12}{35}$

$M_y = \rho\int_0^1 x\left(x^2 - x^3\right) dx = \rho\int_0^1\left(x^3 - x^4\right) dx = \rho\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = \frac{\rho}{20}$

$\bar{x} = \frac{M_y}{m} = \frac{\rho}{20}\left(\frac{12}{\rho}\right) = \frac{3}{5}$

$(\bar{x}, \bar{y}) = \left(\frac{3}{5}, \frac{12}{35}\right)$



18. $m = \rho\int_0^4\left(\sqrt{x} - \frac{x}{2}\right) dx = \rho\left[\frac{2}{3}x^{3/2} - \frac{x^2}{4}\right]_0^4 = \rho\left[\frac{16}{3} - 4\right] = \frac{4}{3}\rho$

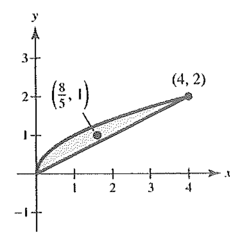
$M_x = \rho\int_0^4 \frac{1}{2}\left(\sqrt{x} + \frac{x}{2}\right)\left(\sqrt{x} - \frac{x}{2}\right) dx = \frac{1}{2}\rho\int_0^4\left(x - \frac{x^2}{4}\right) dx = \frac{\rho}{2}\left[\frac{x^2}{2} - \frac{x^3}{12}\right]_0^4 = \frac{\rho}{2}\left[8 - \frac{16}{3}\right] = \frac{4}{3}\rho$

$\bar{y} = \frac{M_x}{m} = \frac{4/3\rho}{4/3\rho} = 1$

$M_y = \rho\int_0^4 x\left(\sqrt{x} - \frac{x}{2}\right) dx = \rho\left[\frac{2}{5}x^{5/2} - \frac{x^3}{6}\right]_0^4 = \rho\left[\frac{64}{5} - \frac{32}{3}\right] = \frac{32}{15}\rho$

$\bar{x} = \frac{M_y}{m} = \frac{32/15\rho}{4/3\rho} = \frac{8}{5}$

$(\bar{x}, \bar{y}) = (8/5, 1)$



19. $m = \rho\int_0^3\left[\left(-x^2 + 4x + 2\right) - (x + 2)\right] dx = -\rho\left[\frac{x^3}{3} + \frac{3x^2}{2}\right]_0^3 = \frac{9\rho}{2}$

$M_x = \rho\int_0^3\left[\frac{\left(-x^2 + 4x + 2\right) + (x + 2)}{2}\right]\left[\left(-x^2 + 4x + 2\right) - (x + 2)\right] dx$

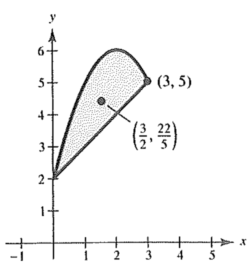
$= \frac{\rho}{2}\int_0^3\left(-x^2 + 5x + 4\right)\left(-x^2 + 3x\right) dx = \frac{\rho}{2}\int_0^3\left(x^4 - 8x^3 + 11x^2 + 12x\right) dx = \frac{\rho}{2}\left[\frac{x^5}{5} - 2x^4 + \frac{11x^3}{3} + 6x^2\right]_0^3 = \frac{99\rho}{5}$

$\bar{y} = \frac{M_x}{m} = \frac{99\rho}{5}\left(\frac{2}{9\rho}\right) = \frac{22}{5}$

$M_y = \rho\int_0^3 x\left[\left(-x^2 + 4x - 2\right) - (x + 2)\right] dx = \rho\int_0^3\left(-x^3 + 3x^2\right) dx = \rho\left[-\frac{x^4}{4} + x^3\right]_0^3 = \frac{27\rho}{4}$

$\bar{x} = \frac{M_y}{m} = \frac{27\rho}{4}\left(\frac{2}{9\rho}\right) = \frac{3}{2}$

$(\bar{x}, \bar{y}) = \left(\frac{3}{2}, \frac{22}{5}\right)$



20. $m = \rho\int_0^9\left[\left(\sqrt{x}+1\right)-\left(\frac{1}{3}x+1\right)\right]dx = \rho\int_0^9\left(\sqrt{x}-\frac{1}{3}x\right)dx = \rho\left[\frac{2}{3}x^{3/2}-\frac{x^2}{6}\right]_0^9 = \rho\left(18-\frac{27}{2}\right) = \frac{9}{2}\rho$

$$M_x = \rho\int_0^9 \frac{\sqrt{x}+1+(1/3)x+1}{2}\left(\sqrt{x}+1-\frac{1}{3}x-1\right)dx = \frac{\rho}{2}\int_0^9\left(\sqrt{x}+\frac{1}{3}x+2\right)\left(\sqrt{x}-\frac{1}{3}x\right)dx$$

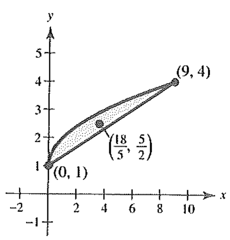
$$= \frac{\rho}{2}\int_0^9\left(x-\frac{1}{3}x^{3/2}+\frac{1}{3}x^{3/2}-\frac{1}{9}x^2+2\sqrt{x}-\frac{2}{3}x\right)dx = \frac{\rho}{2}\int_0^9\left(\frac{1}{3}x-\frac{1}{9}x^2+2\sqrt{x}\right)dx$$

$$= \frac{\rho}{2}\left[\frac{x^2}{6}-\frac{x^3}{27}+\frac{4}{3}x^{3/2}\right]_0^9 = \frac{\rho}{2}\left[\frac{27}{2}-27+36\right] = \frac{45}{4}\rho$$

$$M_y = \rho\int_0^9 x\left[\sqrt{x}+1-\frac{1}{3}x-1\right]dx = \rho\int_0^9\left(x^{3/2}-\frac{1}{3}x^2\right)dx = \rho\left[\frac{2}{5}x^{5/2}-\frac{1}{9}x^3\right]_0^9 = \rho\left[\frac{486}{5}-81\right] = \frac{81}{5}\rho$$

$$\bar{x} = \frac{M_y}{m} = \frac{(81/5)\rho}{(9/2)\rho} = \frac{18}{5};\ \bar{y} = \frac{M_x}{m} = \frac{(45/4)\rho}{(9/2)\rho} = \frac{5}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{18}{5}, \frac{5}{2}\right)$$



21. $m = \rho\int_0^8 x^{2/3}\,dx = \rho\left[\frac{3}{5}x^{5/3}\right]_0^8 = \frac{96\rho}{5}$

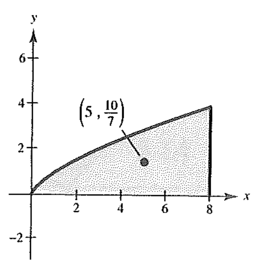
$$M_x = \rho\int_0^8 \frac{x^{2/3}}{2}\left(x^{2/3}\right)dx = \frac{\rho}{2}\left[\frac{3}{7}x^{7/3}\right]_0^8 = \frac{192\rho}{7}$$

$$\bar{y} = \frac{M_x}{m} = \frac{192\rho}{7}\left(\frac{5}{96\rho}\right) = \frac{10}{7}$$

$$M_y = \rho\int_0^8 x\left(x^{2/3}\right)dx = \rho\left[\frac{3}{8}x^{8/3}\right]_0^8 = 96\rho$$

$$\bar{x} = \frac{M_y}{m} = 96\rho\left(\frac{5}{96\rho}\right) = 5$$

$$(\bar{x}, \bar{y}) = \left(5, \frac{10}{7}\right)$$



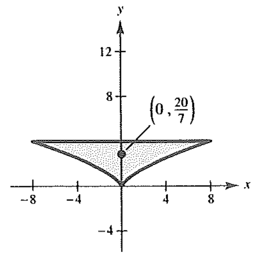
22. $m = 2\rho\int_0^8\left(4 - x^{2/3}\right)dx = 2\rho\left[4x - \frac{3}{5}x^{5/3}\right]_0^8 = \frac{128\rho}{5}$

By symmetry, M_y and $\bar{x} = 0$.

$$M_x = 2\rho\int_0^8\left(\frac{4 + x^{2/3}}{2}\right)\left(4 - x^{2/3}\right)dx = \rho\left[16x - \frac{3}{7}x^{7/3}\right]_0^8 = \frac{512\rho}{7}$$

$$\bar{y} = \frac{512\rho}{7}\left(\frac{5}{128\rho}\right) = \frac{20}{7}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{20}{7}\right)$$



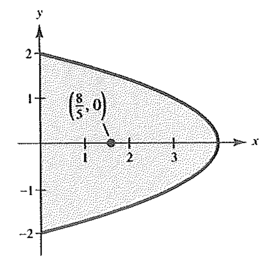
23. $m = 2\rho\int_0^2\left(4 - y^2\right)dy = 2\rho\left[4y - \frac{y^3}{3}\right]_0^2 = \frac{32\rho}{3}$

$$M_y = 2\rho\int_0^2\left(\frac{4 - y^2}{2}\right)\left(4 - y^2\right)dy = \rho\left[16y - \frac{8}{3}y^3 + \frac{y^5}{5}\right]_0^2 = \frac{256\rho}{15}$$

$$\bar{x} = \frac{M_y}{m} = \frac{256\rho}{15}\left(\frac{3}{32\rho}\right) = \frac{8}{5}$$

By symmetry, M_x and $\bar{y} = 0$.

$$(\bar{x}, \bar{y}) = \left(\frac{8}{5}, 0\right)$$



24. $m = \rho\int_0^2\left(2y - y^2\right)dy = \rho\left[y^2 - \frac{y^3}{3}\right]_0^2 = \frac{4\rho}{3}$

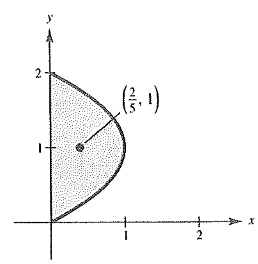
$$M_y = \rho\int_0^2\left(\frac{2y - y^2}{2}\right)\left(2y - y^2\right)dy = \frac{\rho}{2}\left[\frac{4y^3}{3} - y^4 + \frac{y^5}{5}\right]_0^2 = \frac{8\rho}{15}$$

$$\bar{x} = \frac{M_y}{m} = \frac{8\rho}{15}\left(\frac{3}{4\rho}\right) = \frac{2}{5}$$

$$M_x = \rho\int_0^2 y\left(2y - y^2\right)dy = \rho\left[\frac{2y^3}{3} - \frac{y^4}{4}\right]_0^2 = \frac{4\rho}{3}$$

$$\bar{y} = \frac{M_x}{m} = \frac{4\rho}{3}\left(\frac{3}{4\rho}\right) = 1$$

$$(\bar{x}, \bar{y}) = \left(\frac{2}{5}, 1\right)$$



25. $m = \rho\int_0^3\left[\left(2y - y^2\right) - (-y)\right]dy = \rho\left[\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3 = \frac{9\rho}{2}$

$$M_y = \rho\int_0^3\frac{\left[\left(2y - y^2\right) + (-y)\right]}{2}\left[\left(2y - y^2\right) - (-y)\right]dy = \frac{\rho}{2}\int_0^3\left(y - y^2\right)\left(3y - y^2\right)dy$$

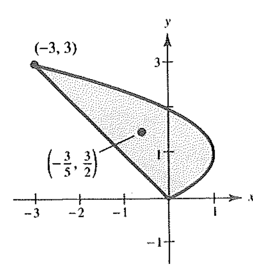
$$= \frac{\rho}{2}\int_0^3\left(y^4 - 4y^3 + 3y^2\right)dy = \frac{\rho}{2}\left[\frac{y^5}{5} - y^4 + y^3\right]_0^3 = -\frac{27\rho}{10}$$

$$\bar{x} = \frac{M_y}{m} = -\frac{27\rho}{10}\left(\frac{2}{9\rho}\right) = -\frac{3}{5}$$

$$M_x = \rho\int_0^3 y\left[\left(2y - y^2\right) - (-y)\right]dy = \rho\int_0^3\left(3y^2 - y^3\right)dy = \rho\left[y^3 - \frac{y^4}{4}\right]_0^3 = \frac{27\rho}{4}$$

$$\bar{y} = \frac{M_x}{m} = \frac{27\rho}{4}\left(\frac{2}{9\rho}\right) = \frac{3}{2}$$

$$(\bar{x}, \bar{y}) = \left(-\frac{3}{5}, \frac{3}{2}\right)$$



26. $m = \rho\int_{-1}^{2}\left[(y+2) - y^2\right] dy = \rho\left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2} = \frac{9\rho}{2}$

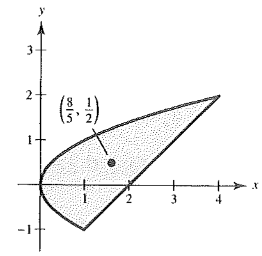
$M_y = \rho\int_{-1}^{2}\frac{\left[(y+2) + y^2\right]}{2}\left[(y+2) - y^2\right] dy = \frac{\rho}{2}\int_{-1}^{2}\left[(y+2)^2 - y^4\right] dy = \frac{\rho}{2}\left[\frac{(y+2)^3}{3} - \frac{y^5}{5}\right]_{-1}^{2} = \frac{36\rho}{5}$

$\bar{x} = \frac{M_y}{m} = \frac{36\rho}{5}\left(\frac{2}{9\rho}\right) = \frac{8}{5}$

$M_x = \rho\int_{-1}^{2} y\left[(y+2) - y^2\right] dy = \rho\int_{-1}^{2}\left(2y + y^2 - y^3\right) dy = \rho\left[y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right]_{-1}^{2} = \frac{9\rho}{4}$

$\bar{y} = \frac{M_x}{m} = \frac{9\rho}{4}\left(\frac{2}{9\rho}\right) = \frac{1}{2}$

$(\bar{x}, \bar{y}) = \left(\frac{8}{5}, \frac{1}{2}\right)$



27. $A = \int_0^2 \left(2x - x^2\right) dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$

$M_x = \frac{1}{2}\int_0^2\left[(2x)^2 - \left(x^2\right)^2\right] dx = \frac{1}{2}\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = \frac{1}{2}\left[\frac{32}{3} - \frac{32}{5}\right] = \frac{32}{15}$

$M_y = \int_0^2 x\left(2x - x^2\right) dx = \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{16}{3} - 4 = \frac{4}{3}$

28. $A = \int_1^4 \frac{1}{x}\, dx = \left[\ln|x|\right]_1^4 = \ln 4$

$M_x = \frac{1}{2}\int_1^4 \frac{1}{x^2}\, dx = \left[\frac{1}{2}\left(-\frac{1}{x}\right)\right]_1^4 = \left(-\frac{1}{8} + \frac{1}{2}\right) = \frac{3}{8}$

$M_y = \int_1^4 x\left(\frac{1}{x}\right) dx = \left[x\right]_1^4 = 3$

29. $A = \int_0^3 (2x + 4)\, dx = \left[x^2 + 4x\right]_0^3 = 9 + 12 = 21$

$M_x = \frac{1}{2}\int_0^3 (2x+4)^2\, dx = \int_0^3\left(2x^2 + 8x + 8\right) dx = \left[\frac{2x^3}{3} + 4x^2 + 8x\right]_0^3 = 18 + 36 + 24 = 78$

$M_y = \int_0^3\left(2x^2 + 4x\right) dx = \left[\frac{2x^3}{3} + 2x^2\right]_0^3 = 18 + 18 = 36$

30. $A = \int_{-2}^{2} -\left(x^2 - 4\right) dx = 2\int_0^2\left(4 - x^2\right) dx = \left[8x - \frac{2x^3}{3}\right]_0^2 = 16 - \frac{16}{3} = \frac{32}{3}$

$M_x = \frac{1}{2}\int_{-2}^{2}\left(x^2 - 4\right)\left(4 - x^2\right) dx = -\frac{1}{2}\int_{-2}^{2}\left(x^4 - 8x^2 + 16\right) dx = -\frac{1}{2}\left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right]_{-2}^{2} = -\left[\frac{32}{5} - \frac{64}{3} + 32\right] = -\frac{256}{15}$

$M_y = 0$ by symmetry.

