31. **(a)** $\frac{dV}{dr} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$

When $r = 2$, $\frac{dV}{dr} = 4\pi(2)^2 = 16\pi$ cubic feet of volume per foot of radius.

(b) The increase in the volume is

$\frac{4}{3}\pi(2.2)^3 - \frac{4}{3}\pi(2)^3 \approx 11.092$ cubic feet.

32. For $t > 0$, the speed of the aircraft in meters per second after t seconds is $\frac{20}{9}t$. Multiplying by $\frac{3600 \text{ sec}}{1 \text{ h}} \cdot \frac{1 \text{ km}}{1000 \text{ m}}$, we find that this is equivalent to $8t$ kilometers per hour. Solving $8t = 200$ gives $t = 25$ seconds. The aircraft takes 25 seconds to become airborne, and the distance it travels during this time is $D(25) \approx 694.444$ meters.

33. Let v_0 be the exit velocity of a particle of lava. Then $s(t) = v_0 t - 16t^2$ feet, so the velocity is $\frac{ds}{dt} = v_0 - 32t$. Solving $\frac{ds}{dt} = 0$ gives $t = \frac{v_0}{32}$. Then the maximum height, in feet, is $s\left(\frac{v_0}{32}\right) = v_0\left(\frac{v_0}{32}\right) - 16\left(\frac{v_0}{32}\right)^2 = \frac{v_0^2}{64}$. Solving $\frac{v_0^2}{64} = 1900$ gives $v_0 \approx \pm 348.712$. The exit velocity was about 348.712 ft/sec. Multiplying by $\frac{3600 \text{ sec}}{1 \text{ h}} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}}$, we find that this is equivalent to about 237.758 mi/h.

34. By estimating the slope of the velocity graph at that point.

35. Since profit = revenue − cost, the Sum and Difference Rule gives $\frac{d}{dx}(\text{profit}) = \frac{d}{dx}(\text{revenue}) - \frac{d}{dx}(\text{cost})$, where x is the number of units produced. This means that marginal profit = marginal revenue − marginal cost.

36. **(a)** It takes 135 seconds.

(b) Average speed $= \frac{\Delta F}{\Delta t} = \frac{5-0}{73-0} = \frac{5}{73}$

≈ 0.068 furlongs/sec.

(c) Using a symmetric difference quotient, the horse's speed is approximately

$\frac{\Delta F}{\Delta t} = \frac{4-2}{59-33} = \frac{2}{26} = \frac{1}{13} \approx 0.077$ furlongs/sec.

(d) The horse is running the fastest during the last furlong (between 9th and 10th furlong markers). This furlong takes only 11 seconds to run, which is the least amount of time for a furlong.

(e) The horse accelerates the fastest during the first furlong (between markers 0 and 1).

37. **(a)** Assume that f is even. Then,

$$\begin{aligned} f'(-x) &= \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} \\ &= \lim_{h\to 0} \frac{f(x-h) - f(x)}{h}, \text{ and substituting } k = -h, \\ &= \lim_{k\to 0} \frac{f(x+k) - f(x)}{-k} \\ &= -\lim_{k\to 0} \frac{f(x+k) - f(x)}{k} = -f'(x) \end{aligned}$$

So, f' is an odd function.

(b) Assume that f is odd. Then,

$$\begin{aligned} f'(-x) &= \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} \\ &= \lim_{h\to 0} \frac{-f(x-h) + f(x)}{h}, \end{aligned}$$

and substituting $k = -h$,

$$\begin{aligned} &= \lim_{k\to 0} \frac{-f(x+k) + f(x)}{-k} \\ &= \lim_{k\to 0} \frac{f(x+k) - f(x)}{k} = f'(x) \end{aligned}$$

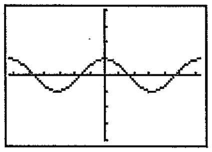
So, f' is an even function.

38.
$$\begin{aligned} \frac{d}{dx}(fgh) &= \frac{d}{dx}[f(gh)] = f \cdot \frac{d}{dx}(gh) + gh \cdot \frac{d}{dx}(f) \\ &= f\left(g \cdot \frac{dh}{dx} + h \cdot \frac{dg}{dx}\right) + gh \cdot \frac{df}{dx} \\ &= \left(\frac{df}{dx}\right)gh + f\left(\frac{dg}{dx}\right)h + fg\left(\frac{dh}{dx}\right) \end{aligned}$$

■ Section 3.5 Derivatives of Trigonometric Functions (pp. 134–141)

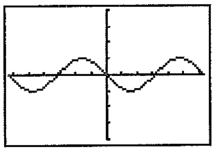
Exploration 1 Making a Conjecture with NDER

1. When the graph of $\sin x$ is increasing, the graph of NDER ($\sin x$) is positive (above the x-axis).
2. When the graph of $\sin x$ is decreasing, the graph of NDER ($\sin x$) is negative (below the x-axis).
3. When the graph of $\sin x$ stops increasing and starts decreasing, the graph of NDER ($\sin x$) crosses the x-axis from above to below.
4. The slope of the graph of $\sin x$ matches the value of NDER ($\sin x$) at these points.
5. We conjecture that NDER ($\sin x$) = $\cos x$. The graphs coincide, supporting our conjecture.



$[-2\pi, 2\pi]$ by $[-4, 4]$

6. When the graph of $\cos x$ is increasing, the graph of NDER $(\cos x)$ is positive (above the x-axis).
When the graph of $\cos x$ is decreasing, the graph of NDER $(\cos x)$ is negative (below the x-axis).
When the graph of $\cos x$ stops increasing and starts decreasing, the graph of NDER $(\cos x)$ crosses the x-axis from above to below.
The slope of the graph of $\cos x$ matches the value of NDER $(\cos x)$ at these points.
We conjecture that NDER $(\cos x) = -\sin x$. The graphs coincide, supporting our conjecture.



$[-2\pi, 2\pi]$ by $[-4, 4]$

Quick Review 3.5

1. $135° \cdot \frac{\pi}{180°} = \frac{3\pi}{4} \approx 2.356$

2. $1.7 \cdot \frac{180°}{\pi} = \left(\frac{306}{\pi}\right)^{\circ} \approx 97.403°$

3. $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

4. Domain: All reals
Range: $[-1, 1]$

5. Domain: $x \neq \frac{k\pi}{2}$ for odd integers k
Range: All reals

6. $\cos a = \pm\sqrt{1 - \sin^2 a} = \pm\sqrt{1 - (-1)^2} = \pm\sqrt{0} = 0$

7. If $\tan a = -1$, then $a = \frac{3\pi}{4} + k\pi$ for some integer k, so $\sin a = \pm\frac{1}{\sqrt{2}}$.

8. $$\frac{1 - \cos h}{h} = \frac{(1 - \cos h)(1 + \cos h)}{h(1 + \cos h)} = \frac{1 - \cos^2 h}{h(1 + \cos h)} = \frac{\sin^2 h}{h(1 + \cos h)}$$

9. $y'(x) = 6x^2 - 14x$
$y'(3) = 12$
The tangent line has slope 12 and passes through $(3, 1)$, so its equation is $y = 12(x - 3) + 1$, or $y = 12x - 35$.

10. $a(t) = v'(t) = 6t^2 - 14t$
$a(3) = 12$

Section 3.5 Exercises

1. $\frac{d}{dx}(1 + x - \cos x) = 0 + 1 - (-\sin x) = 1 + \sin x$

2. $\frac{d}{dx}(2 \sin x - \tan x) = 2 \cos x - \sec^2 x$

3. $\frac{d}{dx}\left(\frac{1}{x} + 5 \sin x\right) = -\frac{1}{x^2} + 5 \cos x$

4. $$\frac{d}{dx}(x \sec x) = x\frac{d}{dx}(\sec x) + \sec x\frac{d}{dx}(x) = x \sec x \tan x + \sec x$$

5. $$\frac{d}{dx}(4 - x^2 \sin x) = \frac{d}{dx}(4) - \left[x^2\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(x^2)\right] = 0 - [x^2 \cos x + (\sin x)(2x)] = -x^2 \cos x - 2x \sin x$$

6. $$\frac{d}{dx}(3x + x \tan x) = \frac{d}{dx}(3x) + \left[x\frac{d}{dx}(\tan x) + (\tan x)\frac{d}{dx}(x)\right] = 3 + x \sec^2 x + \tan x$$

7. $\frac{d}{dx}\left(\frac{4}{\cos x}\right) = \frac{d}{dx}(4 \sec x) = 4 \sec x \tan x$

8. $$\frac{d}{dx}\frac{x}{1 + \cos x} = \frac{(1 + \cos x)\frac{d}{dx}(x) - x\frac{d}{dx}(1 + \cos x)}{(1 + \cos x)^2} = \frac{1 + \cos x + x \sin x}{(1 + \cos x)^2}$$

9. $$\frac{d}{dx}\frac{\cot x}{1 + \cot x} = \frac{(1 + \cot x)\frac{d}{dx}(\cot x) - (\cot x)\frac{d}{dx}(1 + \cot x)}{(1 + \cot x)^2}$$
$$= \frac{(1 + \cot x)(-\csc^2 x) - (\cot x)(-\csc^2 x)}{(1 + \cot x)^2}$$
$$= -\frac{\csc^2 x}{(1 + \cot x)^2} = -\frac{\csc^2 x \sin^2 x}{(1 + \cot x)^2 \sin^2 x} = -\frac{1}{(\sin x + \cos x)^2}$$

10. $$\frac{d}{dx}\frac{\cos x}{1 + \sin x} = \frac{(1 + \sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(1 + \sin x)}{(1 + \sin x)^2}$$
$$= \frac{(1 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(1 + \sin x)^2}$$
$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2}$$
$$= \frac{-(1 + \sin x)}{(1 + \sin x)^2}$$
$$= -\frac{1}{1 + \sin x}$$

11. $y'(x) = \frac{d}{dx}(\sin x + 3) = \cos x$
$y'(\pi) = \cos \pi = -1$

The tangent line has slope -1 and passes through $(\pi, \sin \pi + 3) = (\pi, 3)$.

Its equation is $y = -1(x - \pi) + 3$, or $y = -x + \pi + 3$.

12. $y'(x) = \frac{d}{dx}\frac{\tan x}{x} = \frac{x\frac{d}{dx}(\tan x) - \tan x\frac{d}{dx}(x)}{x^2}$

$= \frac{x\sec^2 x - \tan x}{x^2}$

$y'\left(\frac{\pi}{4}\right) = \frac{\frac{\pi}{4}\cdot(\sqrt{2})^2 - 1}{\left(\frac{\pi}{4}\right)^2} = \frac{8\pi - 16}{\pi^2}$

The normal line has slope $-\frac{\pi^2}{8\pi - 16} = \frac{\pi^2}{16 - 8\pi}$

and passes through $\left(\frac{\pi}{4}, \frac{\tan\left(\frac{\pi}{4}\right)}{\frac{\pi}{4}}\right) = \left(\frac{\pi}{4}, \frac{4}{\pi}\right)$.

Its equation is $y =$

$\frac{\pi^2}{16 - 8\pi}\left(x - \frac{\pi}{4}\right) + \frac{4}{\pi}$, or

$y = \frac{\pi^2}{16 - 8\pi}x - \frac{\pi^3}{64 - 32\pi} + \frac{4}{\pi}$.

Using decimals, this equation is approximately

$y = -1.081x + 2.122$.

13. $y'(x) = \frac{d}{dx}(x^2 \sin x) = x^2\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(x^2)$

$= x^2 \cos x + 2x \sin x$

$y'(3) = 9 \cos 3 + 6 \sin 3$

The tangent line has slope $9 \cos 3 + 6 \sin 3$ and passes through $(3, 9 \sin 3)$. Its equation is
$y = (9 \cos 3 + 6 \sin 3)(x - 3) + 9 \sin 3$, or
$y = (9 \cos 3 + 6 \sin 3)x - 27 \cos 3 - 9 \sin 3$. Using decimals, this equation is approximately
$y = -8.063x + 25.460$.

14. $\lim_{h\to 0} \frac{\cos(x + h) - \cos x}{h}$

$= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h}$

$= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}$

$= \lim_{h\to 0}\left((\cos x)\frac{\cos h - 1}{h} - (\sin x)\frac{\sin h}{h}\right)$

$= (\cos x)\left(\lim_{h\to 0} \frac{\cos h - 1}{h}\right) - (\sin x)\left(\lim_{h\to 0} \frac{\sin h}{h}\right)$

$= (\cos x)(0) - (\sin x)(1) = -\sin x$

15. (a) $\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{(\cos x)\frac{d}{dx}(\sin x) - (\sin x)\frac{d}{dx}(\cos x)}{(\cos x)^2}$

$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

(b) $\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{(\cos x)\frac{d}{dx}(1) - (1)\frac{d}{dx}(\cos x)}{(\cos x)^2}$

$= \frac{(\cos x)(0) - (1)(-\sin x)}{\cos^2 x}$

$= \frac{\sin x}{\cos^2 x} = \sec x \tan x$

16. (a) $\frac{d}{dx}\cot x = \frac{d}{dx}\frac{\cos x}{\sin x}$

$= \frac{(\sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(\sin x)}{(\sin x)^2}$

$= \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x}$

$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$= -\frac{1}{\sin^2 x} = -\csc^2 x$

(b) $\frac{d}{dx}\csc x = \frac{d}{dx}\frac{1}{\sin x}$

$= \frac{(\sin x)\frac{d}{dx}(1) - (1)\frac{d}{dx}(\sin x)}{(\sin x)^2}$

$= \frac{(\sin x)(0) - (1)(\cos x)}{\sin^2 x}$

$= -\frac{\cos x}{\sin^2 x} = -\csc x \cot x$

17. $\frac{d}{dx}\sec x = \sec x \tan x$ which is 0 at $x = 0$, so the slope of the tangent line is 0. $\frac{d}{dx}\cos x = -\sin x$ which is 0 at $x = 0$, so the slope of the tangent line is 0.

18. $\frac{d}{dx}\tan x = \sec^2 x = \frac{1}{\cos^2 x}$, which is never 0.
$\frac{d}{dx}\cot x = -\csc^2 x = -\frac{1}{\sin^2 x}$, which is never 0.

19. $y'(x) = \frac{d}{dx}(\sqrt{2}\cos x) = -\sqrt{2}\sin x$

$y'\left(\frac{\pi}{4}\right) = -\sqrt{2}\sin\frac{\pi}{4} = -\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = -1$

The tangent line has slope -1 and passes through $\left(\frac{\pi}{4}, \sqrt{2}\cos\frac{\pi}{4}\right) = \left(\frac{\pi}{4}, 1\right)$, so its equation is
$y = -1\left(x - \frac{\pi}{4}\right) + 1$, or $y = -x + \frac{\pi}{4} + 1$.

The normal line has slope 1 and passes through $\left(\frac{\pi}{4}, 1\right)$, so its equation is $y = 1\left(x - \frac{\pi}{4}\right) + 1$, or $y = x + 1 - \frac{\pi}{4}$.

20. $y'(x) = \frac{d}{dx}\tan x = \sec^2 x$

$y'(x) = \frac{d}{dx}(2x) = 2$

$\sec^2 x = 2$
$\sec x = \pm\sqrt{2}$

$\cos x = \pm\frac{1}{\sqrt{2}}$

On $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the solutions are $x = \pm\frac{\pi}{4}$. The points on the curve are $\left(-\frac{\pi}{4}, -1\right)$ and $\left(\frac{\pi}{4}, 1\right)$.

21. $$\begin{aligned} y'(x) &= \frac{d}{dx}(4 + \cot x - 2\csc x) \\ &= 0 - \csc^2 x + 2\csc x \cot x \\ &= -\csc^2 x + 2\csc x \cot x \end{aligned}$$

(a) $$\begin{aligned} y'\left(\frac{\pi}{2}\right) &= -\csc^2\frac{\pi}{2} + 2\csc\frac{\pi}{2}\cot\frac{\pi}{2} \\ &= -1^2 + 2(1)(0) = -1 \end{aligned}$$

The tangent line has slope -1 and passes through $P\left(\frac{\pi}{2}, 2\right)$. Its equation is $y = -1\left(x - \frac{\pi}{2}\right) + 2$, or $y = -x + \frac{\pi}{2} + 2$.

(b) $$\begin{aligned} y'(x) &= 0 \\ -\csc^2 x + 2\csc x \cot x &= 0 \\ -\frac{1}{\sin^2 x} + \frac{2\cos x}{\sin^2 x} &= 0 \\ \frac{1}{\sin^2 x}(2\cos x - 1) &= 0 \\ \cos x &= \frac{1}{2} \\ x &= \frac{\pi}{3} \text{ at point } Q \end{aligned}$$

$$\begin{aligned} y\left(\frac{\pi}{3}\right) &= 4 + \cot\frac{\pi}{3} - 2\csc\frac{\pi}{3} \\ &= 4 + \frac{1}{\sqrt{3}} - 2\left(\frac{2}{\sqrt{3}}\right) \\ &= 4 - \frac{3}{\sqrt{3}} = 4 - \sqrt{3} \end{aligned}$$

The coordinates of Q are $\left(\frac{\pi}{3}, 4 - \sqrt{3}\right)$.

The equation of the horizontal line is $y = 4 - \sqrt{3}$.

22. $$\begin{aligned} y'(x) &= \frac{d}{dx}(1 + \sqrt{2}\csc x + \cot x) \\ &= 0 + \sqrt{2}(-\csc x \cot x) + (-\csc^2 x) \\ &= -\sqrt{2}\csc x \cot x - \csc^2 x \end{aligned}$$

(a) $$\begin{aligned} y'\left(\frac{\pi}{4}\right) &= -\sqrt{2}\csc\frac{\pi}{4}\cot\frac{\pi}{4} - \csc^2\frac{\pi}{4} \\ &= -\sqrt{2}(\sqrt{2})(1) - (\sqrt{2})^2 \\ &= -2 - 2 = -4 \end{aligned}$$

The tangent line has slope -4 and passes through $P\left(\frac{\pi}{4}, 4\right)$. Its equation is $y = -4\left(x - \frac{\pi}{4}\right) + 4$, or $y = -4x + \pi + 4$.

(b) $$\begin{aligned} y'(x) &= 0 \\ -\sqrt{2}\csc x \cot x - \csc^2 x &= 0 \\ -\frac{\sqrt{2}\cos x}{\sin^2 x} - \frac{1}{\sin^2 x} &= 0 \\ -\frac{1}{\sin^2 x}(\sqrt{2}\cos x + 1) &= 0 \\ \cos x &= -\frac{1}{\sqrt{2}} \\ x &= \frac{3\pi}{4} \text{ at point } Q \end{aligned}$$

$$\begin{aligned} y\left(\frac{3\pi}{4}\right) &= 1 + \sqrt{2}\csc\frac{3\pi}{4} + \cot\frac{3\pi}{4} \\ &= 1 + \sqrt{2}(\sqrt{2}) + (-1) \\ &= 2 \end{aligned}$$

The coordinates of Q are $\left(\frac{3\pi}{4}, 2\right)$.

The equation of the horizontal line is $y = 2$.

23. (a) Velocity: $s'(t) = -2\cos t$ m/sec
Speed: $|s'(t)| = |2\cos t|$ m/sec
Acceleration: $s''(t) = 2\sin t$ m/sec^2
Jerk: $s'''(t) = 2\cos t$ m/sec^3

(b) Velocity: $-2\cos\frac{\pi}{4} = -\sqrt{2}$ m/sec

Speed: $|-\sqrt{2}| = \sqrt{2}$ m/sec

Acceleration: $2\sin\frac{\pi}{4} = \sqrt{2}$ m/sec^2

Jerk: $2\cos\frac{\pi}{4} = \sqrt{2}$ m/sec^3

(c) The body starts at 2, goes to 0 and then oscillates between 0 and 4.

Speed:

Greatest when $\cos t = \pm 1$ (or $t = k\pi$), at the center of the interval of motion.

Zero when $\cos t = 0$ (or $t = \frac{k\pi}{2}$, k odd), at the endpoints of the interval of motion.

Acceleration:

Greatest (in magnitude) when $\sin t = \pm 1$ (or $t = \frac{k\pi}{2}$, k odd)

Zero when $\sin t = 0$ (or $t = k\pi$)

Jerk:

Greatest (in magnitude) when $\cos t = \pm 1$ (or $t = k\pi$)

Zero when $\cos t = 0$ $\left(\text{or } t = \frac{k\pi}{2}, k \text{ odd}\right)$

24. (a) Velocity: $s'(t) = \cos t - \sin t$ m/sec
Speed: $|s'(t)| = |\cos t - \sin t|$ m/sec
Acceleration: $s''(t) = -\sin t - \cos t$ m/sec^2
Jerk: $s'''(t) = -\cos t + \sin t$ m/sec^3

(b) Velocity: $\cos\frac{\pi}{4} - \sin\frac{\pi}{4} = 0$ m/sec

Speed: $|0| = 0$ m/sec

Acceleration: $-\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = -\sqrt{2}$ m/sec^2

Jerk: $-\cos\frac{\pi}{4} + \sin\frac{\pi}{4} = 0$ m/sec^3

(c) The body starts at 1, goes to $\sqrt{2}$ and then oscillates between $\pm\sqrt{2}$.

Speed:

Greatest when $t = \frac{3\pi}{4} + k\pi$

Zero when $t = \frac{\pi}{4} + k\pi$

Acceleration:

Greatest (in magnitude) when $t = \frac{\pi}{4} + k\pi$

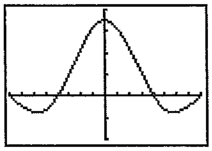
Zero when $t = \frac{3\pi}{4} + k\pi$

Jerk:

Greatest (in magnitude) when $t = \frac{3\pi}{4} + k\pi$

Zero when $t = \frac{\pi}{4} + k\pi$

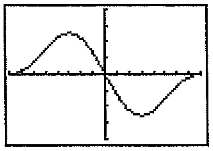
25. (a)



$[-360, 360]$ by $[-0.01, 0.02]$

The limit is $\frac{\pi}{180}$ because this is the conversion factor for changing from degrees to radians.

(b)



$[-360, 360]$ by $[-0.02, 0.02]$

This limit is still 0.

(c) $\frac{d}{dx}\sin x = \lim_{h\to0}\frac{\sin(x+h) - \sin x}{h}$

$= \lim_{h\to0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$

$= \lim_{h\to0}\frac{\sin x(\cos h - 1) + \cos x\sin h}{h}$

$= \left(\lim_{h\to0}\sin x\right)\left(\lim_{h\to0}\frac{\cos h - 1}{h}\right) + \left(\lim_{h\to0}\cos x\right)\left(\lim_{h\to0}\frac{\sin h}{h}\right)$

$= (\sin x)(0) + (\cos x)\left(\frac{\pi}{180}\right)$

$= \frac{\pi}{180}\cos x$

(d) $\frac{d}{dx}\cos x = \lim_{h\to0}\frac{\cos(x+h) - \cos x}{h}$

$= \lim_{h\to0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$

$= \lim_{h\to0}\frac{(\cos x)(\cos h - 1) - \sin x\sin h}{h}$

$= \left(\lim_{h\to0}\cos x\right)\left(\lim_{h\to0}\frac{\cos h - 1}{h}\right) - \left(\lim_{h\to0}\sin x\right)\left(\lim_{h\to0}\frac{\sin h}{h}\right)$

$= (\cos x)(0) - (\sin x)\left(\frac{\pi}{180}\right)$

$= -\frac{\pi}{180}\sin x$

(e) $\frac{d^2}{dx^2}\sin x = \frac{d}{dx}\frac{\pi}{180}\cos x = \frac{\pi}{180}\left(-\frac{\pi}{180}\sin x\right)$

$= -\frac{\pi^2}{180^2}\sin x$

$\frac{d^3}{dx^3}\sin x = \frac{d}{dx}\left(-\frac{\pi^2}{180^2}\sin x\right) = -\frac{\pi^2}{180^2}\left(\frac{\pi}{180}\cos x\right)$

$= -\frac{\pi^3}{180^3}\cos x$

$\frac{d^2}{dx^2}\cos x = \frac{d}{dx}\left(-\frac{\pi}{180}\sin x\right) = -\frac{\pi}{180}\left(\frac{\pi}{180}\cos x\right)$

$= -\frac{\pi^2}{180^2}\cos x$

$\frac{d^3}{dx^3}\cos x = \frac{d}{dx}\left(-\frac{\pi^2}{180^2}\cos x\right) = -\frac{\pi^2}{180^2}\left(-\frac{\pi}{180}\sin x\right)$

$= \frac{\pi^3}{180^3}\sin x$

26. $y' = \frac{d}{dx}\csc x = -\csc x\cot x$

$y'' = \frac{d}{dx}(-\csc x\cot x)$

$= -(\csc x)\frac{d}{dx}(\cot x) - (\cot x)\frac{d}{dx}(\csc x)$

$= -(\csc x)(-\csc^2 x) - (\cot x)(-\csc x\cot x)$

$= \csc^3 x + \csc x\cot^2 x$

27. $y' = \frac{d}{d\theta}(\theta\tan\theta)$

$= \theta\frac{d}{d\theta}(\tan\theta) + (\tan\theta)\frac{d}{d\theta}(\theta)$

$= \theta\sec^2\theta + \tan\theta$

$y'' = \frac{d}{d\theta}(\theta\sec^2\theta + \tan\theta)$

$= \theta\frac{d}{d\theta}[(\sec\theta)(\sec\theta)] + (\sec^2\theta)\frac{d}{d\theta}(\theta) + \frac{d}{d\theta}(\tan\theta)$

$= \theta\left[(\sec\theta)\frac{d}{d\theta}(\sec\theta) + (\sec\theta)\frac{d}{d\theta}(\sec\theta)\right] + \sec^2\theta +$

$\sec^2\theta$

$= 2\theta\sec^2\theta\tan\theta + 2\sec^2\theta$

$= (2\theta\tan\theta + 2)(\sec^2\theta)$

or, writing in terms of sines and cosines,

$= \frac{2 + 2\theta\tan\theta}{\cos^2\theta}$

$= \frac{2\cos\theta + 2\theta\sin\theta}{\cos^3\theta}$

28. Continuous:

Note that $g(0) = \lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \cos x = \cos(0) = 1$, and $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} (x + b) = b$. We require $\lim_{x\to 0^-} g(x) = g(0)$, so $b = 1$. The function is continuous if $b = 1$.

Differentiable:
For $b = 1$, the left-hand derivative is 1 and the right-hand derivative is $-\sin(0) = 0$, so the function is not differentiable. For other values of b, the function is discontinuous at $x = 0$ and there is no left-hand derivative. So, there is no value of b that will make the function differentiable at $x = 0$.

29. Observe the pattern:

$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d^2}{dx^2}\cos x = -\cos x$$
$$\frac{d^3}{dx^3}\cos x = \sin x$$
$$\frac{d^4}{dx^4}\cos x = \cos x$$
$$\frac{d^5}{dx^5}\cos x = -\sin x$$
$$\frac{d^6}{dx^6}\cos x = -\cos x$$
$$\frac{d^7}{dx^7}\cos x = \sin x$$
$$\frac{d^8}{dx^8}\cos x = \cos x$$

Continuing the pattern, we see that $\frac{d^n}{dx^n}\cos x = \sin x$ when $n = 4k + 3$ for any whole number k.

Since $999 = 4(249) + 3$, $\frac{d^{999}}{dx^{999}}\cos x = \sin x$.

30. Observe the pattern:

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d^2}{dx^2}\sin x = -\sin x$$
$$\frac{d^3}{dx^3}\sin x = -\cos x$$
$$\frac{d^4}{dx^4}\sin x = \sin x$$
$$\frac{d^5}{dx^5}\sin x = \cos x$$
$$\frac{d^6}{dx^6}\sin x = -\sin x$$
$$\frac{d^7}{dx^7}\sin x = -\cos x$$
$$\frac{d^8}{dx^8}\sin x = \sin x$$

Continuing the pattern, we see that $\frac{d^n}{dx^n}\sin x = \cos x$ when $n = 4k + 1$ for any whole number k.

Since $725 = 4(181) + 1$, $\frac{d^{725}}{dx^{725}}\sin x = \cos x$.

31. The line is tangent to the graph of $y = \sin x$ at $(0, 0)$. Since $y'(0) = \cos(0) = 1$, the line has slope 1 and its equation is $y = x$.

32. **(a)** Using $y = x$, $\sin(0.12) \approx 0.12$.

(b) $\sin(0.12) \approx 0.1197122$; The approximation is within 0.0003 of the actual value.

33.
$$\begin{aligned}\frac{d}{dx}\sin 2x &= \frac{d}{dx}(2\sin x\cos x)\\ &= 2\frac{d}{dx}(\sin x\cos x)\\ &= 2\left[(\sin x)\frac{d}{dx}(\cos x) + (\cos x)\frac{d}{dx}(\sin x)\right]\\ &= 2[(\sin x)(-\sin x) + (\cos x)(\cos x)]\\ &= 2(\cos^2 x - \sin^2 x)\\ &= 2\cos 2x\end{aligned}$$

34.
$$\begin{aligned}\frac{d}{dx}\cos 2x &= \frac{d}{dx}[(\cos x)(\cos x) - (\sin x)(\sin x)]\\ &= \left[(\cos x)\frac{d}{dx}(\cos x) + (\cos x)\frac{d}{dx}(\cos x)\right] -\\ &\quad \left[(\sin x)\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(\sin x)\right]\\ &= 2(\cos x)(-\sin x) - 2(\sin x)(\cos x)\\ &= -4\sin x\cos x\\ &= -2(2\sin x\cos x)\\ &= -2\sin 2x\end{aligned}$$

35.
$$\begin{aligned}\lim_{h\to 0}\frac{(\cos h - 1)}{h} &= \lim_{h\to 0}\frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)}\\ &= \lim_{h\to 0}\frac{\cos^2 h - 1}{h(\cos h + 1)}\\ &= \lim_{h\to 0}\frac{-\sin^2 h}{h(\cos h + 1)}\\ &= -\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{\sin h}{\cos h + 1}\right)\\ &= -(1)\left(\frac{0}{2}\right) = 0\end{aligned}$$

36.
$$y' = \frac{d}{dx}(A\sin x + B\cos x) = A\cos x - B\sin x$$
$$y'' = \frac{d}{dx}(A\cos x - B\sin x) = -A\sin x - B\cos x$$

Solve:
$$\begin{aligned}y'' - y &= \sin x\\ (-A\sin x - B\cos x) - (A\sin x + B\cos x) &= \sin x\\ -2A\sin x - 2B\cos x &= \sin x\end{aligned}$$

At $x = \frac{\pi}{2}$, this gives $-2A = 1$, so $A = -\frac{1}{2}$.

At $x = 0$, we have $-2B = 0$, so $B = 0$.

Thus, $A = -\frac{1}{2}$ and $B = 0$.