31. $m = \rho\int_0^5 10x\sqrt{125 - x^3}\,dx \approx 1033.0\rho$

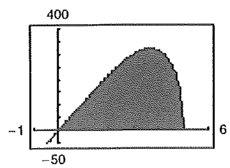
$$M_x = \rho\int_0^5 \left(\frac{10x\sqrt{125 - x^3}}{2}\right)\left(10x\sqrt{125 - x^3}\right)dx = 50\rho\int_0^5 x^2\left(125 - x^3\right)dx = \frac{3{,}124{,}375\rho}{24} \approx 130{,}208\rho$$

$$M_y = \rho\int_0^5 10x^2\sqrt{125 - x^3}\,dx = -\frac{10\rho}{3}\int_0^5 \sqrt{125 - x^3}\left(-3x^2\right)dx = \frac{12{,}500\sqrt{5}\rho}{9} \approx 3105.6\rho$$

$$\bar{x} = \frac{M_y}{m} \approx 3.0$$

$$\bar{y} = \frac{M_x}{m} \approx 126.0$$

Therefore, the centroid is $(3.0, 126.0)$.



32. $m = \rho\int_0^4 xe^{-x/2}\,dx \approx 2.3760\rho$

$$M_x = \rho\int_0^4 \left(\frac{xe^{-x/2}}{2}\right)\left(xe^{-x/2}\right)dx$$

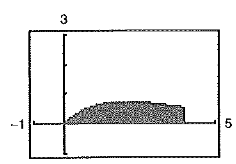
$$= \frac{\rho}{2}\int_0^4 x^2e^{-x}\,dx \approx 0.7619\rho$$

$$M_y = \rho\int_0^4 x^2e^{-x/2}\,dx \approx 5.1732\rho$$

$$\bar{x} = \frac{M_y}{m} \approx 2.2$$

$$\bar{y} = \frac{M_x}{m} \approx 0.3$$

Therefore, the centroid is $(2.2, 0.3)$.



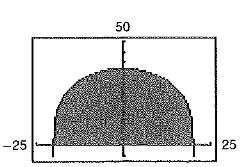
33. $m = \rho\int_{-20}^{20} 5\sqrt[3]{400 - x^2}\,dx \approx 1239.76\rho$

$$M_x = \rho\int_{-20}^{20} \frac{5\sqrt[3]{400 - x^2}}{2}\left(5\sqrt[3]{400 - x^2}\right)dx$$

$$= \frac{25\rho}{2}\int_{-20}^{20} \left(400 - x^2\right)^{2/3}dx \approx 20064.27$$

$$\bar{y} = \frac{M_x}{m} \approx 16.18$$

$\bar{x} = 0$ by symmetry. Therefore, the centroid is $(0, 16.2)$.



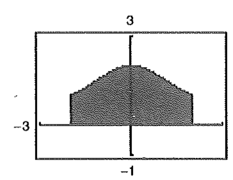
34. $m = \rho\int_{-2}^{2} \frac{8}{x^2 + 4}\,dx \approx 6.2832\rho$

$$M_x = \rho\int_{-2}^{2} \frac{1}{2}\left(\frac{8}{x^2 + 4}\right)\left(\frac{8}{x^2 + 4}\right)dx$$

$$= 32\rho\int_{-2}^{2} \frac{1}{\left(x^2 + 4\right)^2}\,dx \approx 5.14149\rho$$

$$\bar{y} = \frac{M_x}{m} \approx 0.8$$

$\bar{x} = 0$ by symmetry. Therefore, the centroid is $(0, 0.8)$.



35. $A = \frac{1}{2}(2a)c = ac$

$$\frac{1}{A} = \frac{1}{ac}$$

$$\bar{x} = \left(\frac{1}{ac}\right)\frac{1}{2}\int_0^c \left[\left(\frac{b-a}{c}y + a\right)^2 - \left(\frac{b+a}{c}y - a\right)^2\right]dy$$

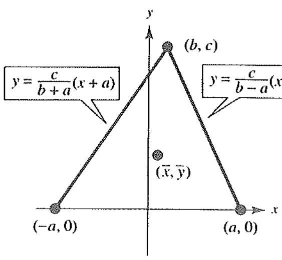
$$= \frac{1}{2ac}\int_0^c \left[\frac{4ab}{c}y - \frac{4ab}{c^2}y^2\right]dy = \frac{1}{2ac}\left[\frac{2ab}{c}y^2 - \frac{4ab}{3c^2}y^3\right]_0^c = \frac{1}{2ac}\left(\frac{2}{3}abc\right) = \frac{b}{3}$$

$$\bar{y} = \frac{1}{ac}\int_0^c y\left[\left(\frac{b-a}{c}y + a\right) - \left(\frac{b+a}{c}y - a\right)\right]dy$$

$$= \frac{1}{ac}\int_0^c y\left(-\frac{2a}{c}y + 2a\right)dy = \frac{2}{c}\int_0^c \left(y - \frac{y^2}{c}\right)dy = \frac{2}{c}\left[\frac{y^2}{2} - \frac{y^3}{3c}\right]_0^c = \frac{c}{3}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{c}{3}\right)$$

From elementary geometry, $(b/3, c/3)$ is the point of intersection of the medians.



36. $A = bh = ac$

$$\frac{1}{A} = \frac{1}{ac}$$

$$\bar{x} = \frac{1}{ac}\frac{1}{2}\int_0^c \left[\left(\frac{b}{c}y + a\right)^2 - \left(\frac{b}{c}y\right)^2\right]dy$$

$$= \frac{1}{2ac}\int_0^c \left(\frac{2ab}{c}y + a^2\right)dy$$

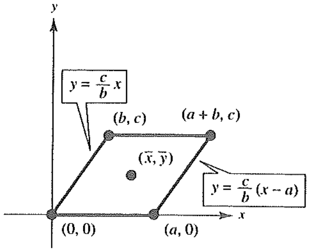
$$= \frac{1}{2ac}\left[\frac{ab}{c}y^2 + a^2y\right]_0^c$$

$$= \frac{1}{2ac}\left[abc + a^2c\right] = \frac{1}{2}(b + a)$$

$$\bar{y} = \frac{1}{ac}\int_0^c y\left[\left(\frac{b}{c}y + a\right) - \left(\frac{b}{c}y\right)\right]dy = \left[\frac{1}{c}\frac{y^2}{2}\right]_0^c = \frac{c}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b+a}{2}, \frac{c}{2}\right)$$

This is the point of intersection of the diagonals.



37. $A = \frac{c}{2}(a + b)$

$\frac{1}{A} = \frac{2}{c(a + b)}$

$$\bar{x} = \frac{2}{c(a+b)}\int_0^c x\left(\frac{b-a}{c}x + a\right)dx = \frac{2}{c(a+b)}\int_0^c \left(\frac{b-a}{c}x^2 + ax\right)dx = \frac{2}{c(a+b)}\left[\frac{b-a}{c}\frac{x^3}{3} + \frac{ax^2}{2}\right]_0^c$$

$$= \frac{2}{c(a+b)}\left[\frac{(b-a)c^2}{3} + \frac{ac^2}{2}\right] = \frac{2}{c(a+b)}\left[\frac{2bc^2 - 2ac^2 + 3ac^2}{6}\right] = \frac{c(2b+a)}{3(a+b)} = \frac{(a+2b)c}{3(a+b)}$$

$$\bar{y} = \frac{2}{c(a+b)}\frac{1}{2}\int_0^c \left(\frac{b-a}{c}x + a\right)^2 dx = \frac{1}{c(a+b)}\int_0^c \left[\left(\frac{b-a}{c}\right)^2 x^2 + \frac{2a(b-a)}{c}x + a^2\right]dx$$

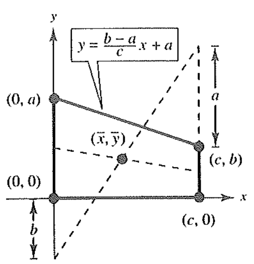
$$= \frac{1}{c(a+b)}\left[\left(\frac{b-a}{c}\right)^2\frac{x^3}{3} + \frac{2a(b-a)}{c}\frac{x^2}{2} + a^2x\right]_0^c = \frac{1}{c(a+b)}\left[\frac{(b-a)^2c}{3} + ac(b-a) + a^2c\right]$$

$$= \frac{1}{3c(a+b)}\left[(b^2 - 2ab + a^2)c + 3ac(b-a) + 3a^2c\right]$$

$$= \frac{1}{3(a+b)}\left[b^2 - 2ab + a^2 + 3ab - 3a^2 + 3a^2\right] = \frac{a^2 + ab + b^2}{3(a+b)}$$

So, $(\bar{x}, \bar{y}) = \left(\frac{(a+2b)c}{3(a+b)}, \frac{a^2+ab+b^2}{3(a+b)}\right)$.

The one line passes through $(0, a/2)$ and $(c, b/2)$. It's equation is $y = \frac{b-a}{2c}x + \frac{a}{2}$. The other line passes through $(0, -b)$ and $(c, a + b)$. It's equation is $y = \frac{a+2b}{c}x - b$. $(\bar{x}, \bar{y})$ is the point of intersection of these two lines.



38. $\bar{x} = 0$ by symmetry.

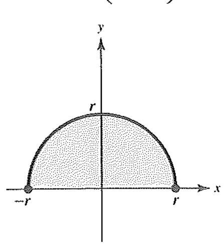
$A = \frac{1}{2}\pi r^2$

$\frac{1}{A} = \frac{2}{\pi r^2}$

$$\bar{y} = \frac{2}{\pi r^2}\frac{1}{2}\int_{-r}^{r}\left(\sqrt{r^2 - x^2}\right)^2 dx$$

$$= \frac{1}{\pi r^2}\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r} = \frac{1}{\pi r^2}\left(\frac{4r^3}{3}\right) = \frac{4r}{3\pi}$$

$(\bar{x}, \bar{y}) = \left(0, \frac{4r}{3\pi}\right)$



39. $\bar{x} = 0$ by symmetry.

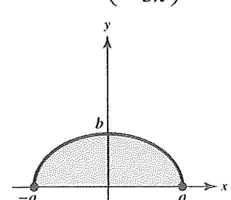
$A = \frac{1}{2}\pi ab$

$\frac{1}{A} = \frac{2}{\pi ab}$

$$\bar{y} = \frac{2}{\pi ab}\frac{1}{2}\int_{-a}^{a}\left(\frac{b}{a}\sqrt{a^2 - x^2}\right)^2 dx$$

$$= \frac{1}{\pi ab}\left(\frac{b^2}{a^2}\right)\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{b}{\pi a^3}\left(\frac{4a^3}{3}\right) = \frac{4b}{3\pi}$$

$(\bar{x}, \bar{y}) = \left(0, \frac{4b}{3\pi}\right)$



40. $A = \int_0^1 \left[1 - \left(2x - x^2\right)\right] dx = \frac{1}{3}$

$\frac{1}{A} = 3$

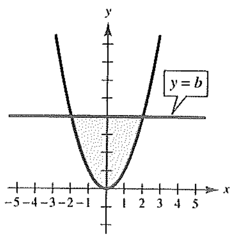
$$\bar{x} = 3\int_0^1 x\left[1 - \left(2x - x^2\right)\right] dx = 3\int_0^1 \left[x - 2x^2 + x^3\right] dx = 3\left[\frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4}\right]_0^1 = \frac{1}{4}$$

$$\bar{y} = 3\int_0^1 \frac{\left[1 + \left(2x - x^2\right)\right]}{2}\left[1 - \left(2x - x^2\right)\right] dx = \frac{3}{2}\int_0^1 \left[1 - \left(2x - x^2\right)^2\right] dx$$

$$= \frac{3}{2}\int_0^1 \left(1 - 4x^2 + 4x^3 - x^4\right) dx = \frac{3}{2}\left[x - \frac{4}{3}x^3 + x^4 - \frac{x^5}{5}\right]_0^1 = \frac{7}{10}$$

$(\bar{x}, \bar{y}) = \left(\frac{1}{4}, \frac{7}{10}\right)$

41. (a)



(b) $\bar{x} = 0$ by symmetry.

(c) $M_y = \int_{-\sqrt{b}}^{\sqrt{b}} x\left(b - x^2\right) dx = 0$ because $bx - x^3$ is odd.

(d) $\bar{y} > \frac{b}{2}$ because there is more area above $y = \frac{b}{2}$ than below.

(e) $$M_x = \int_{-\sqrt{b}}^{\sqrt{b}} \frac{\left(b + x^2\right)\left(b - x^2\right)}{2} dx = \int_{-\sqrt{b}}^{\sqrt{b}} \frac{b^2 - x^4}{2} dx = \frac{1}{2}\left[b^2x - \frac{x^5}{5}\right]_{-\sqrt{b}}^{\sqrt{b}} = b^2\sqrt{b} - \frac{b^2\sqrt{b}}{5} = \frac{4b^2\sqrt{b}}{5}$$

$$A = \int_{-\sqrt{b}}^{\sqrt{b}} \left(b - x^2\right) dx = \left[bx - \frac{x^3}{3}\right]_{-\sqrt{b}}^{\sqrt{b}} = \left(b\sqrt{b} - \frac{b\sqrt{b}}{3}\right)2 = 4\frac{b\sqrt{b}}{3}$$

$$\bar{y} = \frac{M_x}{A} = \frac{4b^2\sqrt{b}/5}{4b\sqrt{b}/3} = \frac{3}{5}b$$

42. (a) $M_y = 0$ by symmetry.

$$M_y = \int_{-2\sqrt[n]{b}}^{2\sqrt[n]{b}} x\left(b - x^{2n}\right) dx = 0$$

because $bx - x^{2n+1}$ is an odd function.