■ Section 3.6 Chain Rule (pp. 141–149)

Quick Review 3.6

1. $f(g(x) = f(x^2 + 1) = \sin(x^2 + 1)$
2. $f(g(h(x))) = f(g(7x) = f((7x)^2 + 1)$
   $= \sin[(7x)^2 + 1] = \sin(49x^2 + 1)$
3. $(g \circ h)(x) = g(h(x)) = g(7x) = (7x)^2 + 1 = 49x^2 + 1$
4. $(h \circ g)(x) = h(g(x)) = h(x^2 + 1) = 7(x^2 + 1) = 7x^2 + 7$
5. $f\left(\frac{g(x)}{h(x)}\right) = f\left(\frac{x^2 + 1}{7x}\right) = \sin\frac{x^2 + 1}{7x}$

6. $\sqrt{\cos x + 2} = g(\cos x) = g(f(x))$

7. $\sqrt{3\cos^2 x + 2} = g(3\cos^2 x) = g(h(\cos x)) = g(h(f(x)))$

8. $3\cos x + 6 = 3(\cos x + 2) = 3(\sqrt{\cos x + 2})^2$
$= h(\sqrt{\cos x + 2}) = h(g(\cos x)) = h(g(f(x)))$

9. $\cos 27x^4 = f(27x^4) = f(3(3x^2)^2) = f(h(3x^2)) = f(h(h(x)))$

10. $\cos\sqrt{2 + 3x^2} = \cos\sqrt{3x^2 + 2} = f(\sqrt{3x^2 + 2})$
$= f(g(3x^2)) = f(g(h(x)))$

Section 3.6 Exercises

1. $\frac{dy}{dx} = \frac{d}{dx}\sin(3x + 1) = [\cos(3x + 1)]\frac{d}{dx}(3x + 1)$
$= [\cos(3x + 1)](3) = 3\cos(3x + 1)$

2. $\frac{dy}{dx} = \frac{d}{dx}\sin(7 - 5x) = [\cos(7 - 5x)]\frac{d}{dx}(7 - 5x)$
$= [\cos(7 - 5x)](-5) = -5\cos(7 - 5x)$

3. $\frac{dy}{dx} = \frac{d}{dx}\cos(\sqrt{3}x) = [-\sin(\sqrt{3}x)]\frac{d}{dx}(\sqrt{3}x)$
$= [-\sin(\sqrt{3}x)](\sqrt{3}) = -\sqrt{3}\sin(\sqrt{3}x)$

4. $\frac{dy}{dx} = \frac{d}{dx}\tan(2x - x^3) = [\sec^2(2x - x^3)]\frac{d}{dx}(2x - x^3)$
$= [\sec^2(2x - x^3)](2 - 3x^2) = (2 - 3x^2)\sec^2(2x - x^3)$

5. $\frac{dy}{dx} = \frac{d}{dx}\left[5\cot\left(\frac{2}{x}\right)\right] = \left[-5\csc^2\left(\frac{2}{x}\right)\right]\frac{d}{dx}(2x^{-1})$
$= \left[-5\csc^2\left(\frac{2}{x}\right)\right](-2x^{-2}) = \frac{10}{x^2}\csc^2\left(\frac{2}{x}\right)$

6. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x}{1 + \cos x}\right)^2 = 2\left(\frac{\sin x}{1 + \cos x}\right)\frac{d}{dx}\left(\frac{\sin x}{1 + \cos x}\right)$
$= 2\left(\frac{\sin x}{1 + \cos x}\right)\left(\frac{(1 + \cos x)\frac{d}{dx}\sin x - \sin x\frac{d}{dx}(1 + \cos x)}{(1 + \cos x)^2}\right)$
$= 2\left(\frac{\sin x}{1 + \cos x}\right)\left(\frac{(1 + \cos x)(\cos x) - (\sin x)(-\sin x)}{(1 + \cos x)^2}\right)$
$= 2\left(\frac{\sin x}{1 + \cos x}\right)\left(\frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2}\right)$
$= 2\left(\frac{\sin x}{1 + \cos x}\right)\left(\frac{1 + \cos x}{(1 + \cos x)^2}\right)$
$= 2\left(\frac{\sin x}{1 + \cos x}\right)\left(\frac{1}{1 + \cos x}\right)$
$= \frac{2\sin x}{(1 + \cos x)^2}$

7. $\frac{dy}{dx} = \frac{d}{dx}\cos(\sin x) = [-\sin(\sin x)]\frac{d}{dx}(\sin x)$
$= -\sin(\sin x)\cos x$

8. $\frac{dy}{dx} = \frac{d}{dx}\sec(\tan x) = \sec(\tan x)\tan(\tan x)\frac{d}{dx}(\tan x)$
$= \sec(\tan x)\tan(\tan x)\sec^2 x$

9. $\frac{dy}{dx} = \frac{d}{dx}(x + \sqrt{x})^{-2} = -2(x + \sqrt{x})^{-3}\frac{d}{dx}(x + \sqrt{x})$
$= -2(x + \sqrt{x})^{-3}\left(1 + \frac{1}{2\sqrt{x}}\right)$

10. $\frac{dy}{dx} = \frac{d}{dx}(\csc x + \cot x)^{-1}$
$= -(\csc x + \cot x)^{-2}\frac{d}{dx}(\csc x + \cot x)$
$= -\frac{1}{(\csc x + \cot x)^2}(-\cot x\csc x - \csc^2 x)$
$= \frac{(\csc x)(\cot x + \csc x)}{(\csc x + \cot x)^2} = \frac{\csc x}{\csc x + \cot x}$

11. $\frac{dy}{dx} = \frac{d}{dx}(\sin^{-5} x - \cos^3 x)$
$= (-5\sin^{-6} x)\frac{d}{dx}(\sin x) - (3\cos^2 x)\frac{d}{dx}(\cos x)$
$= -5\sin^{-6} x\cos x + 3\cos^2 x\sin x$

12. $\frac{dy}{dx} = \frac{d}{dx}[x^3(2x - 5)^4]$
$= (x^3)\frac{d}{dx}(2x - 5)^4 + (2x - 5)^4\frac{d}{dx}(x^3)$
$= (x^3)(4)(2x - 5)^3\frac{d}{dx}(2x - 5) + (2x - 5)^4(3x^2)$
$= (x^3)(4)(2x - 5)^3(2) + 3x^2(2x - 5)^4$
$= 8x^3(2x - 5)^3 + 3x^2(2x - 5)^4$
$= x^2(2x - 5)^3[8x + 3(2x - 5)]$
$= x^2(2x - 5)^3(14x - 15)$

13. $\frac{dy}{dx} = \frac{d}{dx}(\sin^3 x\tan 4x)$
$= (\sin^3 x)\frac{d}{dx}(\tan 4x) + (\tan 4x)\frac{d}{dx}(\sin^3 x)$
$= (\sin^3 x)(\sec^2 4x)\frac{d}{dx}(4x) + (\tan 4x)(3\sin^2 x)\frac{d}{dx}(\sin x)$
$= (\sin^3 x)(\sec^2 4x)(4) + (\tan 4x)(3\sin^2 x)(\cos x)$
$= 4\sin^3 x\sec^2 4x + 3\sin^2 x\cos x\tan 4x$

14. $\frac{dy}{dx} = \frac{d}{dx}(4\sqrt{\sec x + \tan x})$
$= 4\cdot\frac{1}{2\sqrt{\sec x + \tan x}}\frac{d}{dx}(\sec x + \tan x)$
$= \frac{2}{\sqrt{\sec x + \tan x}}(\sec x\tan x + \sec^2 x)$
$= 2\sec x\frac{\sec x + \tan x}{\sqrt{\sec x + \tan x}}$
$= 2\sec x\sqrt{\sec x + \tan x}$

15. $$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\left(\frac{3}{\sqrt{2x+1}}\right)\\&=\frac{(\sqrt{2x+1})\frac{d}{dx}(3)-3\frac{d}{dx}(\sqrt{2x+1})}{(\sqrt{2x+1})^2}\\&=\frac{(\sqrt{2x+1})(0)-3\left(\frac{1}{2\sqrt{2x+1}}\right)\frac{d}{dx}(2x+1)}{2x+1}\\&=\frac{-3\left(\frac{1}{2\sqrt{2x+1}}\right)(2)}{2x+1}\\&=-\frac{3}{(2x+1)\sqrt{2x+1}}\\&=-3(2x+1)^{-3/2}\end{aligned}$$

16. $$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\frac{x}{\sqrt{1+x^2}}\\&=\frac{(\sqrt{1+x^2})\frac{d}{dx}(x)-x\frac{d}{dx}(\sqrt{1+x^2})}{(\sqrt{1+x^2})^2}\\&=\frac{(\sqrt{1+x^2})(1)-x\left(\frac{1}{2\sqrt{1+x^2}}\right)\frac{d}{dx}(1+x^2)}{1+x^2}\\&=\frac{\sqrt{1+x^2}-x\left(\frac{1}{2\sqrt{1+x^2}}\right)(2x)}{1+x^2}\\&=\frac{(1+x^2)-x^2}{(1+x^2)(\sqrt{1+x^2})}\\&=(1+x^2)^{-3/2}\end{aligned}$$

17. The last step here uses the identity $2\sin a\cos a=\sin 2a$.

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\sin^2(3x-2)\\&=2\sin(3x-2)\frac{d}{dx}\sin(3x-2)\\&=2\sin(3x-2)\cos(3x-2)\frac{d}{dx}(3x-2)\\&=2\sin(3x-2)\cos(3x-2)(3)\\&=6\sin(3x-2)\cos(3x-2)\\&=3\sin(6x-4)\end{aligned}$$

18. $$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}(1+\cos 2x)^2=2(1+\cos 2x)\frac{d}{dx}(1+\cos 2x)\\&=2(1+\cos 2x)(-\sin 2x)\frac{d}{dx}(2x)\\&=2(1+\cos 2x)(-\sin 2x)(2)\\&=-4(1+\cos 2x)(\sin 2x)\end{aligned}$$

19. $$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}(1+\cos^2 7x)^3\\&=3(1+\cos^2 7x)^2\frac{d}{dx}(1+\cos^2 7x)\\&=3(1+\cos^2 7x)^2(2\cos 7x)\frac{d}{dx}(\cos 7x)\\&=3(1+\cos^2 7x)^2(2\cos 7x)(-\sin 7x)\frac{d}{dx}(7x)\\&=3(1+\cos^2 7x)^2(2\cos 7x)(-\sin 7x)(7)\\&=-42(1+\cos^2 7x)^2\cos 7x\sin 7x\end{aligned}$$

20. $$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}(\sqrt{\tan 5x})=\frac{1}{2\sqrt{\tan 5x}}\frac{d}{dx}\tan 5x\\&=\frac{1}{2\sqrt{\tan 5x}}(\sec^2 5x)\frac{d}{dx}(5x)\\&=\frac{1}{2\sqrt{\tan 5x}}(\sec^2 5x)(5)\\&=\frac{5\sec^2 5x}{2\sqrt{\tan 5x}}\text{ or }\frac{5}{2}(\tan 5x)^{-1/2}\sec^2 5x\end{aligned}$$

21. $$\begin{aligned}\frac{ds}{dt}&=\frac{d}{dt}\cos\left(\frac{\pi}{2}-3t\right)\\&=\left[-\sin\left(\frac{\pi}{2}-3t\right)\right]\frac{d}{dt}\left(\frac{\pi}{2}-3t\right)\\&=\left[-\sin\left(\frac{\pi}{2}-3t\right)\right](-3)\\&=3\sin\left(\frac{\pi}{2}-3t\right)\end{aligned}$$

22. $$\begin{aligned}\frac{ds}{dt}&=\frac{d}{dt}[t\cos(\pi-4t)]\\&=(t)\frac{d}{dt}[\cos(\pi-4t)]+\cos(\pi-4t)\frac{d}{dt}(t)\\&=t[-\sin(\pi-4t)]\frac{d}{dt}(\pi-4t)+\cos(\pi-4t)(1)\\&=t[-\sin(\pi-4t)](-4)+\cos(\pi-4t)\\&=4t\sin(\pi-4t)+\cos(\pi-4t)\end{aligned}$$

23. $$\begin{aligned}\frac{ds}{dt}&=\frac{d}{dt}\left(\frac{4}{3\pi}\sin 3t+\frac{4}{5\pi}\cos 5t\right)\\&=\frac{4}{3\pi}(\cos 3t)\frac{d}{dt}(3t)+\frac{4}{5\pi}(-\sin 5t)\frac{d}{dt}(5t)\\&=\frac{4}{3\pi}(\cos 3t)(3)+\frac{4}{5\pi}(-\sin 5t)(5)\\&=\frac{4}{\pi}\cos 3t-\frac{4}{\pi}\sin 5t\end{aligned}$$

24. $$\begin{aligned}\frac{ds}{dt}&=\frac{d}{dt}\left[\sin\left(\frac{3\pi}{2}t\right)+\cos\left(\frac{7\pi}{4}t\right)\right]\\&=\cos\left(\frac{3\pi}{2}t\right)\frac{d}{dt}\left(\frac{3\pi}{2}t\right)-\sin\left(\frac{7\pi}{4}t\right)\frac{d}{dt}\left(\frac{7\pi}{4}t\right)\\&=\frac{3\pi}{2}\cos\left(\frac{3\pi}{2}t\right)-\frac{7\pi}{4}\sin\left(\frac{7\pi}{4}t\right)\end{aligned}$$

25. $$\begin{aligned}\frac{dr}{d\theta}&=\frac{d}{d\theta}\tan(2-\theta)=\sec^2(2-\theta)\frac{d}{d\theta}(2-\theta)\\&=\sec^2(2-\theta)(-1)=-\sec^2(2-\theta)\end{aligned}$$

26. $$\begin{aligned}\frac{dr}{d\theta}&=\frac{d}{d\theta}(\sec 2\theta\tan 2\theta)\\&=(\sec 2\theta)\frac{d}{d\theta}(\tan 2\theta)+(\tan 2\theta)\frac{d}{d\theta}(\sec 2\theta)\\&=(\sec 2\theta)(\sec^2 2\theta)\frac{d}{d\theta}(2\theta)+(\tan 2\theta)(\sec 2\theta\tan 2\theta)\frac{d}{d\theta}(2\theta)\\&=2\sec^3 2\theta+2\sec 2\theta\tan^2 2\theta\end{aligned}$$

27. $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}\sqrt{\theta \sin \theta} = \dfrac{1}{2\sqrt{\theta \sin \theta}}\dfrac{d}{d\theta}(\theta \sin \theta)$

$= \dfrac{1}{2\sqrt{\theta \sin \theta}}\left[\theta\dfrac{d}{d\theta}(\sin \theta) + (\sin \theta)\dfrac{d}{d\theta}(\theta)\right]$

$= \dfrac{1}{2\sqrt{\theta \sin \theta}}(\theta \cos \theta + \sin \theta)$

$= \dfrac{\theta \cos \theta + \sin \theta}{2\sqrt{\theta \sin \theta}}$

28. $\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}(2\theta\sqrt{\sec \theta})$

$= (2\theta)\dfrac{d}{d\theta}(\sqrt{\sec \theta}) + (\sqrt{\sec \theta})\dfrac{d}{d\theta}(2\theta)$

$= (2\theta)\left(\dfrac{1}{2\sqrt{\sec \theta}}\right)\dfrac{d}{d\theta}(\sec \theta) + 2\sqrt{\sec \theta}$

$= (2\theta)\left(\dfrac{1}{2\sqrt{\sec \theta}}\right)(\sec \theta \tan \theta) + 2\sqrt{\sec \theta}$

$= \theta(\sqrt{\sec \theta})(\tan \theta) + 2\sqrt{\sec \theta}$

$= \sqrt{\sec \theta}(\theta \tan \theta + 2)$

29. $y' = \dfrac{d}{dx}\tan x = \sec^2 x$

$y'' = \dfrac{d}{dx}\sec^2 x = (2 \sec x)\dfrac{d}{dx}(\sec x)$

$= (2 \sec x)(\sec x \tan x)$

$= 2 \sec^2 x \tan x$

30. $y' = \dfrac{d}{dx}\cot x = -\csc^2 x$

$y'' = \dfrac{d}{dx}(-\csc^2 x) = (-2 \csc x)\dfrac{d}{dx}(\csc x)$

$= (-2 \csc x)(-\csc x \cot x)$

$= 2 \csc^2 x \cot x$

31. $y' = \dfrac{d}{dx}\cot(3x - 1) = -\csc^2(3x - 1)\dfrac{d}{dx}(3x - 1)$

$= -3 \csc^2(3x - 1)$

$y'' = \dfrac{d}{dx}[-3 \csc^2(3x - 1)]$

$= -3[2 \csc(3x - 1)]\dfrac{d}{dx}\csc(3x - 1)$

$= -3[2 \csc(3x - 1)] \cdot$

$[-\csc(3x - 1)\cot(3x - 1)]\dfrac{d}{dx}(3x - 1)$

$= -3[2 \csc(3x - 1)][-\csc(3x - 1)\cot(3x - 1)](3)$

$= 18 \csc^2(3x - 1)\cot(3x - 1)$

32. $y' = \dfrac{d}{dx}\left[9 \tan\left(\dfrac{x}{3}\right)\right] = 9 \sec^2\left(\dfrac{x}{3}\right)\dfrac{d}{dx}\left(\dfrac{x}{3}\right)$

$= 3 \sec^2\left(\dfrac{x}{3}\right)$

$y'' = \dfrac{d}{dx}\left[3 \sec^2\left(\dfrac{x}{3}\right)\right] = 3\left[2 \sec\left(\dfrac{x}{3}\right)\right]\dfrac{d}{dx}\sec\left(\dfrac{x}{3}\right)$

$= 6\left[\sec\left(\dfrac{x}{3}\right)\right]\left[\sec\left(\dfrac{x}{3}\right)\tan\left(\dfrac{x}{3}\right)\right]\dfrac{d}{dx}\left(\dfrac{x}{3}\right)$

$= 2 \sec^2\left(\dfrac{x}{3}\right)\tan\left(\dfrac{x}{3}\right)$

33. $f'(u) = \dfrac{d}{du}(u^5 + 1) = 5u^4$

$g'(x) = \dfrac{d}{dx}(\sqrt{x}) = \dfrac{1}{2\sqrt{x}}$

$(f \circ g)'(1) = f'(g(1))g'(1) = f'(1)g'(1) = (5)\left(\dfrac{1}{2}\right) = \dfrac{5}{2}$

34. $f'(u) = \dfrac{d}{du}(1 - u^{-1}) = u^{-2} = \dfrac{1}{u^2}$

$g'(x) = \dfrac{d}{dx}(1 - x)^{-1} = -(1 - x)^{-2}\dfrac{d}{dx}(1 - x)$

$= -(1 - x)^{-2}(-1) = \dfrac{1}{(1 - x)^2}$

$(f \circ g)'(-1) = f'(g(-1))g'(-1) = f'\left(\dfrac{1}{2}\right)g'(-1)$

$= (4)\left(\dfrac{1}{4}\right) = 1$

35. $f'(u) = \dfrac{d}{du}\left(\cot \dfrac{\pi u}{10}\right) = -\csc^2\left(\dfrac{\pi u}{10}\right)\dfrac{d}{du}\left(\dfrac{\pi u}{10}\right)$

$= -\dfrac{\pi}{10}\csc^2\left(\dfrac{\pi u}{10}\right)$

$g'(x) = \dfrac{d}{dx}(5\sqrt{x}) = \dfrac{5}{2\sqrt{x}}$

$(f \circ g)'(1) = f'(g(1))g'(1) = f'(5)g'(1)$

$= -\dfrac{\pi}{10}\left[\csc^2\left(\dfrac{\pi}{2}\right)\right]\left(\dfrac{5}{2}\right)$

$= -\dfrac{\pi}{10}(1)\left(\dfrac{5}{2}\right) = -\dfrac{\pi}{4}$

36. $f'(u) = \dfrac{d}{du}[u + (\cos u)^{-2}]$

$= 1 - 2(\cos u)^{-3}\dfrac{d}{du}\cos u$

$= 1 + \dfrac{2 \sin u}{\cos^3 u}$

$g'(x) = \dfrac{d}{dx}(\pi x) = \pi$

$(f \circ g)'\left(\dfrac{1}{4}\right) = f'(g(\tfrac{1}{4}))g'\left(\dfrac{1}{4}\right)$

$= f'\left(\dfrac{\pi}{4}\right)g'\left(\dfrac{1}{4}\right)$

$$= \left(1 + \dfrac{\dfrac{2}{\sqrt{2}}}{\left(\dfrac{1}{\sqrt{2}}\right)^3}\right)(\pi)$$

$= 5\pi$

37. $f'(u) = \dfrac{d}{du}\dfrac{2u}{u^2 + 1} = \dfrac{(u^2 + 1)\dfrac{d}{du}(2u) - (2u)\dfrac{d}{du}(u^2 + 1)}{(u^2 + 1)^2}$

$= \dfrac{(u^2 + 1)(2) - (2u)(2u)}{(u^2 + 1)^2} = \dfrac{-2u^2 + 2}{(u^2 + 1)^2}$

$g'(x) = \dfrac{d}{dx}(10x^2 + x + 1) = 20x + 1$

$(f \circ g)'(0) = f'(g(0))g'(0) = f'(1)g'(0) = (0)(1) = 0$

38. $$f'(u) = \frac{d}{du}\left(\frac{u-1}{u+1}\right)^2 = 2\left(\frac{u-1}{u+1}\right)\frac{d}{du}\left(\frac{u-1}{u+1}\right)$$
$$= 2\left(\frac{u-1}{u+1}\right)\frac{(u+1)\frac{d}{du}(u-1) - (u-1)\frac{d}{du}(u+1)}{(u+1)^2}$$
$$= 2\left(\frac{u-1}{u+1}\right)\frac{(u+1)-(u-1)}{(u+1)^2} = \frac{4(u-1)}{(u+1)^3}$$

$$g'(x) = \frac{d}{dx}(x^{-2} - 1) = -2x^{-3}$$
$$\begin{aligned}(f \circ g)'(-1) &= f'(g(-1))g'(-1)\\ &= f'(0)g'(-1)\\ &= (-4)(2) = -8\end{aligned}$$

39. **(a)** $$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= \frac{d}{du}(\cos u)\frac{d}{dx}(6x+2)\\ &= (-\sin u)(6)\\ &= -6\sin u\\ &= -6\sin(6x+2)\end{aligned}$$

(b) $$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= \frac{d}{du}(\cos 2u)\frac{d}{dx}(3x+1)\\ &= (-\sin 2u)(2)\cdot(3)\\ &= -6\sin 2u\\ &= -6\sin(6x+2)\end{aligned}$$

40. **(a)** $$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= \frac{d}{du}\sin(u+1)\frac{d}{dx}(x^2)\\ &= \cos(u+1)(1)\cdot 2x\\ &= 2x\cos(u+1)\\ &= 2x\cos(x^2+1)\end{aligned}$$

(b) $$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= \frac{d}{du}(\sin u)\frac{d}{dx}(x^2+1)\\ &= (\cos u)(2x)\\ &= 2x\cos u\\ &= 2x\cos(x^2+1)\end{aligned}$$

41. $$\frac{dx}{dt} = \frac{d}{dt}(2\cos t) = -2\sin t$$
$$\frac{dy}{dt} = \frac{d}{dt}(2\sin t) = 2\cos t$$
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t}{-2\sin t} = -\cot t$$

The line passes through $\left(2\cos\frac{\pi}{4}, 2\sin\frac{\pi}{4}\right) = (\sqrt{2}, \sqrt{2})$ and has slope $-\cot\frac{\pi}{4} = -1$. Its equation is $y = -(x - \sqrt{2}) + \sqrt{2}$, or $y = -x + 2\sqrt{2}$.

42. $$\frac{dx}{dt} = \frac{d}{dt}(\sin 2\pi t) = (\cos 2\pi t)\frac{d}{dt}(2\pi t) = 2\pi\cos 2\pi t$$
$$\frac{dy}{dt} = \frac{d}{dt}(\cos 2\pi t) = (-\sin 2\pi t)\frac{d}{dt}(2\pi t) = -2\pi\sin 2\pi t$$
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\pi\sin 2\pi t}{2\pi\cos 2\pi t} = -\tan 2\pi t$$

The line passes through $\left(\sin\frac{2\pi}{-6}, \cos\frac{2\pi}{-6}\right) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and has slope $-\tan\frac{2\pi}{-6} = \sqrt{3}$. Its equation is $y = \sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{1}{2}$, or $y = \sqrt{3}x + 2$.

43. $$\begin{aligned}\frac{dx}{dt} = \frac{d}{dt}(\sec^2 t - 1) &= (2\sec t)\frac{d}{dt}(\sec t)\\ &= (2\sec t)(\sec t\tan t)\\ &= 2\sec^2 t\tan t\end{aligned}$$
$$\frac{dy}{dt} = \frac{d}{dt}\tan t = \sec^2 t$$
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec^2 t}{2\sec^2 t\tan t} = \frac{1}{2}\cot t.$$

The line passes through $\left(\sec^2\left(-\frac{\pi}{4}\right) - 1, \tan\left(-\frac{\pi}{4}\right)\right) = (1, -1)$ and has slope $\frac{1}{2}\cot\left(-\frac{\pi}{4}\right) = -\frac{1}{2}$. Its equation is $y = -\frac{1}{2}(x-1) - 1$, or $y = -\frac{1}{2}x - \frac{1}{2}$.

44. $$\frac{dx}{dt} = \frac{d}{dt}\sec t = \sec t\tan t$$
$$\frac{dy}{dt} = \frac{d}{dt}\tan t = \sec^2 t$$
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec^2 t}{\sec t\tan t} = \frac{\sec t}{\tan t} = \frac{1}{\sin t} = \csc t$$

The line passes through $\left(\sec\frac{\pi}{6}, \tan\frac{\pi}{6}\right) = \left(\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and has slope $\csc\frac{\pi}{6} = 2$. Its equation is $y = 2\left(x - \frac{2}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}}$, or $y = 2x - \sqrt{3}$.

45. $\frac{dx}{dt} = \frac{d}{dt}t = 1$

$\frac{dy}{dt} = \frac{d}{dt}\sqrt{t} = \frac{1}{2\sqrt{t}}$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1/(2\sqrt{t})}{1} = \frac{1}{2\sqrt{t}}$

The line passes through $\left(\frac{1}{4}, \sqrt{\frac{1}{4}}\right) = \left(\frac{1}{4}, \frac{1}{2}\right)$ and has slope $\frac{1}{2\sqrt{\frac{1}{4}}} = 1$. Its equation is $y = 1\left(x - \frac{1}{4}\right) + \frac{1}{2}$, or $y = x + \frac{1}{4}$.

46. $\frac{dx}{dt} = \frac{d}{dt}(2t^2 + 3) = 4t$

$\frac{dy}{dt} = \frac{d}{dt}(t^4) = 4t^3$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t^3}{4t} = t^2$

The line passes through $(2(-1)^2 + 3, (-1)^4) = (5, 1)$ and has slope $(-1)^2 = 1$. Its equation is $y = 1(x - 5) + 1$, or $y = x - 4$.

47. $\frac{dx}{dt} = \frac{d}{dt}(t - \sin t) = 1 - \cos t$

$\frac{dy}{dt} = \frac{d}{dt}(1 - \cos t) = \sin t$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sin t}{1 - \cos t}$

The line passes through

$\left(\frac{\pi}{3} - \sin\frac{\pi}{3}, 1 - \cos\frac{\pi}{3}\right) = \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and has slope $\frac{\sin\left(\frac{\pi}{3}\right)}{1 - \cos\left(\frac{\pi}{3}\right)} = \sqrt{3}$. Its equation is

$y = \sqrt{3}\left(x - \frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) + \frac{1}{2}$, or

$y = \sqrt{3}x + 2 - \frac{\pi}{\sqrt{3}}$.

48. $\frac{dx}{dt} = \frac{d}{dt}\cos t = -\sin t$

$\frac{dy}{dt} = \frac{d}{dt}(1 + \sin t) = \cos t$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{-\sin t} = -\cot t$

The line passes through $\left(\cos\frac{\pi}{2}, 1 + \sin\frac{\pi}{2}\right) = (0, 2)$ and has slope $-\cot\left(\frac{\pi}{2}\right) = 0$. Its equation is $y = 2$.

49. (a) $\frac{dx}{dt} = \frac{d}{dt}(t^2 + t) = 2t + 1$

$\frac{dy}{dt} = \frac{d}{dt}\sin t = \cos t$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{2t + 1}$

(b) $\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\frac{\cos t}{2t + 1}$

$= \frac{(2t + 1)\frac{d}{dt}(\cos t) - (\cos t)\frac{d}{dt}(2t + 1)}{(2t + 1)^2}$

$= \frac{(2t + 1)(-\sin t) - (\cos t)(2)}{(2t + 1)^2}$

$= -\frac{(2t + 1)(\sin t) + 2\cos t}{(2t + 1)^2}$

(c) Let $u = \frac{dy}{dx}$.

Then $\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt}$, so $\frac{du}{dx} = \frac{du}{dt} \div \frac{dx}{dt}$. Therefore,

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \div \frac{dx}{dt}$

$= -\frac{(2t + 1)(\sin t) + 2\cos t}{(2t + 1)^2} \div (2t + 1)$

$= -\frac{(2t + 1)(\sin t) + 2\cos t}{(2t + 1)^3}$

(d) The expression in part (c).

50. Since the radius passes through (0, 0) and $(2\cos t, 2\sin t)$, it has slope given by $\tan t$. But the slope of the tangent line is $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t}{-2\sin t} = -\cot t$, which is the negative reciprocal of $\tan t$. This means that the radius and the tangent line are perpendicular. (The preceding argument breaks down when $t = \frac{k\pi}{2}$, where k is an integer. At these values, either the radius is horizontal and the tangent line is vertical or the radius is vertical and the tangent line is horizontal, so the result still holds.)

51. $\frac{ds}{dt} = \frac{ds}{d\theta}\frac{d\theta}{dt} = \frac{d}{d\theta}(\cos\theta)\frac{d\theta}{dt}$

$= (-\sin\theta)\left(\frac{d\theta}{dt}\right)$

When $\theta = \frac{3\pi}{2}$ and $\frac{d\theta}{dt} = 5$, $\frac{ds}{dt} = \left(-\sin\frac{3\pi}{2}\right)(5) = 5$.

52. $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{d}{dx}(x^2 + 7x - 5)\frac{dx}{dt}$

$= (2x + 7)\left(\frac{dx}{dt}\right)$

When $x = 1$ and $\frac{dx}{dt} = \frac{1}{3}$, $\frac{dy}{dt} = [2(1) + 7]\left(\frac{1}{3}\right) = 3$.

53. $\frac{dy}{dx} = \frac{d}{dx}\sin\frac{x}{2} = \left(\cos\frac{x}{2}\right)\frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}\cos\frac{x}{2}$

Since the range of the function $f(x) = \frac{1}{2}\cos\frac{x}{2}$ is $\left[-\frac{1}{2}, \frac{1}{2}\right]$, the largest possible value of $\frac{dy}{dx}$ is $\frac{1}{2}$.

54. $\frac{dy}{dx} = \frac{d}{dx}(\sin mx) = (\cos mx)\frac{d}{dx}(mx) = m\cos mx$

The desired line has slope $y'(0) = m\cos 0 = m$ and passes through $(0, 0)$, so its equation is $y = mx$.

55. $\frac{dy}{dx} = \frac{d}{dx}2\tan\frac{\pi x}{4} = \left(2\sec^2\frac{\pi x}{4}\right)\frac{d}{dx}\left(\frac{\pi x}{4}\right)$

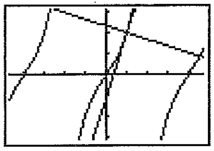
$= \frac{\pi}{2}\sec^2\left(\frac{\pi x}{4}\right)$

$y'(1) = \frac{\pi}{2}\sec^2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}(\sqrt{2})^2 = \pi.$

The tangent line has slope π and passes through $\left(1, 2\tan\frac{\pi}{4}\right) = (1, 2)$. Its equation is $y = \pi(x - 1) + 2$, or $y = \pi x - \pi + 2$.