(b) $\bar{y} > \frac{b}{2}$ because there is more area above $y = \frac{b}{2}$ than below.

(c) $$M_x = \int_{-2\sqrt[n]{b}}^{2\sqrt[n]{b}} \frac{\left(b + x^{2n}\right)\left(b - x^{2n}\right)}{2} dx = \int_{-2\sqrt[n]{b}}^{2\sqrt[n]{b}} \frac{1}{2}\left(b^2 - x^{4n}\right) dx$$

$$= \frac{1}{2}\left(b^2x - \frac{x^{4n+1}}{4n+1}\right)\Bigg]_{-2\sqrt[n]{b}}^{2\sqrt[n]{b}} = b^2b^{1/2n} - \frac{b^{(4n+1)/2n}}{4n+1} = \frac{4n}{4n+1}b^{(4n+1)/2n}$$

$$A = \int_{-2\sqrt[n]{b}}^{2\sqrt[n]{b}} \left(b - x^{2n}\right) dx = 2\left[bx - \frac{x^{2n+1}}{2n+1}\right]_0^{2\sqrt[n]{b}} = 2\left[b \cdot b^{1/2n} - \frac{b^{(2n+1)/2n}}{2n+1}\right] = \frac{4n}{2n+1}b^{(2n+1)/2n}$$

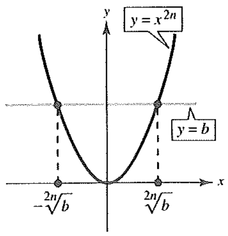
$$\bar{y} = \frac{M_x}{A} = \frac{4nb^{(4n+1)/2n}/(4n+1)}{4nb^{(24n+1)/2n}/(2n+1)} = \frac{2n+1}{4n+1}b$$

(d)

n	1	2	3	4
$\bar{y}$	$\frac{3}{5}b$	$\frac{5}{9}b$	$\frac{7}{13}b$	$\frac{9}{17}b$

(e) $\lim_{n\to\infty} \bar{y} = \lim_{n\to\infty} \frac{2n+1}{4n+1}b = \frac{1}{2}b$

(f) As $n \to \infty$, the figure gets narrower.



43. (a) $\bar{x} = 0$ by symmetry.

$$A = 2\int_0^{40} f(x)\,dx = \frac{2(40)}{3(4)}\left[30 + 4(29) + 2(26) + 4(20) + 0\right] = \frac{20}{3}(278) = \frac{5560}{3}$$

$$M_x = \int_{-40}^{40} \frac{f(x)^2}{2}\,dx = \frac{40}{3(4)}\left[30^2 + 4(29)^2 + 2(26)^2 + 4(20)^2 + 0\right] = \frac{10}{3}(7216) = \frac{72{,}160}{3}$$

$$\bar{y} = \frac{M_x}{A} = \frac{72{,}160/3}{5560/3} = \frac{72{,}160}{5560} \approx 12.98$$

$(\bar{x}, \bar{y}) = (0, 12.98)$

(b) $y = (-1.02 \times 10^{-5})x^4 - 0.0019x^2 + 29.28$ (Use nine data points.)

(c) $\bar{y} = \frac{M_x}{A} \approx \frac{23{,}697.68}{1843.54} \approx 12.85$

$(\bar{x}, \bar{y}) = (0, 12.85)$

44. Let $f(x)$ be the top curve, given by $l + d$. The bottom curve is $d(x)$.

x	0	0.5	1.0	1.5	2
f	2.0	1.93	1.73	1.32	0
d	0.50	0.48	0.43	0.33	0

(a) Area $= 2\int_0^2 \left[f(x) - d(x)\right]dx$

$$\approx 2\frac{2}{3(4)}\left[1.50 + 4(1.45) + 2(1.30) + 4(.99) + 0\right] = \frac{1}{3}[13.86] = 4.62$$

$$M_x = \int_{-2}^{2} \frac{f(x) + d(x)}{2}\left(f(x) - d(x)\right)dx$$

$$= \int_0^2 \left[f(x)^2 - d(x)^2\right]dx$$

$$= \frac{2}{3(4)}\left[3.75 + 4(3.4945) + 2(2.808) + 4(1.6335) + 0\right] = \frac{1}{6}[29.878] = 4.9797$$

$$\bar{y} = \frac{M_x}{A} = \frac{4.9797}{4.62} = 1.078$$

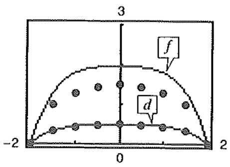
$(\bar{x}, \bar{y}) = (0, 1.078)$

(b) $f(x) = -0.1061x^4 - 0.06126x^2 + 1.9527$

$d(x) = -0.02648x^4 - 0.01497x^2 + .4862$

(c) $\bar{y} = \dfrac{M_x}{A} \approx \dfrac{4.9133}{4.59998} = 1.068$

$(\bar{x}, \bar{y}) = (0, 1.068)$



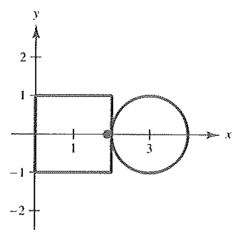
45. Centroids of the given regions: $(1, 0)$ and $(3, 0)$

Area: $A = 4 + \pi$

$$\bar{x} = \frac{4(1) + \pi(3)}{4 + \pi} = \frac{4 + 3\pi}{4 + \pi}$$

$$\bar{y} = \frac{4(0) + \pi(0)}{4 + \pi} = 0$$

$$(\bar{x}, \bar{y}) = \left(\frac{4 + 3\pi}{4 + \pi}, 0\right) \approx (1.88, 0)$$



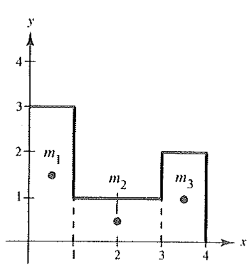
46. Centroids of the given regions: $\left(\frac{1}{2}, \frac{3}{2}\right)$, $\left(2, \frac{1}{2}\right)$, and $\left(\frac{7}{2}, 1\right)$

Area: $A = 3 + 2 + 2 = 7$

$$\bar{x} = \frac{3(1/2) + 2(2) + 2(7/2)}{7} = \frac{25/2}{7} = \frac{25}{14}$$

$$\bar{y} = \frac{3(3/2) + 2(1/2) + 2(1)}{7} = \frac{15/2}{7} = \frac{15}{14}$$

$$(\bar{x}, \bar{y}) = \left(\frac{25}{14}, \frac{15}{14}\right)$$



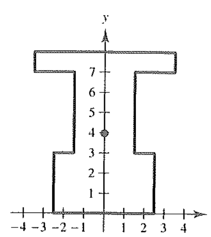
47. Centroids of the given regions: $\left(0, \dfrac{3}{2}\right)$, $(0, 5)$, and $\left(0, \dfrac{15}{2}\right)$

Area: $A = 15 + 12 + 7 = 34$

$$\bar{x} = \frac{15(0) + 12(0) + 7(0)}{34} = 0$$

$$\bar{y} = \frac{15(3/2) + 12(5) + 7(15/2)}{34} = \frac{135}{34}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{135}{34}\right) \approx (0, 3.97)$$



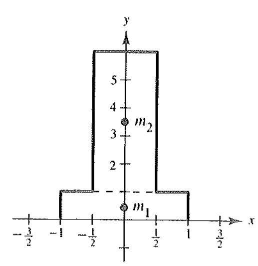
48. $m_1 = \dfrac{7}{8}(2) = \dfrac{7}{4}$, $P_1 = \left(0, \dfrac{7}{16}\right)$

$m_2 = \dfrac{7}{8}\left(6 - \dfrac{7}{8}\right) = \dfrac{287}{64}$, $P_2 = \left(0, \dfrac{55}{16}\right)$

By symmetry, $\bar{x} = 0$.

$$\bar{y} = \frac{(7/4)(7/16) + (287/64)(55/16)}{(7/4) + (287/64)} = \frac{16{,}569}{6384} = \frac{789}{304}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{789}{304}\right) \approx (0, 2.595)$$



49. Centroids of the given regions: $(1, 0)$ and $(3, 0)$

Mass: $4 + 2\pi$

$$\bar{x} = \frac{4(1) + 2\pi(3)}{4 + 2\pi} = \frac{2 + 3\pi}{2 + \pi}$$

$$\bar{y} = 0$$

$$(\bar{x}, \bar{y}) = \left(\frac{2 + 3\pi}{2 + \pi}, 0\right) \approx (2.22, 0)$$

50. Centroids of the given regions: $(3, 0)$ and $(1, 0)$

Mass: $8 + \pi$

$$\bar{y} = 0$$

$$\bar{x} = \frac{8(1) + \pi(3)}{8 + \pi} = \frac{8 + 3\pi}{8 + \pi}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8 + 3\pi}{8 + \pi}, 0\right) \approx (1.56, 0)$$

51. $r = 5$ is distance between center of circle and y-axis.

$A \approx \pi(4)^2 = 16\pi$ is area of circle. So,

$V = 2\pi rA = 2\pi(5)(16\pi) = 160\pi^2 \approx 1579.14.$

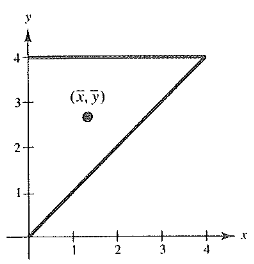
52. $V = 2\pi rA = 2\pi(3)(4\pi) = 24\pi^2$

53. $A = \frac{1}{2}(4)(4) = 8$

$$\bar{y} = \left(\frac{1}{8}\right)\frac{1}{2}\int_0^4 (4 + x)(4 - x)\,dx = \frac{1}{16}\left[16x - \frac{x^3}{3}\right]_0^4 = \frac{8}{3}$$

$$r = \bar{y} = \frac{8}{3}$$

$$V = 2\pi rA = 2\pi\left(\frac{8}{3}\right)(8) = \frac{128\pi}{3} \approx 134.04$$



54. $A = \int_2^6 2\sqrt{x - 2}\,dx = \frac{4}{3}(x - 2)^{3/2}\Big]_2^6 = \frac{32}{3}$

$$M_y = \int_2^6 (x)2\sqrt{x - 2}\,dx = 2\int_2^6 x\sqrt{x - 2}\,dx$$

Let $u = x - 2, x = u + 2, du = dx$:

$$M_y = 2\int_0^4 (u + 2)\sqrt{u}\,du$$

$$= 2\int_0^4 \left(u^{3/2} + 2u^{1/2}\right) du$$

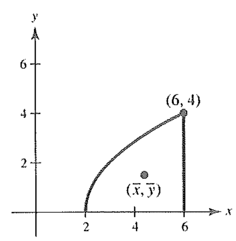
$$= 2\left[\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2}\right]_0^4$$

$$= 2\left(\frac{64}{5} + \frac{32}{3}\right) = \frac{704}{15}$$

$$\bar{x} = \frac{M_y}{A} = \frac{704/15}{32/3} = \frac{22}{5}$$

$$r = \bar{x} = \frac{22}{5}$$

$$V = 2\pi rA = 2\pi\left(\frac{22}{5}\right)\left(\frac{32}{3}\right) = \frac{1408\pi}{15} \approx 294.89$$



55. The center of mass $(\bar{x}, \bar{y})$ is $\bar{x} = M_y/m$ and $\bar{y} = M_x/m$, where:

1. $m = m_1 + m_2 + \cdots + m_n$ is the total mass of the system.
2. $M_y = m_1x_1 + m_2x_2 + \cdots + m_nx_n$ is the moment about the y-axis.
3. $M_x = m_1y_1 + m_2y_2 + \cdots + m_ny_n$ is the moment about the x-axis.

56. A planar lamina is a thin flat plate of constant density. The center of mass $(\bar{x}, \bar{y})$ is the balancing point on the lamina.

57. Let R be a region in a plane and let L be a line such that L does not intersect the interior of R. If r is the distance between the centroid of R and L, then the volume V of the solid of revolution formed by revolving R about L is $V = 2\pi rA$ where A is the area of R.

58. (a) Yes. $(\bar{x}, \bar{y}) = \left(\frac{5}{6}, \frac{5}{18} + 2\right) = \left(\frac{5}{6}, \frac{41}{18}\right)$

(b) Yes. $(\bar{x}, \bar{y}) = \left(\frac{5}{6} + 2, \frac{5}{18}\right) = \left(\frac{17}{6}, \frac{5}{18}\right)$

(c) Yes. $(\bar{x}, \bar{y}) = \left(\frac{5}{6}, -\frac{5}{18}\right)$

(d) No

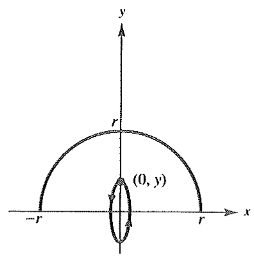
59. The surface area of the sphere is $S = 4\pi r^2$. The arc length of C is $s = \pi r$. The distance traveled by the centroid is

$$d = \frac{S}{s} = \frac{4\pi r^2}{\pi r} = 4r.$$

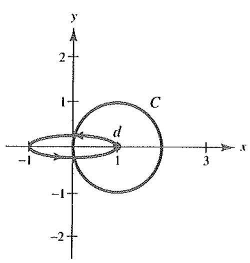
This distance is also the circumference of the circle of radius y.

$$d = 2\pi y$$

So, $2\pi y = 4r$ and you have $y = 2r/\pi$. Therefore, the centroid of the semicircle $y = \sqrt{r^2 - x^2}$ is $(0, 2r/\pi)$.



60. The centroid of the circle is $(1, 0)$. The distance traveled by the centroid is 2π. The arc length of the circle is also 2π. Therefore, $S = (2\pi)(2\pi) = 4\pi^2$.



61. $A = \int_0^1 x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}$

$$m = \rho A = \frac{\rho}{n+1}$$

$$M_x = \frac{\rho}{2}\int_0^1 (x^n)^2\,dx = \left[\frac{\rho}{2}\cdot\frac{x^{2n+1}}{2n+1}\right]_0^1 = \frac{\rho}{2(2n+1)}$$

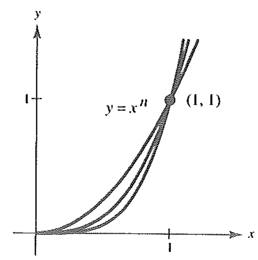
$$M_y = \rho\int_0^1 x(x^n)\,dx = \left[\rho\cdot\frac{x^{n+2}}{n+2}\right]_0^1 = \frac{\rho}{n+2}$$

$$\bar{x} = \frac{M_y}{m} = \frac{n+1}{n+2}$$

$$\bar{y} = \frac{M_x}{m} = \frac{n+1}{2(2n+1)} = \frac{n+1}{4n+2}$$

Centroid: $\left(\frac{n+1}{n+2}, \frac{n+1}{4n+2}\right)$

As $n \to \infty$, $(\bar{x}, \bar{y}) \to \left(1, \frac{1}{4}\right)$. The graph approaches the x-axis and the line $x = 1$ as $n \to \infty$.