The normal line has slope $-\frac{1}{\pi}$ and passes through $(1, 2)$. Its equation is $y = -\frac{1}{\pi}(x - 1) + 2$, or $y = -\frac{1}{\pi}x + \frac{1}{\pi} + 2$.

Graphical support:



[−4.7, 4.7] by [−3.1, 3.1]

56. (a) $\frac{d}{dx}[2f(x)] = 2f'(x)$

At $x = 2$, the derivative is $2f'(2) = 2\left(\frac{1}{3}\right) = \frac{2}{3}$.

(b) $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

At $x = 3$, the derivative is $f'(3) + g'(3) = 2\pi + 5$.

(c) $\frac{d}{dx}[f(x) \cdot g(x)] = f(x)g'(x) + g(x)f'(x)$

At $x = 3$, the derivative is

$$f(3)g'(3) + g(3)f'(3) = (3)(5) + (-4)(2\pi)$$
$$= 15 - 8\pi.$$

(d) $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

At $x = 2$, the derivative is

$$\frac{g(2)f'(2) - f(2)g'(2)}{[g(2)]^2} = \frac{(2)\left(\frac{1}{3}\right) - (8)(-3)}{(2)^2}$$
$$= \frac{\frac{74}{3}}{4} = \frac{37}{6}.$$

(e) $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

At $x = 2$, the derivative is

$f'(g(2))g'(2) = f'(2)g'(2) = \left(\frac{1}{3}\right)(-3) = -1.$

(f) $\frac{d}{dx}\sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}}\frac{d}{dx}f(x) = \frac{f'(x)}{2\sqrt{f(x)}}$

At $x = 2$, the derivative is

$$\frac{f'(2)}{2\sqrt{f(2)}} = \frac{\frac{1}{3}}{2\sqrt{8}} = \frac{1}{6(2\sqrt{2})} = \frac{1}{12\sqrt{2}}.$$

(g) $\frac{d}{dx}\frac{1}{g^2(x)} = \frac{d}{dx}[g(x)]^{-2} = -2[g(x)]^{-3}\frac{d}{dx}g(x) = -\frac{2g'(x)}{[g(x)]^3}$

At $x = 3$, the derivative is

$$-\frac{2g'(3)}{[g(3)]^3} = -\frac{2(5)}{(-4)^3} = -\frac{10}{-64} = \frac{5}{32}.$$

(h) $\frac{d}{dx}\sqrt{f^2(x) + g^2(x)} = \frac{1}{2\sqrt{f^2(x) + g^2(x)}}\frac{d}{dx}[f^2(x) + g^2(x)]$

$$= \frac{1}{2\sqrt{f^2(x) + g^2(x)}}\left[2f(x)\frac{d}{dx}f(x) + 2g(x)\frac{d}{dx}g(x)\right]$$
$$= \frac{f(x)f'(x) + g(x)g'(x)}{\sqrt{f^2(x) + g^2(x)}}$$

At $x = 2$, the derivative is

$$\frac{f(2)f'(2) + g(2)g'(2)}{\sqrt{f^2(2) + g^2(2)}} = \frac{(8)\left(\frac{1}{3}\right) + (2)(-3)}{\sqrt{8^2 + 2^2}}$$
$$= \frac{-\frac{10}{3}}{\sqrt{68}} = -\frac{\frac{10}{3}}{2\sqrt{17}} = -\frac{5}{3\sqrt{17}}$$

57. (a) $\frac{d}{dx}[5f(x) - g(x)] = 5f'(x) - g'(x)$

At $x = 1$, the derivative is

$5f'(1) - g'(1) = 5\left(-\frac{1}{3}\right) - \left(-\frac{8}{3}\right) = 1.$

(b) $\frac{d}{dx}f(x)g^3(x) = f(x)\frac{d}{dx}g^3(x) + g^3(x)\frac{d}{dx}f(x)$

$= f(x)[3g^2(x)]\frac{d}{dx}g(x) + g^3(x)f'(x)$

$= 3f(x)g^2(x)g'(x) + g^3(x)f'(x)$

At $x = 0$, the derivative is $3f(0)g^2(0)g'(0) + g^3(0)f'(0)$

$= 3(1)(1)^2\left(\frac{1}{3}\right) + (1)^3(5) = 6.$

(c) $\frac{d}{dx}\frac{f(x)}{g(x)+1} = \frac{[g(x)+1]\frac{d}{dx}f(x) - f(x)\frac{d}{dx}[g(x)+1]}{[g(x)+1]^2}$

$$= \frac{[g(x)+1]f'(x) - f(x)g'(x)}{[g(x)+1]^2}$$

At $x = 1$, the derivative is

$$\frac{[g(1)+1]f'(1) - f(1)g'(1)}{[g(1)+1]^2} = \frac{(-4+1)\left(-\frac{1}{3}\right) - (3)\left(-\frac{8}{3}\right)}{(-4+1)^2}$$

$$= \frac{9}{9} = 1.$$

(d) $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

At $x = 0$, the derivative is

$$f'(g(0))g'(0) = f'(1)g'(0) = \left(-\frac{1}{3}\right)\left(\frac{1}{3}\right) = -\frac{1}{9}.$$

(e) $\frac{d}{dx}g(f(x)) = g'(f(x))f'(x)$

At $x = 0$, the derivative is

$$g'(f(0))f'(0) = g'(1)f'(0) = \left(-\frac{8}{3}\right)(5) = -\frac{40}{3}$$

(f) $\frac{d}{dx}[g(x)+f(x)]^{-2} = -2[g(x)+f(x)]^{-3}\frac{d}{dx}[g(x)+f(x)]$

$$= -\frac{2[g'(x)+f'(x)]}{[g(x)+f(x)]^3}$$

At $x = 1$, the derivative is

$$-\frac{2[g'(1)+f'(1)]}{[g(1)+f(1)]^3} = -\frac{2\left(-\frac{8}{3}-\frac{1}{3}\right)}{(-4+3)^3} = -\frac{-6}{-1} = -6.$$

(g) $\frac{d}{dx}[f(x+g(x))] = f'(x+g(x))\frac{d}{dx}[x+g(x)]$

$= f'(x+g(x))(1+g'(x))$

At $x = 0$, the derivative is

$$f'(0+g(0))(1+g'(0)) = f'(0+1)\left(1+\frac{1}{3}\right)$$

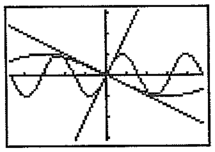
$$= f'(1)\left(\frac{4}{3}\right)$$

$$= \left(-\frac{1}{3}\right)\left(\frac{4}{3}\right) = -\frac{4}{9}.$$

58. For $y = \sin 2x$, $y' = (\cos 2x)\frac{d}{dx}(2x) = 2\cos 2x$ and the slope at the origin is 2.

For $y = -\sin\frac{x}{2}$, $y' = \left(-\cos\frac{x}{2}\right)\frac{d}{dx}\left(\frac{x}{2}\right) = -\frac{1}{2}\cos\frac{x}{2}$ and the slope at the origin is $-\frac{1}{2}$. Since the slopes of the two tangent lines are 2 and $-\frac{1}{2}$, the lines are perpendicular and the curves are orthogonal.

A graph of the two curves along with the tangents $y = 2x$ and $y = -\frac{1}{2}x$ is shown.



[−4.7, 4.7] by [−3.1, 3.1]

59. Because the symbols $\frac{dy}{dx}$, $\frac{dy}{du}$, and $\frac{du}{dx}$ are not fractions. The individual symbols dy, dx, and du do not have numerical values.

60. Velocity: $s'(t) = -2\pi bA\sin(2\pi bt)$
acceleration: $s''(t) = -4\pi^2b^2A\cos(2\pi bt)$
jerk: $s'''(t) = 8\pi^3b^3A\sin(2\pi bt)$
The velocity, amplitude, and jerk are proportional to b, b^2, and b^3, respectively. If the frequency b is doubled, then the amplitude of the velocity is doubled, the amplitude of the acceleration is quadrupled, and the amplitude of the jerk is multiplied by 8.

61. (a) $y'(t) = \frac{d}{dt}37\sin\left[\frac{2\pi}{365}(x-101)\right] + \frac{d}{dt}(25)$

$$= 37\cos\left[\frac{2\pi}{365}(x-101)\right]\cdot\frac{d}{dx}\left[\frac{2\pi}{365}(x-101)\right] + 0$$

$$= 37\cos\left[\frac{2\pi}{365}(x-101)\right]\cdot\frac{2\pi}{365}$$

$$= \frac{74\pi}{365}\cos\left[\frac{2\pi}{365}(x-101)\right]$$

Since $\cos u$ is greatest when $u = 0, \pm 2\pi$, and so on, $y'(t)$ is greatest when $\frac{2\pi}{365}(x-101) = 0$, or $x = 101$. The temperature is increasing the fastest on day 101 (April 11).

(b) The rate of increase is

$y'(101) = \frac{74\pi}{365} \approx 0.637$ degrees per day.

62. Velocity: $s'(t) = \frac{d}{dt}\sqrt{1+4t} = \frac{1}{2\sqrt{1+4t}}\frac{d}{dt}(1+4t)$

$$= \frac{4}{2\sqrt{1+4t}} = \frac{2}{\sqrt{1+4t}}$$

At $t = 6$, the velocity is $\frac{2}{\sqrt{1+4(6)}} = \frac{2}{5}$ m/sec

Acceleration: $s''(t) = \frac{d}{dt}\frac{2}{\sqrt{1+4t}}$

$$= \frac{(\sqrt{1+4t})\frac{d}{dt}(2) - 2\frac{d}{dt}\sqrt{1+4t}}{(\sqrt{1+4t})^2}$$

$$= \frac{-2\left(\frac{1}{2\sqrt{1+4t}}\right)\frac{d}{dt}(1+4t)}{1+4t}$$

$$= \frac{\frac{-4}{\sqrt{1+4t}}}{1+4t} = -\frac{4}{(1+4t)^{3/2}}$$

At $t = 6$, the acceleration is $-\frac{4}{[1+4(6)]^{3/2}} = -\frac{4}{125}$ m/sec^2

63. Acceleration $= \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \left(\frac{dv}{ds}\right)(v) = \left[\frac{d}{ds}(k\sqrt{s})\right](k\sqrt{s})$
$= \left(\frac{k}{2\sqrt{s}}\right)(k\sqrt{s}) = \frac{k^2}{2}$, a constant.

64. Note that this Exercise concerns itself with the slowing down caused by the earth's atmosphere, *not* the acceleration caused by gravity.

Given: $v = \frac{k}{\sqrt{s}}$

Acceleration $= \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \left(\frac{dv}{ds}\right)(v) = (v)\left(\frac{dv}{ds}\right)$

$$= \left(\frac{k}{\sqrt{s}}\right)\frac{d}{ds}\frac{k}{\sqrt{s}}$$

$$= \left(\frac{k}{\sqrt{s}}\right)\left(\frac{\sqrt{s}\frac{d}{ds}(k) - k\frac{d}{ds}\sqrt{s}}{(\sqrt{s})^2}\right)$$

$$= \left(\frac{k}{\sqrt{s}}\right)\left(\frac{\frac{-k}{(2\sqrt{s})}}{s}\right)$$

$$= -\frac{k^2}{2s^2},\ s \geq 0$$

Thus, the acceleration is inversely proportional to s^2.

65. Acceleration $= \frac{dv}{dt} = \frac{df(x)}{dt} = \frac{df(x)}{dx}\frac{dx}{dt} = f'(x)f(x)$

66. $\frac{dT}{du} = \frac{dT}{dL}\frac{dL}{du} = \left(\frac{d}{dL}2\pi\sqrt{\frac{L}{g}}\right)(kL)$

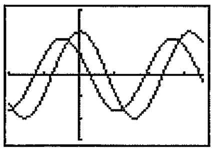
$$= \left(2\pi\frac{1}{2\sqrt{\frac{L}{g}}}\right)\left(\frac{d}{dL}\frac{L}{g}\right)(kL)$$

$$= \left(\frac{\pi}{\sqrt{\frac{L}{g}}}\right)\left(\frac{1}{g}\right)(kL) = k\pi\sqrt{\frac{L}{g}} = \frac{kT}{2}$$

67. No, this does not contradict the Chain Rule. The Chain Rule states that if two functions are differentiable at the appropriate points, then their composite must also be differentiable. It does not say: If a composite is differentiable, then the functions which make up the composite must all be differentiable.

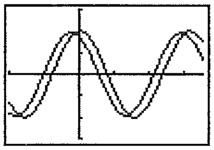
68. Yes. Note that $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$. If the graph of $y = f(g(x))$ has a horizontal tangent at $x = 1$, then $f'(g(1))g'(1) = 0$, so either $g'(1) = 0$ or $f'(g(1)) = 0$. This means that either the graph of $y = g(x)$ has a horizontal tangent at $x = 1$, or the graph of $y = f(u)$ has a horizontal tangent at $u = g(1)$.

69. For $h = 1$:



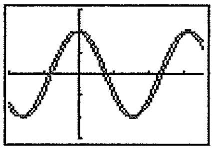
[−2, 3.5] by [−3, 3]

For $h = 0.5$:



[−2, 3.5] by [−3, 3]

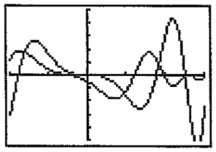
For $h = 0.2$:



[−2, 3.5] by [−3, 3]

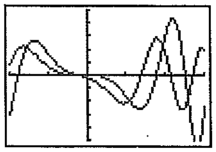
As $h \to 0$, the second curve (the difference quotient) approaches the first ($y = 2\cos 2x$). This is because $2\cos 2x$ is the derivative of sin 2x, and the second curve is the difference quotient used to define the derivative of $\sin 2x$. As $h \to 0$, the difference quotient expression should be approaching the derivative.

70. For $h = 1$:



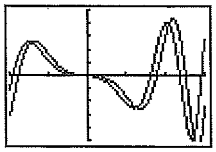
$[-2, 3]$ by $[-5, 5]$

For $h = 0.7$:



$[-2, 3]$ by $[-5, 5]$

For $h = 0.3$:



$[-2, 3]$ by $[-5, 5]$

As $h \to 0$, the second curve (the difference quotient) approaches the first ($y = -2x \sin(x^2)$). This is because $-2x \sin(x^2)$ is the derivative of $\cos(x^2)$, and the second curve is the difference quotient used to define the derivative of $\cos(x^2)$. As $h \to 0$, the difference quotient expression should be approaching the derivative.

71. **(a)** Let $f(x) = |x|$.

Then

$$\frac{d}{dx}|u| = \frac{d}{dx}f(u) = f'(u)\frac{du}{dx} = \left(\frac{d}{du}|u|\right)\left(\frac{du}{dx}\right) = \frac{u}{|u|}u'.$$

The derivative of the absolute value function is $+1$ for positive values, -1 for negative values, and undefined at 0. So $f'(u) = \begin{cases} -1, & u < 0 \\ 1, & u > 0. \end{cases}$

But this is exactly how the expression $\frac{u}{|u|}$ evaluates.

(b) $f'(x) = \left[\frac{d}{dx}(x^2 - 9)\right] \cdot \frac{x^2 - 9}{|x^2 - 9|} = \frac{(2x)(x^2 - 9)}{|x^2 - 9|}$

$$\begin{aligned} g'(x) &= \frac{d}{dx}(|x| \sin x) \\ &= |x|\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}|x| \\ &= |x| \cos x + \frac{x \sin x}{|x|} \end{aligned}$$

Note: The expression for $g'(x)$ above is undefined at $x = 0$, but actually

$$g'(0) = \lim_{h \to 0} \frac{g(0 + h) - g(0)}{h} = \lim_{h \to 0} \frac{|h| \sin h}{h} = 0.$$

Therefore, we may express the derivative as

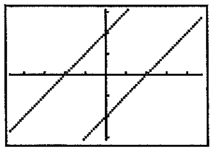
$$g'(x) = \begin{cases} |x| \cos x + \frac{x \sin x}{|x|}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

72.
$$\begin{aligned} \frac{dG}{dx} &= \frac{d}{dx}\sqrt{uv} = \frac{d}{dx}\sqrt{x(x + c)} = \frac{d}{dx}\sqrt{x^2 + cx} \\ &= \frac{1}{2\sqrt{x^2 + cx}}\frac{d}{dx}(x^2 + cx) = \frac{2x + c}{2\sqrt{x^2 + cx}} = \frac{x + (x + c)}{2\sqrt{x(x + c)}} \\ &= \frac{u + v}{2\sqrt{uv}} = \frac{A}{G} \end{aligned}$$

Section 3.7 Implicit Differentiation
(pp. 149–157)

Exploration 1 An Unexpected Derivative

1. $2x - 2y - 2xy' + 2yy' = 0$. Solving for y', we find that $\frac{dy}{dx} = 1$ (provided $y \neq x$).
2. With a constant derivative of 1, the graph would seem to be a line with slope 1.
3. Letting $x = 0$ in the original equation, we find that $y = \pm 2$. This would seem to indicate that this equation defines two lines implicitly, both with slope 1. The two lines are $y = x + 2$ and $y = x - 2$.
4. Factoring the original equation, we have
$[(x - y) - 2][(x - y) + 2] = 0$
$\therefore x - y - 2 = 0$ or $x - y + 2 = 0$
$\therefore y = x - 2$ or $y = x + 2$.
The graph is shown below.

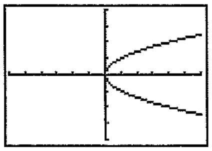


$[-4.7, 4.7]$ by $[-3.1, 3.1]$

5. At each point (x, y) on either line, $\frac{dy}{dx} = 1$. The condition $y \neq x$ is true because both lines are parallel to the line $y = x$. The derivative is surprising because it does not depend on x or y, but there are no inconsistencies.

Quick Review 3.7

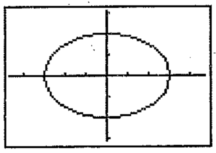
1.
$$\begin{aligned} x - y^2 &= 0 \\ x &= y^2 \\ \pm\sqrt{x} &= y \\ y_1 &= \sqrt{x}, y_2 = -\sqrt{x} \end{aligned}$$



$[-6, 6]$ by $[-4, 4]$

2. $4x^2 + 9y^2 = 36$

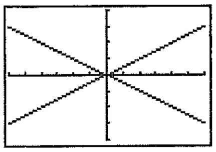
$$9y^2 = 36 - 4x^2$$
$$y^2 = \frac{36 - 4x^2}{9} = \frac{4}{9}(9 - x^2)$$
$$y = \pm\frac{2}{3}\sqrt{9 - x^2}$$
$$y_1 = \frac{2}{3}\sqrt{9 - x^2},\ y_2 = -\frac{2}{3}\sqrt{9 - x^2}$$



[−4.7, 4.7] by [−3.1, 3.1]

3. $x^2 - 4y^2 = 0$

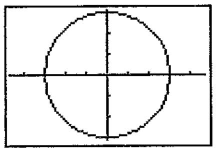
$$(x + 2y)(x - 2y) = 0$$
$$y = \pm\frac{x}{2}$$
$$y_1 = \frac{x}{2},\ y_2 = -\frac{x}{2}$$



[−6, 6] by [−4, 4]

4. $x^2 + y^2 = 9$

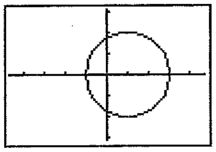
$$y^2 = 9 - x^2$$
$$y = \pm\sqrt{9 - x^2}$$
$$y_1 = \sqrt{9 - x^2},\ y_2 = -\sqrt{9 - y^2}$$



[−4.7, 4.7] by [−3.1, 3.1]

5. $x^2 + y^2 = 2x + 3$

$$y^2 = 2x + 3 - x^2$$
$$y = \pm\sqrt{2x + 3 - x^2}$$
$$y_1 = \sqrt{2x + 3 - x^2},\ y_2 = -\sqrt{2x + 3 - x^2}$$



[−4.7, 4.7] by [−3.1, 3.1]

6. $x^2y' - 2xy = 4x - y$

$$x^2y' = 4x - y + 2xy$$
$$y' = \frac{4x - y + 2xy}{x^2}$$

7. $y' \sin x - x \cos x = xy' + y$

$$y' \sin x - xy' = y + x \cos x$$
$$(\sin x - x)y' = y + x \cos x$$
$$y' = \frac{y + x \cos x}{\sin x - x}$$

8. $x(y^2 - y') = y'(x^2 - y)$

$$xy^2 = y'(x^2 - y + x)$$
$$y' = \frac{xy^2}{x^2 - y + x}$$

9. $\sqrt{x}(x - \sqrt[3]{x}) = x^{1/2}(x - x^{1/3})$

$$= x^{1/2}x - x^{1/2}x^{1/3}$$
$$= x^{3/2} - x^{5/6}$$

10. $\dfrac{x + \sqrt[3]{x^2}}{\sqrt{x^3}} = \dfrac{x + x^{2/3}}{x^{3/2}}$

$$= \frac{x}{x^{3/2}} + \frac{x^{2/3}}{x^{3/2}}$$
$$= x^{-1/2} + x^{-5/6}$$

Section 3.7 Exercises

1. $\dfrac{dy}{dx} = \dfrac{d}{dx}x^{9/4} = \dfrac{9}{4}x^{(9/4)-1} = \dfrac{9}{4}x^{5/4}$

2. $\dfrac{dy}{dx} = \dfrac{d}{dx}x^{-3/5} = -\dfrac{3}{5}x^{(-3/5)-1} = -\dfrac{3}{5}x^{-8/5}$

3. $\dfrac{dy}{dx} = \dfrac{d}{dx}\sqrt[3]{x} = \dfrac{d}{dx}x^{1/3} = \dfrac{1}{3}x^{(1/3)-1} = \dfrac{1}{3}x^{-2/3}$

4. $\dfrac{dy}{dx} = \dfrac{d}{dx}\sqrt[4]{x} = \dfrac{d}{dx}x^{1/4} = \dfrac{1}{4}x^{(1/4)-1} = \dfrac{1}{4}x^{-3/4}$

5. $\dfrac{dy}{dx} = \dfrac{d}{dx}(2x + 5)^{-1/2} = -\dfrac{1}{2}(2x + 5)^{(-1/2)-1}\dfrac{d}{dx}(2x + 5)$

$$= -\frac{1}{2}(2x + 5)^{-3/2}(2) = -(2x + 5)^{-3/2}$$

6. $\dfrac{dy}{dx} = \dfrac{d}{dx}(1 - 6x)^{2/3}$

$$= \frac{2}{3}(1 - 6x)^{(2/3)-1}\frac{d}{dx}(1 - 6x)$$
$$= \frac{2}{3}(1 - 6x)^{-1/3}(-6)$$
$$= -4(1 - 6x)^{-1/3}$$

7. $\dfrac{dy}{dx} = \dfrac{d}{dx}(x\sqrt{x^2 + 1})$

$$= x\frac{d}{dx}\sqrt{x^2 + 1} + \sqrt{x^2 + 1}\,\frac{d}{dx}(x)$$
$$= x\frac{d}{dx}(x^2 + 1)^{1/2} + (x^2 + 1)^{1/2}$$
$$= x \cdot \frac{1}{2}(x^2 + 1)^{-1/2}(2x) + (x^2 + 1)^{1/2}$$
$$= x^2(x^2 + 1)^{-1/2} + (x^2 + 1)^{1/2}$$

Note: This answer is equivalent to $\dfrac{2x^2 + 1}{\sqrt{x^2 + 1}}$.

8. $$\frac{dy}{dx} = \frac{d}{dx}\frac{x}{\sqrt{x^2+1}} = \frac{(x^2+1)^{1/2}\frac{d}{dx}x - x\frac{d}{dx}(x^2+1)^{1/2}}{x^2+1}$$

$$= \frac{(x^2+1)^{1/2} - x \cdot \frac{1}{2}(x^2+1)^{-1/2}(2x)}{x^2+1}$$

$$= \frac{x^2+1-x^2}{(x^2+1)(x^2+1)^{1/2}}$$

$$= \frac{1}{(x^2+1)^{3/2}}$$

$$= (x^2+1)^{-3/2}$$

9.
$$\begin{aligned}
x^2y + xy^2 &= 6 \\
\frac{d}{dx}(x^2y) + \frac{d}{dx}(xy^2) &= \frac{d}{dx}(6) \\
x^2\frac{dy}{dx} + y(2x) + x(2y)\frac{dy}{dx} + y^2(1) &= 0 \\
x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} &= -(2xy + y^2) \\
(2xy + x^2)\frac{dy}{dx} &= -(2xy + y^2) \\
\frac{dy}{dx} &= -\frac{2xy + y^2}{2xy + x^2}
\end{aligned}$$

10.
$$\begin{aligned}
x^3 + y^3 &= 18xy \\
\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) &= \frac{d}{dx}(18xy) \\
3x^2 + 3y^2\frac{dy}{dx} &= 18x\frac{dy}{dx} + 18y\,(1) \\
3y^2\frac{dy}{dx} - 18x\frac{dy}{dx} &= 18y - 3x^2 \\
(3y^2 - 18x)\frac{dy}{dx} &= 18y - 3x^2 \\
\frac{dy}{dx} &= \frac{18y - 3x^2}{3y^2 - 18x} \\
\frac{dy}{dx} &= \frac{6y - x^2}{y^2 - 6x}
\end{aligned}$$

11.
$$\begin{aligned}
y^2 &= \frac{x-1}{x+1} \\
\frac{d}{dx}y^2 &= \frac{d}{dx}\frac{x-1}{x+1} \\
2y\frac{dy}{dx} &= \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2} \\
2y\frac{dy}{dx} &= \frac{2}{(x+1)^2} \\
\frac{dy}{dx} &= \frac{1}{y(x+1)^2}
\end{aligned}$$

12.
$$\begin{aligned}
x^2 &= \frac{x-y}{x+y} \\
\frac{d}{dx}(x^2) &= \frac{d}{dx}\frac{x-y}{x+y} \\
2x &= \frac{(x+y)\left(1 - \frac{dy}{dx}\right) - (x-y)\left(1 + \frac{dy}{dx}\right)}{(x+y)^2} \\
2x &= \frac{\left[x - x\frac{dy}{dx} + y - y\frac{dy}{dx}\right] - \left[x + x\frac{dy}{dx} - y - y\frac{dy}{dx}\right]}{(x+y)^2} \\
2x &= \frac{2y - 2x\frac{dy}{dx}}{(x+y)^2} \\
x(x+y)^2 &= y - x\frac{dy}{dx} \\
x\frac{dy}{dx} &= y - x(x+y)^2 \\
\frac{dy}{dx} &= \frac{y - x(x+y)^2}{x} = \frac{y}{x} - (x+y)^2
\end{aligned}$$

Alternate solution:

$$\begin{aligned}
x^2 &= \frac{x-y}{x+y} \\
x^2(x+y) &= x - y \\
x^3 + x^2y &= x - y \\
\frac{d}{dx}(x^3) + \frac{d}{dx}(x^2y) &= \frac{d}{dx}(x) - \frac{d}{dx}(y) \\
3x^2 + x^2\frac{dy}{dx} + y(2x) &= 1 - \frac{dy}{dx} \\
(x^2+1)\frac{dy}{dx} &= 1 - 3x^2 - 2xy \\
\frac{dy}{dx} &= \frac{1 - 3x^2 - 2xy}{x^2+1}
\end{aligned}$$

13.
$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(1 - x^{1/2})^{1/2} \\
&= \frac{1}{2}(1 - x^{1/2})^{-1/2}\frac{d}{dx}(1 - x^{1/2}) \\
&= \frac{1}{2}(1 - x^{1/2})^{-1/2}\left(-\frac{1}{2}x^{-1/2}\right) \\
&= -\frac{1}{4}(1 - x^{1/2})^{-1/2}x^{-1/2}
\end{aligned}$$

14.
$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}3(2x^{-1/2} + 1)^{-1/3} \\
&= -(2x^{-1/2} + 1)^{-4/3}\,\frac{d}{dx}(2x^{-1/2} + 1) \\
&= -(2x^{-1/2} + 1)^{-4/3}(-x^{-3/2}) \\
&= x^{-3/2}(2x^{-1/2} + 1)^{-4/3}
\end{aligned}$$

15.
$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}3(\csc x)^{3/2} \\
&= \frac{9}{2}(\csc x)^{1/2}\frac{d}{dx}(\csc x) \\
&= \frac{9}{2}(\csc x)^{1/2}(-\csc x \cot x) \\
&= -\frac{9}{2}(\csc x)^{3/2}\cot x
\end{aligned}$$

16.
$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[\sin(x+5)]^{5/4}\\ &= \frac{5}{4}[\sin(x+5)]^{1/4}\frac{d}{dx}\sin(x+5)\\ &= \frac{5}{4}[\sin(x+5)]^{1/4}\cos(x+5)\end{aligned}$$

17.
$$\begin{aligned}x &= \tan y\\ \frac{d}{dx}(x) &= \frac{d}{dx}(\tan y)\\ 1 &= \sec^2 y\frac{dy}{dx}\\ \frac{dy}{dx} &= \frac{1}{\sec^2 y} = \cos^2 y\end{aligned}$$

18.
$$\begin{aligned}x &= \sin y\\ \frac{d}{dx}(x) &= \frac{d}{dx}(\sin y)\\ 1 &= \cos y\frac{dy}{dx}\\ \frac{dy}{dx} &= \frac{1}{\cos y} = \sec y\end{aligned}$$

19.
$$\begin{aligned}x + \tan xy &= 0\\ \frac{d}{dx}(x) + \frac{d}{dx}(\tan xy) &= \frac{d}{dx}(0)\\ 1 + \sec^2(xy)\frac{d}{dx}(xy) &= 0\\ 1 + (\sec^2 xy)\left[x\frac{dy}{dx} + (y)(1)\right] &= 0\\ (\sec^2 xy)(x)\frac{dy}{dx} &= -1 - (\sec^2 xy)(y)\\ \frac{dy}{dx} &= \frac{-1 - y\sec^2 xy}{x\sec^2 xy}\\ \frac{dy}{dx} &= -\frac{1}{x}\cos^2 xy - \frac{y}{x}\end{aligned}$$

0.
$$\begin{aligned}x + \sin y &= xy\\ \frac{d}{dx}(x) + \frac{d}{dx}(\sin y) &= \frac{d}{dx}(xy)\\ 1 + (\cos y)\frac{dy}{dx} &= x\frac{dy}{dx} + (y)(1)\\ (\cos y - x)\frac{dy}{dx} &= -1 + y\\ \frac{dy}{dx} &= \frac{-1+y}{\cos y - x} = \frac{1-y}{x - \cos y}\end{aligned}$$

(a) If $f(x) = \frac{3}{2}x^{2/3} - 3$, then

$f'(x) = x^{-1/3}$ and $f''(x) = -\frac{1}{3}x^{-4/3}$

which contradicts the given equation $f''(x) = x^{-1/3}$.

) If $f(x) = \frac{9}{10}x^{5/3} - 7$, then

$f'(x) = \frac{3}{2}x^{2/3}$ and $f''(x) = x^{-1/3}$,

which matches the given equation.