62. Let T be the shaded triangle with vertices $(-1, 4)$, $(1, 4)$, and $(0, 3)$. Let U be the large triangle with vertices $(-4, 4)$, $(4, 4)$, and $(0, 0)$. V consists of the region U minus the region T.

Centroid of T: $\left(0, \frac{11}{3}\right)$; Area $= 1$

Centroid of U: $\left(0, \frac{8}{3}\right)$; Area $= 16$

Area: $V = 16 - 1 = 15$

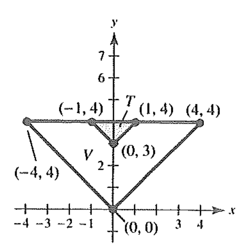
$\bar{x} = 0$ by symmetry.

$$15\bar{y} + 1\left(\frac{11}{3}\right) = 16\left(\frac{8}{3}\right)$$

$$15\bar{y} = \frac{117}{3}$$

$$\bar{y} = \frac{13}{5}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{13}{5}\right)$$



Section 7.7 Fluid Pressure and Fluid Force

1. $F = PA = [62.4(8)]3 = 1497.6 \text{ lb}$

2. $F = PA = [62.4(8)]16 = 7987.2 \text{ lb}$

3. $F = PA = [62.4(8)]10 = 4992 \text{ lb}$

4. $F = PA = [62.4(8)]22 = 10{,}982.4 \text{ lb}$

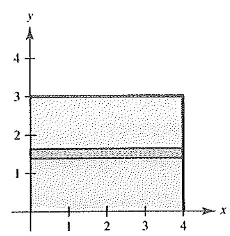
5. $F = 62.4(h+2)(6) - (62.4)(h)(6)$
$= 62.4(2)(6) = 748.8 \text{ lb}$

6. $F = 62.4(h+4)(48) - (62.4)(h)(48)$
$= 62.4(4)(48) = 11{,}980.8 \text{ lb}$

7. $h(y) = 3 - y$

$L(y) = 4$

$$F = 62.4\int_0^3 (3-y)(4)\,dy$$
$$= 249.6\int_0^3 (3-y)\,dy$$
$$= 249.6\left[3y - \frac{y^2}{2}\right]_0^3 = 1123.2 \text{ lb}$$

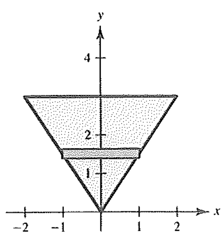


8. $h(y) = 3 - y$

$L(y) = \frac{4}{3}y$

$$F = 62.4\int_0^3 (3-y)\left(\frac{4}{3}y\right)dy$$
$$= \frac{4}{3}(62.4)\int_0^3 (3y - y^2)\,dy$$
$$= \frac{4}{3}(62.4)\left[\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3 = 374.4 \text{ lb}$$

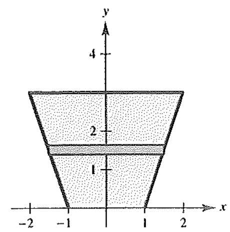
Force is one-third that of Exercise 7.



9. $h(y) = 3 - y$

$L(y) = 2\left(\frac{y}{3} + 1\right)$

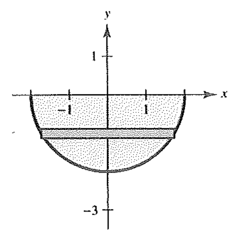
$$F = 2(62.4)\int_0^3 (3-y)\left(\frac{y}{3}+1\right)dy$$
$$= 124.8\int_0^3\left(3 - \frac{y^2}{3}\right)dy$$
$$= 124.8\left[3y - \frac{y^3}{9}\right]_0^3 = 748.8 \text{ lb}$$



10. $h(y) = -y$

$L(y) = 2\sqrt{4 - y^2}$

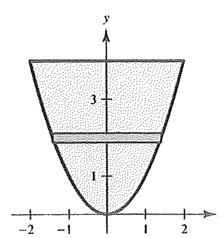
$$F = 62.4\int_{-2}^0 (-y)(2)\sqrt{4-y^2}\,dy$$
$$= \left[62.4\left(\tfrac{2}{3}\right)\left(4-y^2\right)^{3/2}\right]_{-2}^0 = 332.8 \text{ lb}$$



11. $h(y) = 4 - y$

$L(y) = 2\sqrt{y}$

$$F = 2(62.4)\int_0^4 (4-y)\sqrt{y}\,dy$$
$$= 124.8\int_0^4\left(4y^{1/2} - y^{3/2}\right)dy$$
$$= 124.8\left[\frac{8y^{3/2}}{3} - \frac{2y^{5/2}}{5}\right]_0^4 = 1064.96 \text{ lb}$$



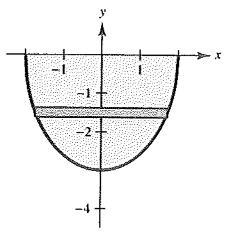
12. $h(y) = -y$

$L(y) = \frac{4}{3}\sqrt{9 - y^2}$

$$F = 62.4\int_{-3}^{0} (-y)\frac{4}{3}\sqrt{9 - y^2}\, dy$$

$$= 62.4\left(\frac{2}{3}\right)\int_{-3}^{0} (9 - y^2)^{1/2}(-2y)\, dy$$

$$= \left[62.4\left(\frac{4}{9}\right)(9 - y^2)^{3/2}\right]_{-3}^{0} = 748.8 \text{ lb}$$

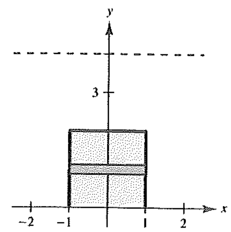


13. $h(y) = 4 - y$

$L(y) = 2$

$$F = 9800\int_{0}^{2} 2(4 - y)\, dy$$

$$= 9800\left[8y - y^2\right]_{0}^{2} = 117{,}600 \text{ newtons}$$



14. $h(y) = \left(1 + 3\sqrt{2}\right) - y$

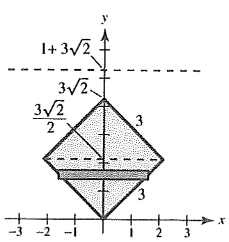
$L_1(y) = 2y$ (lower part)

$L_2(y) = 2\left(3\sqrt{2} - y\right)$ (upper part)

$$F = 2(9800)\left[\int_{0}^{3\sqrt{2}/2} \left(1 + 3\sqrt{2} - y\right)y\, dy + \int_{3\sqrt{2}/2}^{3\sqrt{2}} \left(1 + 3\sqrt{2} - y\right)\left(3\sqrt{2} - y\right) dy\right]$$

$$= 19{,}600\left[\left[\frac{y^2}{2} - 3\sqrt{2}y - \frac{y^3}{3}\right]_{0}^{3\sqrt{2}/2} + \left[3\sqrt{2}y + 18y + \frac{y^3}{3} - \frac{6\sqrt{2} + 1}{2}y\right]_{3\sqrt{2}/2}^{3\sqrt{2}}\right]$$

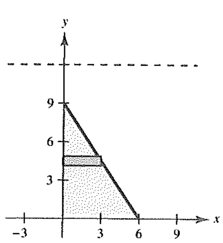
$$= 19{,}600\left[\frac{9\left(2\sqrt{2} + 1\right)}{4} + \frac{9\left(\sqrt{2} + 1\right)}{4}\right] = 44{,}100\left(3\sqrt{2} + 2\right) \text{ newtons}$$



15. $h(y) = 12 - y$

$L(y) = 6 - \frac{2y}{3}$

$$F = 9800\int_{0}^{9} (12 - y)\left(6 - \frac{2y}{3}\right) dy = 9800\left[72y - 7y^2 + \frac{2y^3}{9}\right]_{0}^{9} = 2{,}381{,}400 \text{ newtons}$$



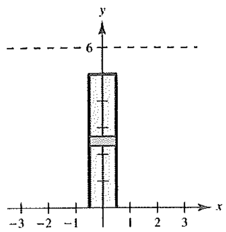
16. $h(y) = 6 - y$

$L(y) = 1$

$$F = 9800\int_0^5 1(6 - y)\,dy$$

$$= 9800\left[6y - \frac{y^2}{2}\right]_0^5$$

$$= 171{,}500 \text{ newtons}$$



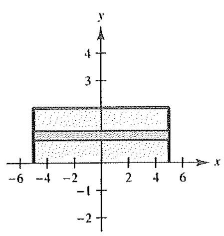
17. $h(y) = 2 - y$

$L(y) = 10$

$$F = 140.7\int_0^2 (2 - y)(10)\,dy$$

$$= 1407\int_0^2 (2 - y)\,dy$$

$$= 1407\left[2y - \frac{y^2}{2}\right]_0^2 = 2814 \text{ lb}$$



18. $h(y) = -y$

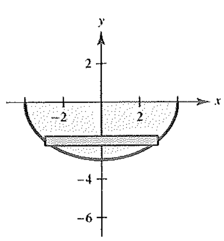
$L(y) = 2\left(\frac{4}{3}\sqrt{9 - y^2}\right)$

$$F = 140.7\int_{-3}^0 (-y)(2)\left(\frac{4}{3}\sqrt{9 - y^2}\right) dy$$

$$= \frac{(140.7)(4)}{3}\int_{-3}^0 \sqrt{9 - y^2}(-2y)\,dy$$

$$= \left[\frac{(140.7)(4)}{3}\left(\frac{2}{3}\right)(9 - y^2)^{3/2}\right]_{-3}^0$$

$$= 3376.8 \text{ lb}$$



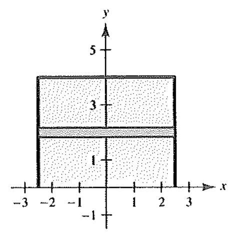
19. $h(y) = 4 - y$

$L(y) = 6$

$$F = 140.7\int_0^4 (4 - y)(6)\,dy$$

$$= 844.2\int_0^4 (4 - y)\,dy$$

$$= 844.2\left[4y - \frac{y^2}{2}\right]_0^4 = 6753.6 \text{ lb}$$



20. $h(y) = -y$

$L(y) = 5 + \frac{5}{3}y$

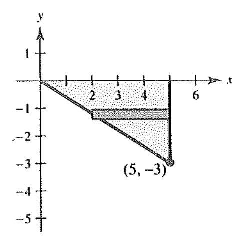
$$F = 140.7\int_{-3}^0 (-y)\left(5 + \frac{5}{3}y\right) dy$$

$$= 140.7\int_{-3}^0 \left(-5y - \frac{5}{3}y^2\right) dy$$

$$= 140.7\left[-\frac{5}{2}y^2 - \frac{5}{9}y^3\right]_{-3}^0$$

$$= 140.7\left[\frac{45}{2} - 15\right]$$

$$= 1055.25 \text{ lb}$$



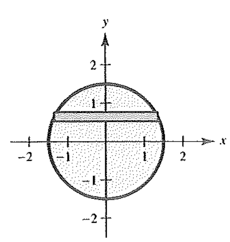
21. $h(y) = -y$

$L(y) = 2\left(\frac{1}{2}\right)\sqrt{9 - 4y^2}$

$$F = 42\int_{-3/2}^0 (-y)\sqrt{9 - 4y^2}\,dy$$

$$= \frac{42}{8}\int_{-3/2}^0 (9 - 4y^2)^{1/2}(-8y)\,dy$$

$$= \left[\left(\frac{21}{4}\right)\left(\frac{2}{3}\right)(9 - 4y^2)^{3/2}\right]_{-3/2}^0 = 94.5 \text{ lb}$$



22. $h(y) = \frac{3}{2} - y$

$L(y) = 2\left(\frac{1}{2}\right)\sqrt{9 - 4y^2}$

$$F = 42\int_{-3/2}^{3/2}\left(\frac{3}{2} - y\right)\sqrt{9 - 4y^2}\,dy = 63\int_{-3/2}^{3/2}\sqrt{9 - 4y^2}\,dy + \frac{21}{4}\int_{-3/2}^{3/2}\sqrt{9 - 4y^2}(-8y)\,dy$$

The second integral is zero because it is an odd function and the limits of integration are symmetric to the origin. The first integral is twice the area of a semicircle of radius $\frac{3}{2}$.

$$\left(\sqrt{9 - 4y^2} = 2\sqrt{(9/4) - y^2}\right)$$

So, the force is $63\left(\frac{9}{4}\pi\right) = 141.75\pi \approx 445.32$ lb.