(c) Differentiating both sides of the given equation

$f''(x) = x^{-1/3}$ gives $f'''(x) = -\frac{1}{3}x^{-4/3}$, so it *must* be true that $f'''(x) = -\frac{1}{3}x^{-4/3}$.

(d) If $f'(x) = \frac{3}{2}x^{2/3} + 6$, then $f''(x) = x^{-1/3}$, which matches the given equation.

Conclusion: (b), (c), and (d) could be true.

22. (a) If $g'(t) = 4\sqrt[4]{t} - 4$, then

$g''(t) = \frac{d}{dx}(4t^{1/4} - 4) = t^{-3/4} = \frac{1}{t^{3/4}}$, which matches the given equation.

(b) Differentiating both sides of the given equation

$g''(t) = \frac{1}{t^{3/4}} = t^{-3/4}$ gives $g'''(t) = -\frac{3}{4}t^{-7/4}$, which is not consistent with $g'''(t) = -\frac{4}{\sqrt[4]{t}}$.

(c) If $g(t) = t - 7 + \frac{16}{5}t^{5/4}$, then $g'(t) = 1 + 4t^{1/4}$ and $g''(t) = t^{-3/4} = \frac{1}{t^{3/4}}$, which matches the given equation.

(d) If $g'(t) = \frac{1}{4}t^{1/4}$, then $g''(t) = \frac{1}{16}t^{-3/4}$, which contradicts the given equation.

Conclusion: (a) and (c) could be true.

23.
$$\begin{aligned}x^2 + y^2 &= 1\\ \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= \frac{d}{dx}(1)\\ 2x + 2yy' &= 0\\ 2yy' &= -2x\\ y' &= -\frac{x}{y}\\ y'' &= \frac{d}{dx}\left(-\frac{x}{y}\right)\\ &= -\frac{(y)(1) - (x)(y')}{y^2}\\ &= -\frac{y - x\left(-\frac{x}{y}\right)}{y^2}\\ &= -\frac{x^2 + y^2}{y^3}\end{aligned}$$

Since our original equation was $x^2 + y^2 = 1$, we may substitute 1 for $x^2 + y^2$, giving $y'' = -\frac{1}{y^3}$.

24. $x^{2/3} + y^{2/3} = 1$

$$\frac{d}{dx}(x^{2/3}) + \frac{d}{dx}(y^{2/3}) = \frac{d}{dx}(1)$$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = -\frac{x^{-1/3}}{y^{-1/3}} = -\left(\frac{y}{x}\right)^{1/3}$$

$$y'' = \frac{d}{dx}\left[-\left(\frac{y}{x}\right)^{1/3}\right]$$

$$= -\frac{1}{3}\left(\frac{y}{x}\right)^{-2/3}\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$= -\frac{1}{3}\left(\frac{y}{x}\right)^{-2/3}\frac{xy' - (y)(1)}{x^2}$$

$$= -\frac{1}{3}\,\frac{-(x)\left(\frac{y}{x}\right)^{1/3} - y}{x^{4/3}y^{2/3}}$$

$$= \frac{1}{3}\,\frac{x^{2/3}y^{1/3} + y}{x^{4/3}y^{2/3}}$$

$$= \frac{x^{2/3} + y^{2/3}}{3x^{4/3}y^{1/3}}$$

Since our original equation was $x^{2/3} + y^{2/3} = 1$, we may substitute 1 for $x^{2/3} + y^{2/3}$, giving $y'' = \frac{1}{3x^{4/3}y^{1/3}}$.

25. $y^2 = x^2 + 2x$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(2x)$$

$$2yy' = 2x + 2$$

$$y' = \frac{2x+2}{2y} = \frac{x+1}{y}$$

$$y'' = \frac{d}{dx}\left(\frac{x+1}{y}\right)$$

$$= \frac{(y)(1) - (x+1)y'}{y^2}$$

$$= \frac{y - (x+1)\left(\frac{x+1}{y}\right)}{y^2}$$

$$= \frac{y^2 - (x+1)^2}{y^3}$$

Since our original equation was $y^2 = x^2 + 2x$, we may write $y^2 - (x+1)^2 = (x^2 + 2x) - (x^2 + 2x + 1) = -1$, which gives $y'' = -\frac{1}{y^3}$.

26. $y^2 + 2y = 2x + 1$

$$\frac{d}{dx}(y^2 + 2y) = \frac{d}{dx}(2x + 1)$$

$$(2y + 2)y' = 2$$

$$y' = \frac{1}{y+1}$$

$$y'' = \frac{d}{dx}\frac{1}{y+1}$$

$$= -(y+1)^{-2}y'$$

$$= -(y+1)^{-2}\left(\frac{1}{y+1}\right)$$

$$= -\frac{1}{(y+1)^3}$$

27. $x^2 + xy - y^2 = 1$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) - \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$

$$2x + x\frac{dy}{dx} + (y)(1) - 2y\frac{dy}{dx} = 0$$

$$(x - 2y)\frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x - y}{x - 2y} = \frac{2x + y}{2y - x}$$

Slope at (2, 3): $\frac{2(2) + 3}{2(3) - 2} = \frac{7}{4}$

(a) Tangent: $y = \frac{7}{4}(x - 2) + 3$ or $y = \frac{7}{4}x - \frac{1}{2}$

(b) Normal: $y = -\frac{4}{7}(x - 2) + 3$ or $y = -\frac{4}{7}x + \frac{29}{7}$

28. $x^2 + y^2 = 25$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Slope at (3, −4): $-\frac{3}{-4} = \frac{3}{4}$

(a) Tangent: $y = \frac{3}{4}(x - 3) + (-4)$ or $y = \frac{3}{4}x - \frac{25}{4}$

(b) Normal: $y = -\frac{4}{3}(x - 3) + (-4)$ or $y = -\frac{4}{3}x$

29. $x^2y^2 = 9$

$$\frac{d}{dx}(x^2y^2) = \frac{d}{dx}(9)$$

$$(x^2)(2y)\frac{dy}{dx} + (y^2)(2x) = 0$$

$$2x^2y\frac{dy}{dx} = -2xy^2$$

$$\frac{dy}{dx} = -\frac{2xy^2}{2x^2y} = -\frac{y}{x}$$

Slope at (−1, 3): $-\frac{3}{-1} = 3$

(a) Tangent: $y = 3(x + 1) + 3$ or $y = 3x + 6$

(b) Normal: $y = -\frac{1}{3}(x + 1) + 3$ or $y = -\frac{1}{3}x + \frac{8}{3}$

30. $y^2 - 2x - 4y - 1 = 0$

$$\frac{d}{dx}(y^2) - \frac{d}{dx}(2x) - \frac{d}{dx}(4y) - \frac{d}{dx}(1) = \frac{d}{dx}(0)$$

$$2y\frac{dy}{dx} - 2 - 4\frac{dy}{dx} - 0 = 0$$

$$(2y - 4)\frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = \frac{1}{y - 2}$$

Slope at (−2, 1): $\frac{1}{1 - 2} = -1$

(a) Tangent: $y = -(x + 2) + 1$ or $y = -x - 1$

(b) Normal: $y = 1(x + 2) + 1$ or $y = x + 3$

31. $$6x^2 + 3xy + 2y^2 + 17y - 6 = 0$$

$$\frac{d}{dx}(6x^2) + \frac{d}{dx}(3xy) + \frac{d}{dx}(2y^2) + \frac{d}{dx}(17y) - \frac{d}{dx}(6) = \frac{d}{dx}(0)$$

$$12x + 3x\frac{dy}{dx} + (3y)(1) + 4y\frac{dy}{dx} + 17\frac{dy}{dx} - 0 = 0$$

$$3x\frac{dy}{dx} + 4y\frac{dy}{dx} + 17\frac{dy}{dx} = -12x - 3y$$

$$(3x + 4y + 17)\frac{dy}{dx} = -12x - 3y$$

$$\frac{dy}{dx} = \frac{-12x - 3y}{3x + 4y + 17}$$

Slope at $(-1, 0)$: $\dfrac{-12(-1) - 3(0)}{3(-1) + 4(0) + 17} = \dfrac{12}{14} = \dfrac{6}{7}$

(a) Tangent: $y = \frac{6}{7}(x + 1) + 0$ or $y = \frac{6}{7}x + \frac{6}{7}$

(b) Normal: $y = -\frac{7}{6}(x + 1) + 0$ or $y = -\frac{7}{6}x - \frac{7}{6}$

32. $$x^2 - \sqrt{3}xy + 2y^2 = 5$$

$$\frac{d}{dx}(x^2) - \sqrt{3}\frac{d}{dx}(xy) + 2\frac{d}{dx}(y^2) = \frac{d}{dx}(5)$$

$$2x - \sqrt{3}(x)\frac{dy}{dx} - \sqrt{3}(y)(1) + 4y\frac{dy}{dx} = 0$$

$$(-x\sqrt{3} + 4y)\frac{dy}{dx} = y\sqrt{3} - 2x$$

$$\frac{dy}{dx} = \frac{y\sqrt{3} - 2x}{-x\sqrt{3} + 4y}$$

Slope at $(\sqrt{3}, 2)$: $\dfrac{2\sqrt{3} - 2\sqrt{3}}{-\sqrt{3}\sqrt{3} + 4(2)} = 0$

(a) Tangent: $y = 2$

(b) Normal: $x = \sqrt{3}$

33. $$2xy + \pi \sin y = 2\pi$$

$$2\frac{d}{dx}(xy) + \pi\frac{d}{dx}(\sin y) = \frac{d}{dx}(2\pi)$$

$$2x\frac{dy}{dx} + 2y(1) + \pi \cos y\frac{dy}{dx} = 0$$

$$(2x + \pi \cos y)\frac{dy}{dx} = -2y$$

$$\frac{dy}{dx} = -\frac{2y}{2x + \pi \cos y}$$

Slope at $\left(1, \frac{\pi}{2}\right)$: $-\dfrac{2(\pi/2)}{2(1) + \pi\cos(\pi/2)} = -\dfrac{\pi}{2}$

(a) Tangent: $y = -\frac{\pi}{2}(x - 1) + \frac{\pi}{2}$ or $y = -\frac{\pi}{2}x + \pi$

(b) Normal: $y = \frac{2}{\pi}(x - 1) + \frac{\pi}{2}$ or $y = \frac{2}{\pi}x - \frac{2}{\pi} + \frac{\pi}{2}$

34. $x \sin 2y = y \cos 2x$

$$\frac{d}{dx}(x \sin 2y) = \frac{d}{dx}(y \cos 2x)$$

$$(x)(\cos 2y)(2)\frac{dy}{dx} + (\sin 2y)(1) =$$

$$(y)(-\sin 2x)(2) + (\cos 2x)\left(\frac{dy}{dx}\right)$$

$$(2x \cos 2y)\frac{dy}{dx} - (\cos 2x)\frac{dy}{dx} = -2y \sin 2x - \sin 2y$$

$$\frac{dy}{dx} = -\frac{2y \sin 2x + \sin 2y}{2x \cos 2y - \cos 2x}$$

Slope at $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$: $-\dfrac{2\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) + \sin(\pi)}{2\left(\frac{\pi}{4}\right)\cos(\pi) - \cos\left(\frac{\pi}{2}\right)}$

$$= -\frac{(\pi)(1) + 0}{\left(\frac{\pi}{2}\right)(-1) - 0} = 2$$

(a) Tangent: $y = 2\left(x - \frac{\pi}{4}\right) + \frac{\pi}{2}$ or $y = 2x$

(b) Normal: $y = -\frac{1}{2}\left(x - \frac{\pi}{4}\right) + \frac{\pi}{2}$ or $y = -\frac{1}{2}x + \frac{5\pi}{8}$

35. $$y = 2 \sin(\pi x - y)$$

$$\frac{dy}{dx} = \frac{d}{dx}2 \sin(\pi x - y)$$

$$\frac{dy}{dx} = 2 \cos(\pi x - y)\left(\pi - \frac{dy}{dx}\right)$$

$$[1 + 2 \cos(\pi x - y)]\frac{dy}{dx} = 2\pi \cos(\pi x - y)$$

$$\frac{dy}{dx} = \frac{2\pi \cos(\pi x - y)}{1 + 2 \cos(\pi x - y)}$$

Slope at $(1, 0)$: $\dfrac{2\pi \cos \pi}{1 + 2 \cos \pi} = \dfrac{2\pi(-1)}{1 + 2(-1)} = 2\pi$

(a) Tangent: $y = 2\pi(x - 1) + 0$ or $y = 2\pi x - 2\pi$

(b) Normal: $y = -\frac{1}{2\pi}(x - 1) + 0$ or $y = -\frac{x}{2\pi} + \frac{1}{2\pi}$

36. $x^2 \cos^2 y - \sin y = 0$

$$\frac{d}{dx}(x^2 \cos^2 y) - \frac{d}{dx}(\sin y) = \frac{d}{dx}(0)$$

$$(x^2)(2 \cos y)(-\sin y)\left(\frac{dy}{dx}\right) + (\cos^2 y)(2x) - (\cos y)\frac{dy}{dx} = 0$$

$$-(2x^2 \cos y \sin y + \cos y)\frac{dy}{dx} = -2x \cos^2 y$$

$$\frac{dy}{dx} = \frac{2x \cos^2 y}{\cos y + 2x^2 \cos y \sin y} = \frac{2x \cos y}{1 + 2x^2 \sin y}$$

Slope at $(0, \pi)$: $\dfrac{2(0) \cos \pi}{1 + 2(0)^2 \sin \pi} = 0$

(a) Tangent: $y = \pi$

(b) Normal: $x = 0$

37. (a)

$$y^4 = y^2 - x^2$$

$$\frac{d}{dx}(y^4) = \frac{d}{dx}(y^2) - \frac{d}{dx}x^2$$

$$4y^3\frac{dy}{dx} = 2y\frac{dy}{dx} - 2x$$

$$(4y^3 - 2y)\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{4y^3 - 2y} = \frac{x}{y - 2y^3}$$

At $\left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)$:

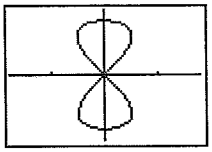
$$\text{Slope } = \frac{\frac{\sqrt{3}}{4}}{\frac{\sqrt{3}}{2} - 2\left(\frac{\sqrt{3}}{2}\right)^3}$$

$$= \frac{\frac{\sqrt{3}}{4}}{\frac{\sqrt{3}}{2} - \frac{3\sqrt{3}}{4}} \cdot \frac{\frac{4}{\sqrt{3}}}{\frac{4}{\sqrt{3}}} = \frac{1}{2-3} = -1$$

At $\left(\frac{\sqrt{3}}{4}, \frac{1}{2}\right)$:

$$\text{Slope} = \frac{\frac{\sqrt{3}}{4}}{\frac{1}{2} - 2\left(\frac{1}{2}\right)^3} = \frac{\frac{\sqrt{3}}{4}}{\frac{1}{2} - \frac{1}{4}} \cdot \frac{4}{4} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

(b)



[−1.8, 1.8] by [−1.2, 1.2]

Parameter interval: $-1 \le t \le 1$

38. (a)

$$y^2(2 - x) = x^3$$

$$\frac{d}{dx}[y^2(2 - x)] = \frac{d}{dx}(x^3)$$

$$(y^2)(-1) + (2 - x)(2y)\frac{dy}{dx} = 3x^2$$

$$2y(2 - x)\frac{dy}{dx} = 3x^2 + y^2$$

$$\frac{dy}{dx} = \frac{3x^2 + y^2}{2y(2 - x)}$$

Slope at (1, 1): $\frac{3(1)^2 + (1)^2}{2(1)(2 - 1)} = \frac{4}{2} = 2$

Tangent: $y = 2(x - 1) + 1$ or $y = 2x - 1$

Normal: $y = -\frac{1}{2}(x - 1) + 1$ or $y = -\frac{1}{2}x + \frac{3}{2}$

(b) One way is to graph the equations $y = \pm\sqrt{\frac{x^3}{2 - x}}$.