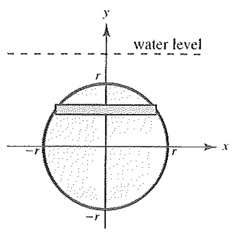
23. $h(y) = k - y$

$L(y) = 2\sqrt{r^2 - y^2}$

$$F = w\int_{-r}^{r}(k - y)\sqrt{r^2 - y^2}(2)\,dy = w\left[2k\int_{-r}^{r}\sqrt{r^2 - y^2}\,dy + \int_{-r}^{r}\sqrt{r^2 - y^2}(-2y)\,dy\right]$$

The second integral is zero because its integrand is odd and the limits of integration are symmetric to the origin. The first integral is the area of a semicircle with radius r.

$$F = w\left[(2k)\frac{\pi r^2}{2} + 0\right] = wk\pi r^2$$



24. (a) $F = wk\pi r^2 = (62.4)(7)(\pi 2^2) = 1747.2\pi$ lb

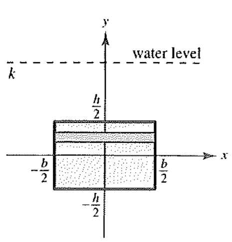
(b) $F = wk\pi r^2 = (62.4)(5)(\pi 3^2) = 2808\pi$ lb

25. $h(y) = k - y$

$L(y) = b$

$$F = w\int_{-h/2}^{h/2}(k - y)b\,dy$$

$$= wb\left[ky - \frac{y^2}{2}\right]_{-h/2}^{h/2} = wb(hk) = wkhb$$



26. (a) $F = wkhb$

$= (62.4)\left(\frac{11}{2}\right)(3)(5) = 5148$ lb

(b) $F = wkhb$

$= (62.4)\left(\frac{17}{2}\right)(5)(10) = 26{,}520$ lb

27. From Exercise 25:

$F = 64(15)(1)(1) = 960$ lb

28. From Exercise 23:

$F = 64(15)\pi\left(\frac{1}{2}\right)^2 \approx 753.98$ lb

29. $h(y) = 4 - y$

$F = 62.4 \int_0^4 (4 - y)L(y)\,dy$

Using Simpson's Rule with $n = 8$ you have:

$F \approx 62.4\left(\frac{4-0}{3(8)}\right)\left[0 + 4(3.5)(3) + 2(3)(5) + 4(2.5)(8) + 2(2)(9) + 4(1.5)(10) + 2(1)(10.25) + 4(0.5)(10.5) + 0\right]$

$= 3010.8$ lb

30. $h(y) = 3 - y$

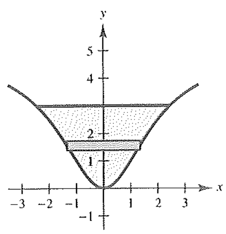
Solving $y = 5x^2/(x^2 + 4)$ for x, you obtain

$x = \sqrt{4y/(5 - y)}.$

$L(y) = 2\sqrt{\frac{4y}{5 - y}}$

$F = 62.4(2)\int_0^3 (3 - y)\sqrt{\frac{4y}{5 - y}}\,dy$

$= 2(124.8)\int_0^3 (3 - y)\sqrt{\frac{y}{5 - y}}\,dy \approx 546.265$ lb

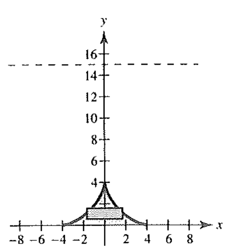


31. $h(y) = 15 - y$

$L(y) = 2\left(4^{2/3} - y^{2/3}\right)^{3/2}$

$F = 62.4\int_0^4 2(15 - y)\left(4^{2/3} - y^{2/3}\right)^{3/2}\,dy$

≈ 8213.04 lb



32. $h(y) = 15 - y$

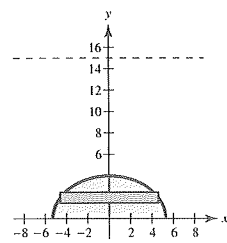
$\frac{x^2}{28} = 1 - \frac{y_2}{16} = \frac{16 - y^2}{16}$

$x^2 = \frac{7}{4}(16 - y^2)$

$x = \frac{1}{2}\sqrt{7(16 - y^2)}$

$L(y) = \sqrt{7(16 - y^2)}$

$F = 62.4\sqrt{7}\int_0^4 (15 - y)\sqrt{16 - y^2}\,dy \approx 27{,}597.63$ lb



33. If the fluid force is one-half of 1123.2 lb, and the height of the water is b, then

$h(y) = b - y$

$L(y) = 4$

$F = 62.4\int_0^b (b - y)(4)\,dy = \frac{1}{2}(1123.2)$

$\int_0^b (b - y)\,dy = 2.25$

$\left[by - \frac{y^2}{2}\right]_0^b = 2.25$

$b^2 - \frac{b^2}{2} = 2.25$

$b^2 = 4.5 \Rightarrow b \approx 2.12$ ft.

The pressure increases with increasing depth.

34. (a) Fluid pressure is the force per unit of area exerted by a fluid over the surface of a body.

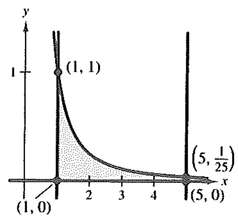
(b) $F = Fw = w\int_c^d h(y)L(y)\,dy$, see page 510.

35. You use horizontal representative rectangles because you are measuring total force against a region between two depths.

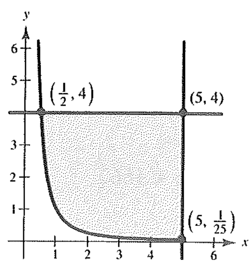
36. The left window experiences the greater fluid force because its centroid is lower.

Review Exercises for Chapter 7

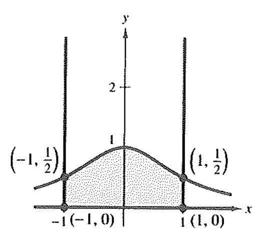
1. $A = \int_1^5 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^5 = \frac{4}{5}$



2. $A = \int_{1/2}^5 \left(4 - \frac{1}{x^2}\right)dx = \left[4x + \frac{1}{x}\right]_{1/2}^5 = \frac{81}{5}$



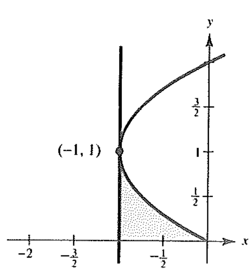
3. $A = \int_{-1}^1 \frac{1}{x^2+1}\,dx = \left[\arctan x\right]_{-1}^1 = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$



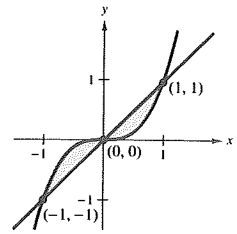
4. $A = \int_0^1 \left[\left(y^2 - 2y\right) - (-1)\right]dy$

$= \int_0^1 \left(y^2 - 2y + 1\right)dy$

$= \int_0^1 (y-1)^2\,dy = \left[\frac{(y-1)^3}{3}\right]_0^1 = \frac{1}{3}$

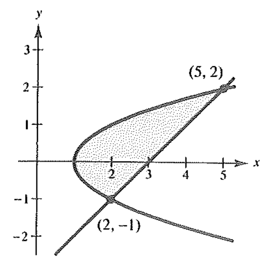


5. $A = 2\int_0^1 \left(x - x^3\right)dx = 2\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1 = \frac{1}{2}$

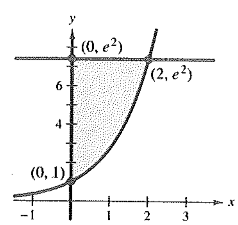


6. $A = \int_{-1}^2 \left[(y+3) - \left(y^2+1\right)\right]dy$

$= \int_{-1}^2 \left(2 + y - y^2\right)dy = \left[2y + \frac{1}{2}y^2 - \frac{1}{3}y^3\right]_{-1}^2 = \frac{9}{2}$



7. $A = \int_0^2 \left(e^2 - e^x\right)dx = \left[xe^2 - e^x\right]_0^2 = e^2 + 1$

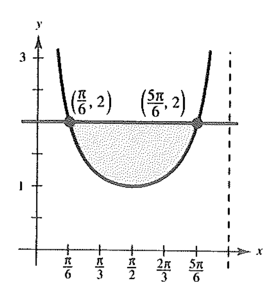


8. $A = 2\int_{\pi/6}^{\pi/2} (2 - \csc x)\,dx$

$= 2\left[2x - \ln|\csc x - \cot x|\right]_{\pi/6}^{\pi/2}$

$= 2\left(\left[\pi - 0\right] - \left[\frac{\pi}{3} - \ln\left(2 - \sqrt{3}\right)\right]\right)$

$= 2\left[\frac{2\pi}{3} + \ln\left(2 - \sqrt{3}\right)\right] \approx 1.555$

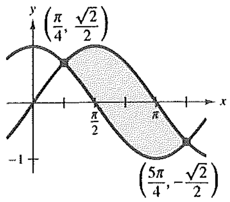


9. $A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx$

$= \left[-\cos x - \sin x\right]_{\pi/4}^{5\pi/4}$

$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$

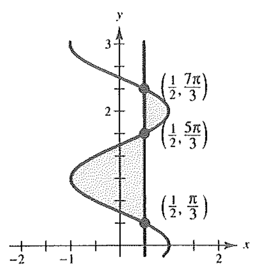
$= \frac{4}{\sqrt{2}} = 2\sqrt{2}$



10. $A = \int_{\pi/3}^{5\pi/3} \left(\frac{1}{2} - \cos y\right) dy + \int_{5\pi/3}^{7\pi/3} \left(\cos y - \frac{1}{2}\right) dy$

$= \left[\frac{y}{2} - \sin y\right]_{\pi/3}^{5\pi/3} + \left[\sin y - \frac{y}{2}\right]_{5\pi/3}^{7\pi/3}$

$= \frac{\pi}{3} + 2\sqrt{3}$



11. Points of intersection:

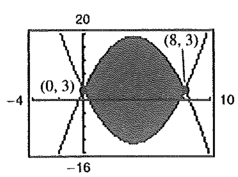
$x^2 - 8x + 3 = 3 + 8x - x^2$

$2x^2 - 16x = 0$ when $x = 0, 8$

$A = \int_0^8 \left[\left(3 + 8x - x^2\right) - \left(x^2 - 8x + 3\right)\right] dx$

$= \int_0^8 \left(16x - 2x^2\right) dx$

$= \left[8x^2 - \frac{2}{3}x^3\right]_0^8 = \frac{512}{3} \approx 170.667$



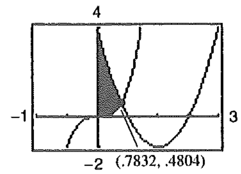
12. Point of intersection:

$x^3 - x^2 + 4x - 3 = 0 \Rightarrow x \approx 0.783.$

$A \approx \int_0^{0.783} \left(3 - 4x + x^2 - x^3\right) dx$

$= \left[3x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^{0.783}$

≈ 1.189

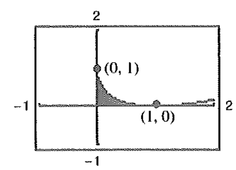


13. $y = \left(1 - \sqrt{x}\right)^2$

$A = \int_0^1 \left(1 - \sqrt{x}\right)^2 dx$

$= \int_0^1 \left(1 - 2x^{1/2} + x\right) dx$

$= \left[x - \frac{4}{3}x^{3/2} + \frac{1}{2}x^2\right]_0^1 = \frac{1}{6} \approx 0.1667$



14. Points of intersection:

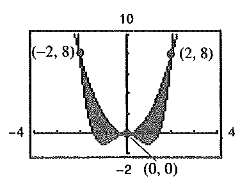
$x^4 - 2x^2 = 2x^2$

$x^4 - 4x^2 = 0$ when $x = 0, \pm 2$

$A = 2\int_0^2 \left[2x^2 - \left(x^4 - 2x^2\right)\right] dx$

$= 2\int_0^2 \left(4x^2 - x^4\right) dx$

$= 2\left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2 = \frac{128}{15} \approx 8.5333$



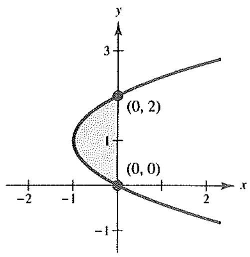
15. $x = y^2 - 2y \Rightarrow x + 1 = (y-1)^2 \Rightarrow y = 1 \pm \sqrt{x+1}$

$$A = \int_{-1}^{0} \left[\left(1 + \sqrt{x+1}\right) - \left(1 - \sqrt{x+1}\right)\right] dx$$

$$= \int_{-1}^{0} 2\sqrt{x+1}\, dx$$

$$A = \int_{0}^{2} \left[0 - \left(y^2 - 2y\right)\right] dy$$

$$= \int_{0}^{2} \left(2y - y^2\right) dy = \left[y^2 - \tfrac{1}{3}y^3\right]_0^2 = \tfrac{4}{3}$$



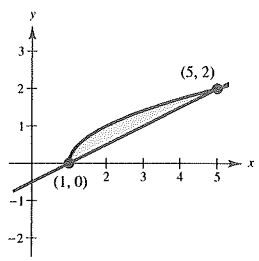
16. $y = \sqrt{x-1} \Rightarrow x = y^2 + 1$

$$y = \frac{x-1}{2} \Rightarrow x = 2y + 1$$

$$A = \int_{0}^{2} \left[(2y+1) - \left(y^2 + 1\right)\right] dy$$

$$A = \int_{1}^{5} \left[\sqrt{x-1} - \frac{x-1}{2}\right] dx$$

$$= \left[\frac{2}{3}(x-1)^{3/2} - \frac{1}{4}(x-1)^2\right]_1^5 = \frac{4}{3}$$



17. $A = \int_{0}^{2} \left[1 - \left(1 - \dfrac{x}{2}\right)\right] dx + \int_{2}^{3} \left[1 - (x-2)\right] dx$

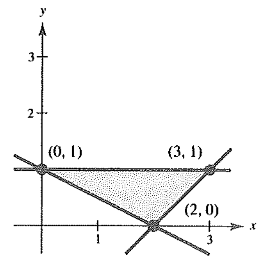
$$= \int_{0}^{2} \frac{x}{2}\, dx + \int_{2}^{3} (3 - x)\, dx$$

$$y = 1 - \frac{x}{2} \Rightarrow x = 2 - 2y$$

$$y = x - 2 \Rightarrow x = y + 2,\ y = 1$$

$$A = \int_{0}^{1} \left[(y+2) - (2 - 2y)\right] dy$$

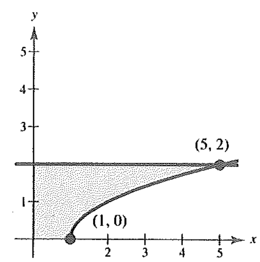
$$= \int_{0}^{1} 3y\, dy = \left[\frac{3}{2}y^2\right]_0^1 = \frac{3}{2}$$



18. $A = \int_{0}^{1} 2\, dx + \int_{1}^{5} \left[2 - \sqrt{x-1}\right] dx$

$$x = y^2 + 1$$

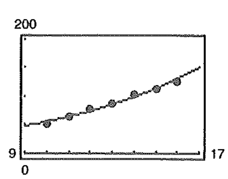
$$A = \int_{0}^{2} \left(y^2 + 1\right) dy = \left[\tfrac{1}{3}y^3 + y\right]_0^2 = \tfrac{14}{3}$$



19. (a) Trapezoidal: Area $\approx \frac{160}{2(8)}\left[0 + 2(50) + 2(54) + 2(82) + 2(82) + 2(73) + 2(75) + 2(80) + 0\right] = 9920 \text{ ft}^2$

(b) Simpson's: Area $\approx \frac{160}{3(8)}\left[0 + 4(50) + 2(54) + 4(82) + 2(82) + 4(73) + 2(75) + 4(80) + 0\right] = 10{,}413\tfrac{1}{3} \text{ ft}^2$

20. (a) $y = 13.2945(1.1539)^t = 13.2945\, e^{0.1432t}$

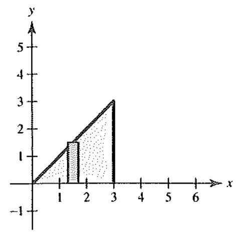


(b) $R_2 = 6 + 13.9e^{0.14t}$

Difference: $\int_{20}^{25} (y - R_2)\, dt \approx 17.7$ billion dollars

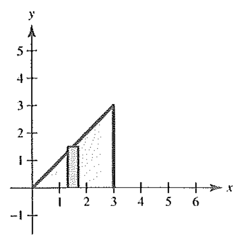
21. (a) **Disk**

$$V = \pi \int_0^3 x^2\, dx = \left[\frac{\pi x^3}{3}\right]_0^3 = 9\pi$$



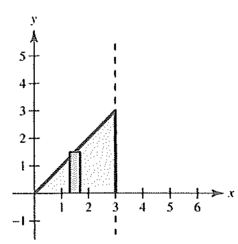
(b) **Shell**

$$V = 2\pi \int_0^3 x(x)\, dx = 2\pi \left[\frac{x^3}{3}\right]_0^3 = 18\pi$$



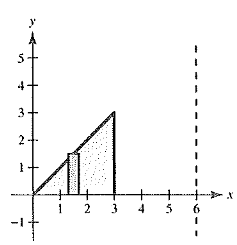
(c) **Shell**

$$V = 2\pi \int_0^3 (3 - x)x\, dx = 2\pi \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = 9\pi$$



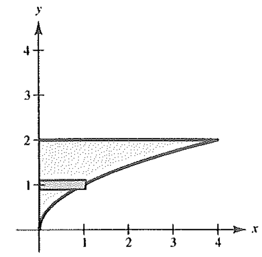
(d) **Shell**

$$V = 2\pi \int_0^3 (6 - x)x\, dx = 2\pi \left[3x^2 - \frac{x^3}{3}\right]_0^3 = 36\pi$$



22. (a) **Shell**

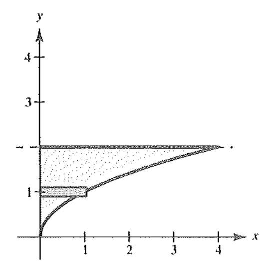
$$V = 2\pi \int_0^2 y^3\, dy = \left[\frac{\pi}{2}y^4\right]_0^2 = 8\pi$$



(b) **Shell**

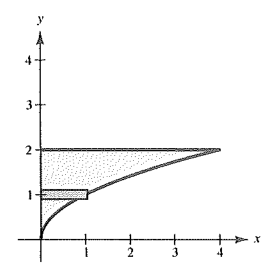
$$V = 2\pi \int_0^2 (2 - y)y^2\, dy$$

$$= 2\pi \int_0^2 \left(2y^2 - y^3\right) dy = 2\pi \left[\frac{2}{3}y^3 - \frac{1}{4}y^4\right]_0^2 = \frac{8\pi}{3}$$



(c) **Disk**

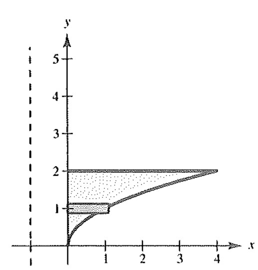
$$V = \pi \int_0^2 y^4\, dy = \left[\frac{\pi}{5}y^5\right]_0^2 = \frac{32\pi}{5}$$



(d) **Disk**

$$V = \pi \int_0^2 \left[\left(y^2 + 1\right)^2 - 1^2\right] dy$$

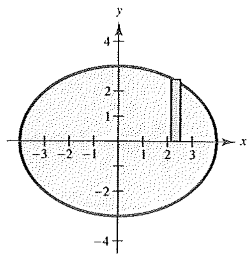
$$= \pi \int_0^2 \left(y^4 + 2y^2\right) dy = \pi \left[\frac{1}{5}y^5 + \frac{2}{3}y^3\right]_0^2 = \frac{176\pi}{15}$$



23. (a) **Shell**

$$V = 4\pi \int_0^4 x\left(\frac{3}{4}\right)\sqrt{16 - x^2}\, dx$$

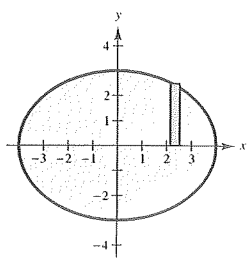
$$= \left[3\pi\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(16 - x^2\right)^{3/2}\right]_0^4 = 64\pi$$



(b) **Disk**

$$V = 2\pi \int_0^4 \left[\frac{3}{4}\sqrt{16 - x^2}\right]^2 dx$$

$$= \frac{9\pi}{8}\left[16x - \frac{x^3}{3}\right]_0^4 = 48\pi$$

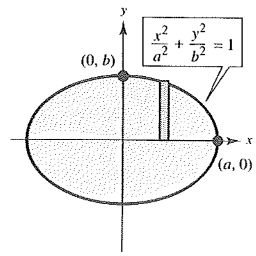


24. (a) **Shell**

$$V = 4\pi \int_0^a (x)\frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

$$= \frac{-2\pi b}{a}\int_0^a \left(a^2 - x^2\right)^{1/2}(-2x)\, dx$$

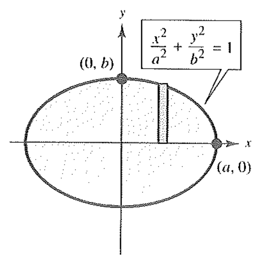
$$= \left[\frac{-4\pi b}{3a}\left(a^2 - x^2\right)^{3/2}\right]_0^a = \frac{4}{3}\pi a^2 b$$



(b) **Disk**

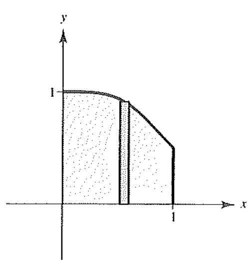
$$V = 2\pi \int_0^a \frac{b^2}{a^2}\left(a^2 - x^2\right) dx$$

$$= \frac{2\pi b^2}{a^2}\left[a^2 x - \frac{1}{3}x^3\right]_0^a = \frac{4}{3}\pi a b^2$$



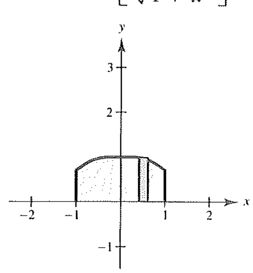
25. Shell

$$V = 2\pi \int_0^1 \frac{x}{x^4 + 1}\, dx = \pi \int_0^1 \frac{(2x)}{\left(x^2\right)^2 + 1}\, dx = \left[\pi \arctan\left(x^2\right)\right]_0^1 = \pi\left(\frac{\pi}{4} - 0\right) = \frac{\pi^2}{4}$$

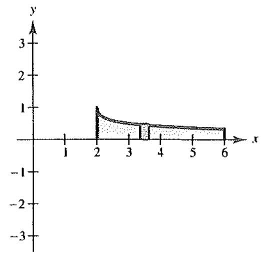


26. Disk

$$V = 2\pi \int_0^1 \left[\frac{1}{\sqrt{1 + x^2}}\right]^2 dx = \left[2\pi \arctan x\right]_0^1 = 2\pi\left(\frac{\pi}{4} - 0\right) = \frac{\pi^2}{2}$$



27. Shell: $V = 2\pi \int_2^6 \frac{x}{1 + \sqrt{x - 2}}\, dx$



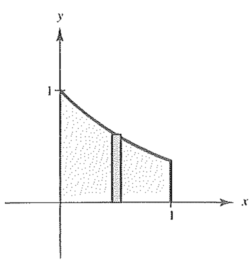
$$u = \sqrt{x - 2}$$
$$x = u^2 + 2$$
$$dx = 2u\, du$$

$$V = 2\pi \int_2^6 \frac{x}{1 + \sqrt{x - 2}}\, dx = 4\pi \int_0^2 \frac{(u^2 + 2)u}{1 + u}\, du$$

$$= 4\pi \int_0^2 \frac{u^3 + 2u}{1 + u}\, du = 4\pi \int_0^2 \left(u^2 - u + 3 - \frac{3}{1 + u}\right) du = 4\pi\left[\frac{1}{3}u^3 - \frac{1}{2}u^2 + 3u - 3\ln(1 + u)\right]_0^2 = \frac{4\pi}{3}(20 - 9\ln 3) \approx 42.359$$

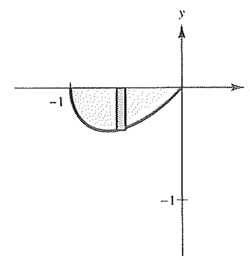
28. Disk

$$V = \pi \int_0^1 \left(e^{-x}\right)^2 dx$$
$$= \pi \int_0^1 e^{-2x}\, dx$$
$$= \left[-\frac{\pi}{2}e^{-2x}\right]_0^1$$
$$= \left(\frac{-\pi}{2e^2} + \frac{\pi}{2}\right) = \frac{\pi}{2}\left(1 - \frac{1}{e^2}\right)$$



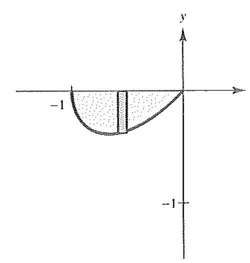
29. (a) Because $y \le 0$, $A = -\int_{-1}^0 x\sqrt{x + 1}\, dx$.

$$u = x + 1$$
$$x = u - 1$$
$$dx = du$$
$$A = -\int_0^1 (u - 1)\sqrt{u}\, du$$
$$= -\int_0^1 \left(u^{3/2} - u^{1/2}\right) du$$
$$= -\left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right]_0^1$$
$$= \tfrac{4}{15}$$



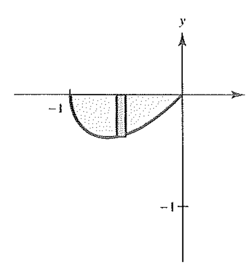
(b) **Disk**

$$V = \pi \int_{-1}^0 x^2(x + 1)\, dx$$
$$= \pi \int_{-1}^0 \left(x^3 + x^2\right) dx$$
$$= \pi\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^0$$
$$= \frac{\pi}{12}$$



(c) **Shell**

$$u = \sqrt{x + 1}$$
$$x = u^2 - 1$$
$$dx = 2u\, du$$
$$V = 2\pi \int_{-1}^0 x^2\sqrt{x + 1}\, dx$$
$$= 4\pi \int_0^1 \left(u^2 - 1\right)^2 u^2\, du$$
$$= 4\pi \int_0^1 \left(u^6 - 2u^4 + u^2\right) du$$
$$= 4\pi\left[\frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3\right]_0^1 = \frac{32\pi}{105}$$



30. (a) **Disk:**

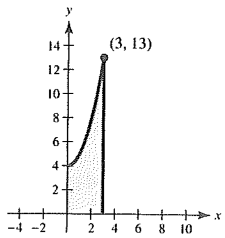
$$V = \pi\int_0^3 \left(x^2 + 4\right)^2 dx$$

(b) **Shell:** $y = x^2 + 4 \Rightarrow x = \sqrt{4 - y}$

$$V = 2\pi\int_0^4 3y\,dy + 2\pi\int_4^{13} y\left(3 - \sqrt{y - 4}\right) dy$$

(c) No. The integral in (a) is respect to x, while those in (b) are with respect to y.

Note: The volume is $843\pi/5$.



31. From Exercise 23(a) you have: $V = 64\pi \text{ ft}^3$

$$\tfrac{1}{4}V = 16\pi$$

Disk:

$$\pi\int_{-3}^{y_0} \tfrac{16}{9}\left(9 - y^2\right) dy = 16\pi$$

$$\tfrac{1}{9}\int_{-3}^{y_0} \left(9 - y^2\right) dy = 1$$

$$\left[9y - \tfrac{1}{3}y^3\right]_{-3}^{y_0} = 9$$

$$\left(9y_0 - \tfrac{1}{3}y_0^3\right) - \left(-27 + 9\right) = 9$$

$$y_0^3 - 27y_0 - 27 = 0$$

By Newton's Method, $y_0 \approx -1.042$ and the depth of the gasoline is $3 - 1.042 = 1.958$ feet.

32. $A(x) = \dfrac{1}{2}bh = \dfrac{1}{2}\left(2\sqrt{a^2 - x^2}\right)\left(\sqrt{3}\sqrt{a^2 - x^2}\right)$

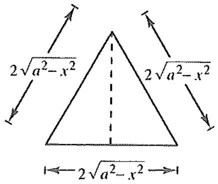
$$= \sqrt{3}\left(a^2 - x^2\right)$$

$$V = \sqrt{3}\int_{-a}^{a} \left(a^2 - x^2\right) dx = \sqrt{3}\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a}$$

$$= \sqrt{3}\left(\frac{4a^3}{3}\right)$$

Because $\left(4\sqrt{3}a^3\right)/3 = 10$, you have

$a^3 = \left(5\sqrt{3}\right)/2$. So, $a = \sqrt[3]{\dfrac{5\sqrt{3}}{2}} \approx 1.630$ meters.



33.

$$f(x) = \tfrac{4}{5}x^{5/4}$$

$$f'(x) = x^{1/4}$$

$$1 + \left[f'(x)\right]^2 = 1 + \sqrt{x}$$

$$u = 1 + \sqrt{x}$$

$$x = (u - 1)^2$$

$$dx = 2(u - 1)\,du$$

$$s = \int_0^4 \sqrt{1 + \sqrt{x}}\,dx = 2\int_1^3 \sqrt{u}(u - 1)\,du$$

$$= 2\int_1^3 \left(u^{3/2} - u^{1/2}\right) du$$

$$= 2\left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right]_1^3 = \tfrac{4}{15}\left[u^{3/2}(3u - 5)\right]_1^3$$

$$= \tfrac{8}{15}\left(1 + 6\sqrt{3}\right) \approx 6.076$$

34.

$$y = \frac{x^3}{6} + \frac{1}{2x}$$

$$y' = \frac{1}{2}x^2 - \frac{1}{2x^2}$$

$$1 + (y')^2 = \left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)^2$$

$$s = \int_1^3 \left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right) dx = \left[\frac{1}{6}x^3 - \frac{1}{2x}\right]_1^3 = \frac{14}{3}$$

35. $y = 300\cosh\left(\dfrac{x}{2000}\right) - 280, -2000 \le x \le 2000$

$$y' = \frac{3}{20}\sinh\left(\frac{x}{2000}\right)$$

$$s = \int_{-2000}^{2000} \sqrt{1 + \left[\frac{3}{20}\sinh\left(\frac{x}{2000}\right)\right]^2}\,dx$$

$$= \frac{1}{20}\int_{-2000}^{2000} \sqrt{400 + 9\sinh^2\left(\frac{x}{2000}\right)}\,dx$$

$= 4018.2$ ft (by Simpson's Rule or graphing utility)

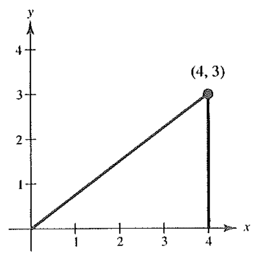
36. Because $f(x) = \tan x$ has $f'(x) = \sec^2 x$, this integral represents the length of the graph of $\tan x$ from $x = 0$ to $x = \pi/4$. This length is a little over 1 unit. Answer (b).

37. $y = \frac{3}{4}x$

$$y' = \frac{3}{4}$$

$$1 + (y')^2 = \frac{25}{16}$$

$$S = 2\pi\int_0^4 \left(\frac{3}{4}x\right)\sqrt{\frac{25}{16}}\,dx = \left[\left(\frac{15\pi}{8}\right)\frac{x^2}{2}\right]_0^4 = 15\pi$$



38. $y = 2\sqrt{x},\ y' = \frac{1}{\sqrt{x}}$

$$1 + (y')^2 = 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$S = 2\pi\int_3^8 2\sqrt{x}\sqrt{\frac{x+1}{x}}\,dx = 4\pi\int_3^8 \sqrt{x+1}\,dx$$

$$= 4\pi\left[\frac{2}{3}(x+1)^{3/2}\right]_3^8 = \frac{152\pi}{3}$$

39. $F = kx$

$$5 = k(1)$$

$$F = 5x$$

$$W = \int_0^5 5x\,dx = \left[\frac{5x^2}{2}\right]_0^5 = \frac{125}{2}\text{ in-lb} \approx 5.21\text{ ft-lb}$$

40. $F = kx$

$$50 = k(1) \Rightarrow k = 50$$

$$W = \int_0^{10} 50x\,dx = \left[25x^2\right]_0^{10} = 2500\text{ in-lb} \approx 208.3\text{ ft-lb}$$

41. Volume of disk: $\pi\left(\frac{1}{3}\right)^2 \Delta y$ $\left[\text{diameter} = \frac{2}{3}\text{ ft}\right]$

Weight of disk: $62.4\pi\left(\frac{1}{3}\right)^2 \Delta y$

Distance: $190 - y$

$$W = \frac{62.4\pi}{9}\int_0^{165}(190 - y)\,dy$$

$$= \frac{62.4\pi}{9}\left[190y - \frac{y^2}{2}\right]_0^{165}$$

$$= \frac{62.4\pi}{9}\left[\frac{35{,}475}{2}\right] = 122{,}980\pi\text{ ft-lb}$$

$$\approx 193.2\text{ foot-tons}$$

42. You know that

$$\frac{dV}{dt} = \frac{4\text{ gal/min} - 12\text{ gal/min}}{7.481\text{ gal/ft}^3} = \frac{-8}{7.481}\text{ ft}^3\text{/min}$$

$$V = \pi r^2 h = \pi\left(\frac{1}{9}\right)h$$

$$\frac{dV}{dt} = \frac{\pi}{9}\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{9}{\pi}\left(\frac{dV}{dt}\right) = \frac{9}{\pi}\left(\frac{-8}{7.481}\right) \approx -3.064\text{ ft/min}$$

Depth of water: $-3.064t + 165$

Time to drain well: $t = \frac{165}{3.064} \approx 53.85$ min

$(53.85)(12) \approx 646.2$ gallons pumped

Volume of water pumped in Exercise 41:

$$\left[165\pi\left(\frac{1}{9}\right)\text{ft}^3\right]\cdot\left[7.481\text{ gal/ft}^3\right] = 430.87\text{ gallons}$$

$$\frac{430.87}{386{,}353.1} = \frac{646.2}{x}$$

$$\Rightarrow x = \frac{646.2}{430.87}(386{,}353.1) \approx 579{,}435.5$$

Work $\approx 579{,}435.5$ ft-lb

43. Weight of section of chain: $4\,\Delta x$

Distance moved: $10 - x$

$$W = 4\int_0^{10}(10 - x)\,dx = 4\left[10x - \frac{x^2}{2}\right]_0^{10}$$

$$= 200\text{ ft-lb}$$

44. (a) Weight of section of cable: $5\,\Delta x$

Distance: $200 - x$

$$W = 5\int_0^{200}(200 - x)\,dx$$

$$= 5\left[200x - \frac{x^2}{2}\right]_0^{200}$$

$$= 100{,}000\text{ ft-lb}$$

(b) Work to move 300 pounds 200 feet vertically:

$300(200) = 60{,}000$ ft-lb.

Total work: $100{,}000 + 60{,}000 = 160{,}000$ ft-lb

45. $W = \int_a^b F(x)\,dx$

$$80 = \int_0^4 ax^2\,dx = \left[\frac{ax^3}{3}\right]_0^4 = \frac{64}{3}a$$

$$a = \frac{3(80)}{64} = \frac{15}{4} = 3.75$$

46. $W = \int_a^b F(x)\,dx$

$$F(x) = \begin{cases} -(2/9)x + 6, & 0 \le x \le 9 \\ -(4/3)x + 16, & 9 \le x \le 12 \end{cases}$$

$$\begin{aligned} W &= \int_0^9 \left(-\tfrac{2}{9}x + 6\right) dx + \int_9^{12} \left(-\tfrac{4}{3}x + 16\right) dx \\ &= \left[-\tfrac{1}{9}x^2 + 6x\right]_0^9 + \left[-\tfrac{2}{3}x^2 + 16x\right]_9^{12} \\ &= (-9 + 54) + (-96 + 192 + 54 - 144) \\ &= 51 \text{ ft-lbs} \end{aligned}$$

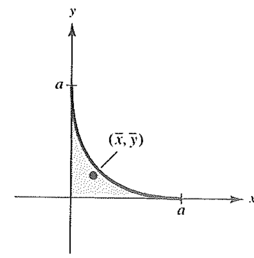
47. $A = \int_0^a \left(\sqrt{a} - \sqrt{x}\right)^2 dx = \int_0^a \left(a - 2\sqrt{a}x^{1/2} + x\right) dx = \left[ax - \frac{4}{3}\sqrt{a}x^{3/2} + \frac{1}{2}x^2\right]_0^a = \frac{a^2}{6}$

$$\frac{1}{A} = \frac{6}{a^2}$$

$$\bar{x} = \frac{6}{a^2}\int_0^a x\left(\sqrt{a} - \sqrt{x}\right)^2 dx = \frac{6}{a^2}\int_0^a \left(ax - 2\sqrt{a}x^{3/2} + x^2\right) dx = \frac{a}{5}$$

$$\begin{aligned} \bar{y} &= \left(\frac{6}{a^2}\right)\frac{1}{2}\int_0^a \left(\sqrt{a} - \sqrt{x}\right)^4 dx \\ &= \frac{3}{a^2}\int_0^a \left(a^2 - 4a^{3/2}x^{1/2} + 6ax - 4a^{1/2}x^{3/2} + x^2\right) dx \\ &= \frac{3}{a^2}\left[a^2x - \frac{8}{3}a^{3/2}x^{3/2} + 3ax^2 - \frac{8}{5}a^{1/2}x^{5/2} + \frac{1}{3}x^3\right]_0^a = \frac{a}{5} \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(\frac{a}{5}, \frac{a}{5}\right)$$



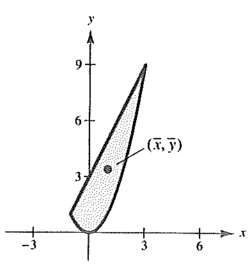
48. $A = \int_{-1}^3 \left[(2x + 3) - x^2\right] dx = \left[x^2 + 3x - \frac{1}{3}x^3\right]_{-1}^3 = \frac{32}{3}$

$$\frac{1}{A} = \frac{3}{32}$$

$$\bar{x} = \frac{3}{32}\int_{-1}^3 x\left(2x + 3 - x^2\right) dx = \frac{3}{32}\int_{-1}^3 \left(3x + 2x^2 - x^3\right) dx = \frac{3}{32}\left[\frac{3}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4\right]_{-1}^3 = 1$$

$$\begin{aligned} \bar{y} &= \left(\frac{3}{32}\right)\frac{1}{2}\int_{-1}^3 \left[(2x + 3)^2 - x^4\right] dx = \frac{3}{64}\int_{-1}^3 \left(9 + 12x + 4x^2 - x^4\right) dx \\ &= \frac{3}{64}\left[9x + 6x^2 + \frac{4}{3}x^3 - \frac{1}{5}x^5\right]_{-1}^3 = \frac{17}{5} \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(1, \frac{17}{5}\right)$$



49. By symmetry, $x = 0$.

$$A = 2\int_0^1 \left(a^2 - x^2\right) dx = 2\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{4a^3}{3}$$

$$\frac{1}{A} = \frac{3}{4a^3}$$

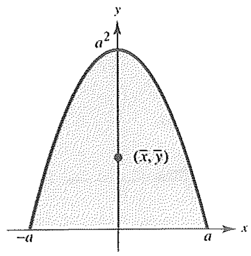
$$\bar{y} = \left(\frac{3}{4a^3}\right)\frac{1}{2}\int_{-a}^{a}\left(a^2 - x^2\right)^2 dx$$

$$= \frac{6}{8a^3}\int_0^a \left(a^4 - 2a^2x^2 + x^4\right) dx$$

$$= \frac{6}{8a^3}\left[a^4x - \frac{2a^2}{3}x^3 + \frac{1}{5}x^5\right]_0^a$$

$$= \frac{6}{8a^3}\left(a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5\right) = \frac{2a^2}{5}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{2a^2}{5}\right)$$



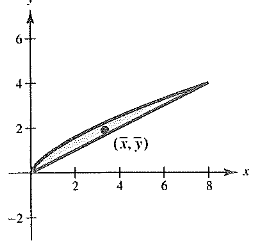
50. $$A = \int_0^8 \left(x^{2/3} - \frac{1}{2}x\right) dx = \left[\frac{3}{5}x^{5/3} - \frac{1}{4}x^2\right]_0^8 = \frac{16}{5}$$

$$\frac{1}{A} = \frac{5}{16}$$

$$\bar{x} = \frac{5}{16}\int_0^8 x\left(x^{2/3} - \frac{1}{2}x\right) dx = \frac{5}{16}\left[\frac{3}{8}x^{8/3} - \frac{1}{6}x^3\right]_0^8 = \frac{10}{3}$$

$$\bar{y} = \left(\frac{5}{16}\right)\frac{1}{2}\int_0^8\left(x^{4/3} - \frac{1}{4}x^2\right) dx = \frac{1}{2}\left(\frac{5}{16}\right)\left[\frac{3}{7}x^{7/3} - \frac{1}{12}x^3\right]_0^8 = \frac{40}{21}$$

$$(\bar{x}, \bar{y}) = \left(\frac{10}{3}, \frac{40}{21}\right)$$



51. $\bar{y} = 0$ by symmetry.

For the trapezoid:

$$m = [(4)(6) - (1)(6)]\rho = 18\rho$$

$$M_y = \rho\int_0^6 x\left[\left(\frac{1}{6}x + 1\right) - \left(-\frac{1}{6}x - 1\right)\right]dx = \rho\int_0^6\left(\frac{1}{3}x^2 + 2x\right)dx = \rho\left[\frac{x^3}{9} + x^2\right]_0^6 = 60\rho$$

For the semicircle:

$$m = \left(\frac{1}{2}\right)(\pi)(2)^2\rho = 2\pi\rho$$

$$M_y = \rho\int_6^8 x\left[\sqrt{4 - (x-6)^2} - \left(-\sqrt{4 - (x-6)^2}\right)\right]dx = 2\rho\int_6^8 x\sqrt{4 - (x-6)^2}\,dx$$

Let $u = x - 6$, then $x = u + 6$ and $dx = du$. When $x = 6, u = 0$. When $x = 8, u = 2$.

$$M_y = 2\rho\int_0^2 (u+6)\sqrt{4-u^2}\,du = 2\rho\int_0^2 u\sqrt{4-u^2}\,du + 12\rho\int_0^2\sqrt{4-u^2}\,du$$

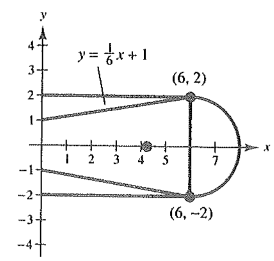
$$= 2\rho\left[\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)(4-u^2)^{3/2}\right]_0^2 + 12\rho\left[\frac{\pi(2)^2}{4}\right] = \frac{16\rho}{3} + 12\pi\rho = \frac{4\rho(4+9\pi)}{3}$$

So, you have:

$$\bar{x}(18\rho + 2\pi\rho) = 60\rho + \frac{4\rho(4+9\pi)}{3}$$

$$\bar{x} = \frac{180\rho + 4\rho(4+9\pi)}{3}\cdot\frac{1}{2\rho(9+\pi)} = \frac{2(9\pi+49)}{3(\pi+9)}$$

The centroid of the blade is $\left(\frac{2(9\pi+49)}{3(\pi+9)}, 0\right)$.



52. Wall at shallow end:

$$F = 62.4\int_0^5 y(20)\,dy = \left[(1248)\frac{y^2}{2}\right]_0^5 = 15{,}600 \text{ lb}$$

Wall at deep end:

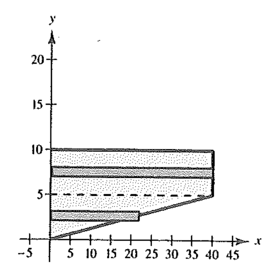
$$F = 62.4\int_0^{10} y(20)\,dy = \left[(624)y^2\right]_0^{10} = 62{,}400 \text{ lb}$$

Side wall:

$$F_1 = 62.4\int_0^5 y(40)\,dy = \left[(1248)y^2\right]_0^5 = 31{,}200 \text{ lb}$$

$$F_2 = 62.4\int_0^5 (10-y)8y\,dy = 62.4\int_0^5\left(80y - 8y^2\right)dy$$

$$F = F_1 + F_2 = 72{,}800 \text{ lb}$$



53. Let D = surface of liquid; ρ = weight per cubic volume.

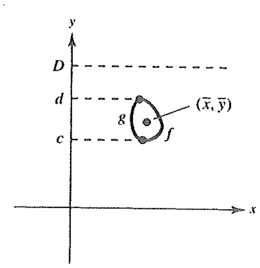
$$F = \rho\int_c^d (D - y)[f(y) - g(y)]\,dy$$

$$= \rho\left[\int_c^d D[f(y) - g(y)]\,dy - \int_c^d y[f(y) - g(y)]\,dy\right]$$

$$= \rho\left[\int_c^d [f(y) - g(y)]\,dy\right]\left[D - \frac{\int_c^d y[f(y) - g(y)]\,dy}{\int_c^d [f(y) - g(y)]\,dy}\right]$$

$$= \rho(\text{Area})(D - \bar{y})$$

$$= \rho(\text{Area})(\text{depth of centroid})$$



54. $F = 62.4(16\pi)10 = 9984\pi$ lb

Problem Solving for Chapter 7

1. $T = \frac{1}{2}c(c^2) = \frac{1}{2}c^3$

$$R = \int_0^c (cx - x^2)\,dx = \left[\frac{cx^2}{2} - \frac{x^3}{3}\right]_0^c = \frac{c^3}{2} - \frac{c^3}{3} = \frac{c^3}{6}$$

$$\lim_{c\to 0^+} \frac{T}{R} = \lim_{c\to 0^+} \frac{\frac{1}{2}c^3}{\frac{1}{6}c^3} = 3$$

2. (a) By symmetry, $M_x = 0$ for L

(b) Because

$(M_y \text{ for } L) + (M_y \text{ for } A) = (M_y \text{ for } B)$,

you have

$(M_y \text{ for } L) = (M_y \text{ for } B) - (M_y \text{ for } A)$

(c) M_y for $B = 0$, because B is a circle at the origin

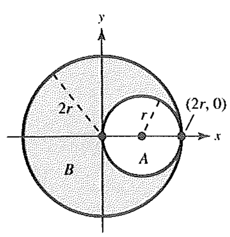
For A, $\bar{x} = \frac{M_y}{\text{Area}} \Rightarrow M_y = r(\pi r^2) = \pi r^3$

So, $(M_y \text{ for } L) = 0 - \pi r^3 = -\pi r^3$

(d) $\bar{y} = 0$ by symmetry.

$$\bar{x} = \frac{M_y \text{ of } L}{\text{Area of } L} = \frac{-\pi r^3}{4\pi r^2 - \pi r^2} = -\frac{r}{3}$$

$$(\bar{x}, \bar{y}) = \left(-\frac{r}{3}, 0\right)$$



3. $R = \int_0^1 x(1 - x)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

Let (c, mc) be the intersection of the line and the parabola.

Then, $mc = c(1 - c) \Rightarrow m = 1 - c$ or $c = 1 - m$.

$$\frac{1}{2}\left(\frac{1}{6}\right) = \int_0^{1-m} (x - x^2 - mx)\,dx$$

$$\frac{1}{12} = \left[\frac{x^2}{2} - \frac{x^3}{3} - m\frac{x^2}{2}\right]_0^{1-m}$$

$$= \frac{(1-m)^2}{2} - \frac{(1-m)^3}{3} - m\frac{(1-m)^2}{2}$$

$$1 = 6(1-m)^2 - 4(1-m)^3 - 6m(1-m)^2$$

$$= (1-m)^2(6 - 4(1-m) - 6m)$$

$$= (1-m)^2(2 - 2m)$$

$$\frac{1}{2} = (1-m)^3$$

$$\left(\frac{1}{2}\right)^{1/3} = 1 - m$$

$$m = 1 - \left(\frac{1}{2}\right)^{1/3} \approx 0.2063$$

So, $y = 0.2063x$.

4. (a) $\frac{1}{2}V = \int_0^1 \left[\pi\left(2 + \sqrt{1-y^2}\right)^2 - \pi\left(2 - \sqrt{1-y^2}\right)^2\right] dy$

$$= \pi\int_0^1 \left[\left(4 + 4\sqrt{1-y^2} + (1-y^2)\right) - \left(4 - 4\sqrt{1-y^2} + (1-y^2)\right)\right] dy$$

$= 8\pi\int_0^1 \sqrt{1-y^2}\,dy$ (Integral represents 1/4 (area of circle))

$$= 8\pi\left(\frac{\pi}{4}\right) = 2\pi^2 \Rightarrow V = 4\pi^2$$

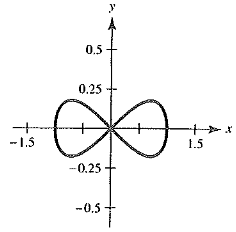
(b) $(x - R)^2 + y^2 = r^2 \Rightarrow x = R \pm \sqrt{r^2 - y^2}$

$$\frac{1}{2}V = \int_0^r \left[\pi\left(R + \sqrt{r^2-y^2}\right)^2 - \pi\left(R - \sqrt{r^2-y^2}\right)^2\right] dy = \pi\int_0^r 4R\sqrt{r^2-y^2}\,dy = \pi(4R)\frac{1}{4}\pi r^2 = \pi^2 r^2 R$$

$$V = 2\pi^2 r^2 R$$

5. $8y^2 = x^2(1 - x^2)$

$$y = \pm\frac{|x|\sqrt{1 - x^2}}{2\sqrt{2}}$$



For $x > 0$, $y' = \dfrac{1 - 2x^2}{2\sqrt{2}\sqrt{1 - x^2}}$

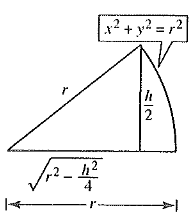
$$S = 2(2\pi)\int_0^1 x\sqrt{1 + \left(\frac{1 - 2x^2}{2\sqrt{2}\sqrt{1 - x^2}}\right)^2}\,dx$$

$$= \frac{5\sqrt{2}\pi}{3}$$

6. $V = 2(2\pi)\int_{\sqrt{r^2 - (h^2/4)}}^{r} x\sqrt{r^2 - x^2}\,dx$

$$= -2\pi\left[\frac{2}{3}(r^2 - x^2)^{3/2}\right]_{\sqrt{r^2 - (h^2/4)}}^{r}$$

$$= \frac{-4\pi}{3}\left[-\frac{h^3}{8}\right] = \frac{\pi h^3}{6} \text{ which does not depend on } r$$



7. By the Theorem of Pappus,

$$V = 2\pi rA$$

$$= 2\pi\left[d + \tfrac{1}{2}\sqrt{w^2 + l^2}\right]lw$$

8. (a) Tangent at A: $y = x^3$, $y' = 3x^2$

$$y - 1 = 3(x - 1)$$

$$y = 3x - 2$$

To find point B:

$$x^3 = 3x - 2$$

$$x^3 - 3x + 2 = 0$$

$$(x - 1)^2(x + 2) = 0 \Rightarrow B = (-2, -8)$$

Tangent at B: $y = x^3$, $y' = 3x^2$

$$y + 8 = 12(x + 2)$$

$$y = 12x + 16$$

To find point C:

$$x^3 = 12x + 16$$

$$x^3 - 12x - 16 = 0$$

$$(x + 2)^2(x - 4) = 0 \Rightarrow C = (4, 64)$$

Area of $R = \int_{-2}^{1}(x^3 - 3x + 2)\,dx = \dfrac{27}{4}$

Area of $S = \int_{-2}^{4}(12x + 16 - x^3)\,dx = 108$

Area of $S = 16(\text{area of } R)$ $\left[\dfrac{\text{area } S}{\text{area } R} = 16\right]$

(b) Tangent at $A(a, a^3)$: $y - a^3 = 3a^2(x - a)$

$$y = 3a^2x - 2a^3$$

To find point B: $x^3 - 3a^2x + 2a^3 = 0$

$$(x - a)^2(x + 2a) = 0$$

$$\Rightarrow B = (-2a, -8a^3)$$

Tangent at B: $y + 8a^3 = 12a^2(x + 2a)$

$$y = 12a^2x + 16a^3$$

To find point C: $x^3 - 12a^2x - 16a^3 = 0$

$$(x + 2a)^2(x - 4a) = 0$$

$$\Rightarrow C = (4a, 64a^3)$$

Area of $R = \int_{-2a}^{a}\left[x^3 - 3a^2x + 2a^3\right]dx = \dfrac{27}{4}a^4$

Area of $S = \int_{-2a}^{4a}\left[12a^2x + 16a^3 - x^3\right]dx = 108a^4$

Area of $S = 16(\text{area of } R)$

9. $f'(x)^2 = e^x$

$$f'(x) = e^{x/2}$$

$$f(x) = 2e^{x/2} + C$$

$$f(0) = 0 \Rightarrow C = -2$$

$$f(x) = 2e^{x/2} - 2$$

10. $s(x) = \int_a^x \sqrt{1 + f'(t)^2}\, dt$

(a) $s'(x) = \dfrac{ds}{dx} = \sqrt{1 + f'(x)^2}$

(b) $ds = \sqrt{1 + f'(x)^2}\, dx$

$$(ds)^2 = \left[1 + f'(x)^2\right](dx)^2$$

$$= \left[1 + \left(\frac{dy}{dx}\right)^2\right](dx)^2 = (dx)^2 + (dy)^2$$

(c) $s(x) = \int_1^x \sqrt{1 + \left(\frac{3}{2}t^{1/2}\right)^2}\, dt = \int_1^x \sqrt{1 + \frac{9}{4}t}\, dt$

(d) $s(2) = \int_1^2 \sqrt{1 + \frac{9}{4}t}\, dt$

$$= \left[\frac{8}{27}\left(1 + \frac{9}{4}t\right)^{3/2}\right]_1^2$$

$$= \frac{22}{27}\sqrt{22} - \frac{13}{27}\sqrt{13} \approx 2.0858$$

This is the length of the curve $y = x^{3/2}$ from $x = 1$ to $x = 2$.

11. Let ρ_f be the density of the fluid and ρ_0 the density of the iceberg. The buoyant force is

$$F = \rho_f g \int_{-h}^0 A(y)\, dy$$

where $A(y)$ is a typical cross section and g is the acceleration due to gravity. The weight of the object is

$$W = \rho_0 g \int_{-h}^{L-h} A(y)\, dy.$$

$$F = W$$

$$\rho_f g \int_{-h}^0 A(y)\, dy = \rho_0 g \int_{-h}^{L-h} A(y)\, dy$$

$$\frac{\rho_0}{\rho_f} = \frac{\text{submerged volume}}{\text{total volume}}$$

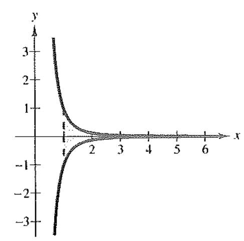
$$= \frac{0.92 \times 10^3}{1.03 \times 10^3} = 0.893 \text{ or } 89.3\%$$

12. (a) $\bar{y} = 0$ by symmetry

$$M_y = \int_1^6 x\left(\frac{1}{x^3} - \left(-\frac{1}{x^3}\right)\right) dx = \int_1^6 \frac{2}{x^2}\, dx = \left[-2\frac{1}{x}\right]_1^6 = \frac{5}{3}$$

$$m = 2\int_1^6 \frac{1}{x^3}\, dx = \left[-\frac{1}{x^2}\right]_1^6 = \frac{35}{36}$$

$$\bar{x} = \frac{5/3}{35/36} = \frac{12}{7} \qquad (\bar{x}, \bar{y}) = \left(\frac{12}{7}, 0\right)$$



(b) $m = 2\int_1^b \frac{1}{x^3}\, dx = \dfrac{b^2 - 1}{b^2}$

$$M_y = 2\int_1^6 \frac{1}{x^2}\, dx = \frac{2(b - 1)}{b}$$

$$\bar{x} = \frac{2(b - 1)/b}{(b^2 - 1)/b^2} = \frac{2b}{b + 1} \qquad (\bar{x}, \bar{y}) = \left(\frac{2b}{b + 1}, 0\right)$$

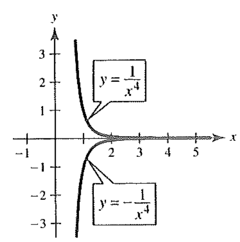
(c) $\lim_{b\to\infty} \bar{x} = \lim_{b\to\infty} \dfrac{2b}{b + 1} = 2 \qquad (\bar{x}, \bar{y}) = (2, 0)$

13. (a) $\bar{y} = 0$ by symmetry

$$M_y = 2\int_1^6 x\frac{1}{x^4}\, dx = 2\int_1^6 \frac{1}{x^3}\, dx = \frac{35}{36}$$

$$m = 2\int_1^6 \frac{1}{x^4}\, dx = \frac{215}{324}$$

$$\bar{x} = \frac{35/36}{215/324} = \frac{63}{43} \qquad (\bar{x}, \bar{y}) = \left(\frac{63}{43}, 0\right)$$



(b) $M_y = 2\int_1^b \frac{1}{x^3}\,dx = \frac{b^2-1}{b^2}$

$m = 2\int_1^b \frac{1}{x^4}\,dx = \frac{2(b^3-1)}{3b^3}$

$\bar{x} = \frac{(b^2-1)/b^2}{2(b^3-1)/3b^3} = \frac{3b(b+1)}{2(b^2+b+1)} \qquad (\bar{x}, \bar{y}) = \left(\frac{3b(b+1)}{2(b^2+b+1)}, 0\right)$

(c) $\lim_{b\to\infty} \bar{x} = \frac{3b(b+1)}{2(b^2+b+1)} = \frac{3}{2} \qquad (\bar{x}, \bar{y}) = \left(\frac{3}{2}, 0\right)$

14. (a) $W = \text{area} = 2 + 4 + 6 = 12$

(b) $W = \text{area} = 3 + (1 + 1) + 2 + \frac{1}{2} = 7\frac{1}{2}$

15. Point of equilibrium: $50 - 0.5x = 0.125x$

$x = 80,\ p = 10$

$(P_0, x_0) = (10, 80)$

Consumer surplus $= \int_0^{80} \left[(50 - 0.5x) - 10\right] dx = 1600$

Producer surplus $= \int_0^{80} (10 - 0.125x)\,dx = 400$

16. Point of equilibrium: $1000 - 0.4x^2 = 42x$

$x = 20,\ p = 840$

$(P_0, x_0) = (840, 20)$

Consumer surplus $= \int_0^{20} \left[(1000 - 0.4x^2) - 840\right] dx$

$= 2133.33$

Producer surplus $= \int_0^{20} (840 - 42x)\,dx = 8400$

17. Use Exercise 25, Section 7.7, which gives $F = wkhb$ for a rectangle plate.

Wall at shallow end

From Exercise 25: $F = 62.4(2)(4)(20) = 9984$ lb

Wall at deep end

From Exercise 25: $F = 62.4(4)(8)(20) = 39{,}936$ lb

Side wall

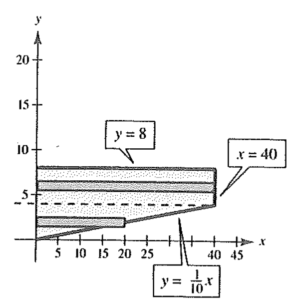
From Exercise 25: $F_1 = 62.4(2)(4)(40) = 19{,}968$ lb

$F_2 = 62.4\int_0^4 (8 - y)(10y)\,dy$

$= 624\int_0^4 (8y - y^2)\,dy = 624\left[4y^2 - \frac{y^3}{3}\right]_0^4$

$= 26{,}624$ lb

Total force: $F_1 + F_2 = 46{,}592$ lb



18. (a) Answers will vary.

$f_1(x) = 6(x - x^2)$

$f_2(x) = \frac{\pi}{2}\sin(\pi x)$

(b) f_1 arc length ≈ 3.2490

f_2 arc length ≈ 3.3655

(c) See the article by Professor Larson Riddle at http://ecademy.agnesscott.edu/~Iriddle/arc/contest.htm

One such function is

$f_3(x) = \frac{8}{\pi}\sqrt{x - x^2} \qquad (\text{arc length} \approx 2.9195)$

CHAPTER 8
Integration Techniques, L'Hôpital's Rule, and Improper Integrals