39. (a) $(-1)^3(1)^2 = \cos(\pi)$ is true since both sides equal -1.

(b)

$$x^3y^2 = \cos(\pi y)$$

$$\frac{d}{dx}(x^3y^2) = \frac{d}{dx}\cos(\pi y)$$

$$(x^3)(2y)\frac{dy}{dx} + (y^2)(3x^2) = (-\sin \pi y)(\pi)\frac{dy}{dx}$$

$$(2x^3y + \pi \sin \pi y)\frac{dy}{dx} = -3x^2y^2$$

$$\frac{dy}{dx} = -\frac{3x^2y^2}{2x^3y + \pi \sin \pi y}.$$

Slope at $(-1, 1)$: $-\frac{3(-1)^2(1)}{2(-1)^3(1) + \pi \sin \pi} = \frac{-3}{-2} = \frac{3}{2}$

The slope of the tangent line is $\frac{3}{2}$.

40. (a) When $x = 2$, we have $y^3 - 2y = -1$, or $y^3 - 2y + 1 = 0$. Clearly, $y = 1$ is one solution, and we may factor $y^3 - 2y + 1$ as $(y - 1)(y^2 + y - 1)$. The solutions of $y^2 + y - 1 = 0$ are $y = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}$. Hence, there are three possible y-values: 1, $\frac{-1 - \sqrt{5}}{2}$, and $\frac{-1 + \sqrt{5}}{2}$.

(b)

$$y^3 - xy = -1$$

$$\frac{d}{dx}(y^3) - \frac{d}{dx}(xy) = \frac{d}{dx}(-1)$$

$$3y^2y' - xy' - (y)(1) = 0$$

$$(3y^2 - x)y' = y$$

$$y' = \frac{y}{3y^2 - x}$$

$$y'' = \frac{d}{dx}\frac{y}{3y^2 - x}$$

$$= \frac{(3y^2 - x)(y') - (y)(6y\,y' - 1)}{(3y^2 - x)^2}$$

$$= \frac{y - xy' - 3y^2y'}{(3y^2 - x)^2}$$

Since we are working with numerical information, there is no need to write a general expression for y'' in terms of x and y.

To evaluate $f'(2)$, evaluate the expression for y' using $x = 2$ and $y = 1$:

$$f'(2) = \frac{1}{3(1)^2 - 2} = \frac{1}{1} = 1$$

To evaluate $f''(2)$, evaluate the expression for y'' using $x = 2$, $y = 1$, and $y' = 1$:

$$f''(2) = \frac{(1) - 2(1) - 3(1)^2(1)}{[3(1)^2 - 2]^2} = \frac{-4}{1} = -4$$

41. Find the two points:
The curve crosses the x-axis when $y = 0$, so the equation becomes $x^2 + 0x + 0 = 7$, or $x^2 = 7$. The solutions are $x = \pm\sqrt{7}$, so the points are $(\pm\sqrt{7}, 0)$.
Show tangents are parallel:

$$\begin{aligned} x^2 + xy + y^2 &= 7 \\ \frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) &= \frac{d}{dx}(7) \\ 2x + x\frac{dy}{dx} + (y)(1) + 2y\frac{dy}{dx} &= 0 \\ (x + 2y)\frac{dy}{dx} &= -(2x + y) \\ \frac{dy}{dx} &= -\frac{2x + y}{x + 2y} \end{aligned}$$

Slope at $(\sqrt{7}, 0)$: $-\dfrac{2\sqrt{7} + 0}{\sqrt{7} + 2(0)} = -2$

Slope at $(-\sqrt{7}, 0)$: $-\dfrac{2(-\sqrt{7}) + 0}{-\sqrt{7} + 2(0)} = -2$

The tangents at these points are parallel because they have the same slope. The common slope is -2.

42.

$$\begin{aligned} x^2 + xy + y^2 &= 7 \\ \frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) &= \frac{d}{dx}(7) \\ 2x + x\frac{dy}{dx} + (y)(1) + 2y\frac{dy}{dx} &= 0 \\ (x + 2y)\frac{dy}{dx} &= -(2x + y) \\ \frac{dy}{dx} &= -\frac{2x + y}{x + 2y} \end{aligned}$$

(a) The tangent is parallel to the x-axis when
$\dfrac{dy}{dx} = -\dfrac{2x + y}{x + 2y} = 0$, or $y = -2x$.
Substituting $-2x$ for y in the original equation, we have

$$\begin{aligned} x^2 + xy + y^2 &= 7 \\ x^2 + (x)(-2x) + (-2x)^2 &= 7 \\ x^2 - 2x^2 + 4x^2 &= 7 \\ 3x^2 &= 7 \\ x &= \pm\sqrt{\frac{7}{3}} \end{aligned}$$

The points are $\left(-\sqrt{\frac{7}{3}}, 2\sqrt{\frac{7}{3}}\right)$ and $\left(\sqrt{\frac{7}{3}}, -2\sqrt{\frac{7}{3}}\right)$.

(b) Since x and y are interchangeable in the original equation, $\dfrac{dx}{dy}$ can be obtained by interchanging x and y in the expression for $\dfrac{dy}{dx}$. That is, $\dfrac{dx}{dy} = -\dfrac{2y + x}{x + 2y}$. The tangent is parallel to the y-axis when $\dfrac{dx}{dy} = 0$, or $x = -2y$. Substituting $-2y$ for x in the original equation, we have:

$$\begin{aligned} x^2 + xy + y^2 &= 7 \\ (-2y)^2 + (-2y)(y) + y^2 &= 7 \\ 4y^2 - 2y^2 + y^2 &= 7 \\ 3y^2 &= 7 \\ y &= \pm\sqrt{\frac{7}{3}} \end{aligned}$$

The points are $\left(-2\sqrt{\frac{7}{3}}, \sqrt{\frac{7}{3}}\right)$ and $\left(2\sqrt{\frac{7}{3}}, -\sqrt{\frac{7}{3}}\right)$.
Note that these are the same points that would be obtained by interchanging x and y in the solution to part (a).

43. First curve:

$$\begin{aligned} 2x^2 + 3y^2 &= 5 \\ \frac{d}{dx}(2x^2) + \frac{d}{dx}(3y^2) &= \frac{d}{dx}(5) \\ 4x + 6y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{4x}{6y} = -\frac{2x}{3y} \end{aligned}$$

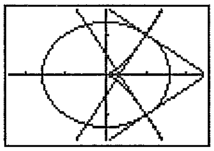
Second curve:

$$\begin{aligned} y^2 &= x^3 \\ \frac{d}{dx}y^2 &= \frac{d}{dx}x^3 \\ 2y\frac{dy}{dx} &= 3x^2 \\ \frac{dy}{dx} &= \frac{3x^2}{2y} \end{aligned}$$

At $(1, 1)$, the slopes are $-\dfrac{2}{3}$ and $\dfrac{3}{2}$ respectively. At $(1, -1)$, the slopes are $\dfrac{2}{3}$ and $-\dfrac{3}{2}$ respectively. In both cases, the tangents are perpendicular. To graph the curves and normal lines, we may use the following parametric equations for $-\pi \le t \le \pi$:

First curve: $x = \sqrt{\frac{5}{2}}\cos t$, $y = \sqrt{\frac{5}{3}}\sin t$
Second curve: $x = \sqrt[3]{t^2}$, $y = t$

Tangents at $(1, 1)$: $x = 1 + 3t$, $y = 1 - 2t$
$x = 1 + 2t$, $y = 1 + 3t$
Tangents at $(1, -1)$: $x = 1 + 3t$, $y = -1 + 2t$
$x = 1 + 2t$, $y = -1 - 3t$



[−2.4, 2.4] by [−1.6, 1.6]

44. $v(t) = s'(t) = \frac{d}{dt}(4 + 6t)^{3/2} = \frac{3}{2}(4 + 6t)^{1/2}(6)$

$= 9(4 + 6t)^{1/2}$

$a(t) = v'(t) = \frac{d}{dt}[9(4 + 6t)^{1/2}]$

$= \frac{9}{2}(4 + 6t)^{-1/2}(6) = 27(4 + 6t)^{-1/2}$

At $t = 2$, the velocity is $v(2) = 36$ m/sec and the acceleration is $a(2) = \frac{27}{4}$ m/sec^2.

45. $$\begin{aligned}\text{Acceleration} = \frac{dv}{dt} &= \frac{d}{dt}[8(s - t)^{1/2} + 1]\\ &= 4(s - t)^{-1/2}\left(\frac{ds}{dt} - 1\right)\\ &= 4(s - t)^{-1/2}(v - 1)\\ &= 4(s - t)^{-1/2}[(8(s - t)^{1/2} + 1) - 1]\\ &= 32(s - t)^{-1/2}(s - t)^{1/2}\\ &= 32 \text{ ft/sec}^2\end{aligned}$$

46. $$y^4 - 4y^2 = x^4 - 9x^2$$

$$\frac{d}{dx}(y^4) - \frac{d}{dx}(4y^2) = \frac{d}{dx}(x^4) - \frac{d}{dx}(9x^2)$$

$$4y^3\frac{dy}{dx} - 8y\frac{dy}{dx} = 4x^3 - 18x$$

$$\frac{dy}{dx} = \frac{4x^3 - 18x}{4y^3 - 8y} = \frac{2x^3 - 9x}{2y^3 - 4y}$$

Slope at (3, 2): $\frac{2(3)^3 - 9(3)}{2(2)^3 - 4(2)} = \frac{27}{8}$

Slope at (−3, 2): $\frac{2(-3)^3 - 9(-3)}{2(2)^3 - 4(2)} = -\frac{27}{8}$

Slope at (−3, −2): $\frac{2(-3)^3 - 9(-3)}{2(-2)^3 - 4(-2)} = \frac{27}{8}$

Slope at (3, −2): $\frac{2(3)^3 - 9(3)}{2(-2)^3 - 4(-2)} = -\frac{27}{8}$

47. (a) $$x^3 + y^3 - 9xy = 0$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - 9\frac{d}{dx}(xy) = \frac{d}{dx}(0)$$

$$3x^2 + 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} - 9(y)(1) = 0$$

$$(3y^2 - 9x)\frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}$$

Slope at (4, 2): $\frac{3(2) - (4)^2}{(2)^2 - 3(4)} = \frac{-10}{-8} = \frac{5}{4}$

Slope at (2, 4): $\frac{3(4) - (2)^2}{(4)^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}$

(b) The tangent is horizontal when

$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x} = 0$, or $y = \frac{x^2}{3}$.

Substituting $\frac{x^2}{3}$ for y in the original equation, we have:

$$x^3 + y^3 - 9xy = 0$$

$$x^3 + \left(\frac{x^2}{3}\right)^3 - 9x\left(\frac{x^2}{3}\right) = 0$$

$$x^3 + \frac{x^6}{27} - 3x^3 = 0$$

$$\frac{x^3}{27}(x^3 - 54) = 0$$

$$x = 0 \text{ or } x = \sqrt[3]{54} = 3\sqrt[3]{2}$$

At $x = 0$, we have $y = \frac{0^2}{3} = 0$, which gives the point (0, 0), which is the origin. At $x = 3\sqrt[3]{2}$, we have $y = \frac{1}{3}(3\sqrt[3]{2})^2 = \frac{1}{3}(9\sqrt[3]{4}) = 3\sqrt[3]{4}$, so the point other than the origin is $(3\sqrt[3]{2}, 3\sqrt[3]{4})$ or approximately (3.780, 4.762).

(c) The equation $x^3 + y^3 - 9xy$ is not affected by interchanging x and y, so its graph is symmetric about the line $y = x$ and we may find the desired point by interchanging the x-value and the y-value in the answer to part (b). The desired point is $(3\sqrt[3]{4}, 3\sqrt[3]{2})$ or approximately (4.762, 3.780).

48. $$x^2 + 2xy - 3y^2 = 0$$

$$\frac{d}{dx}(x^2) + 2\frac{d}{dx}(xy) - \frac{d}{dx}(3y^2) = \frac{d}{dx}(0)$$

$$2x + 2x\frac{dy}{dx} + 2(y)(1) - 6y\frac{dy}{dx} = 0$$

$$(2x - 6y)\frac{dy}{dx} = -2x - 2y$$

$$\frac{dy}{dx} = \frac{-2x - 2y}{2x - 6y} = \frac{x + y}{3y - x}$$

At (1, 1) the curve has slope $\frac{1 + 1}{3(1) - 1} = \frac{2}{2} = 1$, so the normal line is $y = -1(x - 1) + 1$ or $y = -x + 2$.

Substituting $-x + 2$ for y in the original equation, we have:

$$\begin{aligned}x^2 + 2xy - 3y^2 &= 0\\ x^2 + 2x(-x + 2) - 3(-x + 2)^2 &= 0\\ x^2 - 2x^2 + 4x - 3(x^2 - 4x + 4) &= 0\\ -4x^2 + 16x - 12 &= 0\\ -4(x - 1)(x - 3) &= 0\\ x = 1 \text{ or } x &= 3\end{aligned}$$

Since the given point (1, 1) had $x = 1$, we choose $x = 3$ and so $y = -(3) + 2 = -1$. The desired point is (3, −1).

49.
$$xy + 2x - y = 0$$
$$\frac{d}{dx}(xy) + \frac{d}{dx}(2x) - \frac{d}{dx}(y) = \frac{d}{dx}(0)$$
$$x\frac{dy}{dx} + (y)(1) + 2 - \frac{dy}{dx} = 0$$
$$(x - 1)\frac{dy}{dx} = -2 - y$$
$$\frac{dy}{dx} = \frac{-2 - y}{x - 1} = \frac{2 + y}{1 - x}$$

Since the slope of the line $2x + y = 0$ is -2, we wish to find points where the normal has slope -2, that is, where the tangent has slope $\frac{1}{2}$. Thus, we have

$$\frac{2 + y}{1 - x} = \frac{1}{2}$$
$$2(2 + y) = 1 - x$$
$$4 + 2y = 1 - x$$
$$x = -2y - 3$$

Substituting $-2y - 3$ in the original equation, we have:

$$xy + 2x - y = 0$$
$$(-2y - 3)y + 2(-2y - 3) - y = 0$$
$$-2y^2 - 8y - 6 = 0$$
$$-2(y + 1)(y + 3) = 0$$
$$y = -1 \text{ or } y = -3$$

At $y = -1$, $x = -2y - 3 = 2 - 3 = -1$.
At $y = -3$: $x = -2y - 3 = 6 - 3 = 3$.
The desired points are $(-1, -1)$ and $(3, -3)$.
Finally, we find the desired normals to the curve, which are the lines of slope -2 passing through each of these points. At $(-1, -1)$, the normal line is $y = -2(x + 1) - 1$ or $y = -2x - 3$. At $(3, -3)$, the normal line is $y = -2(x - 3) - 3$ or $y = -2x + 3$.

50.
$$x = y^2$$
$$\frac{d}{dx}(x) = \frac{d}{dx}(y^2)$$
$$1 = 2y\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{2y}$$

The normal line at (x, y) has slope $-2y$. Thus, the normal line at (b^2, b) is $y = -2b(x - b^2) + b$, or $y = -2bx + 2b^3 + b$. This line intersects the x-axis at $x = \frac{2b^3 + b}{2b} = b^2 + \frac{1}{2}$, which is the value of a and must be greater than $\frac{1}{2}$ if $b \neq 0$.

The two normals at $(b^2, \pm b)$ will be perpendicular when they have slopes ± 1, which gives $-2y = \pm 1$ or $y = \pm\frac{1}{2}\left(\text{or } b = \pm\frac{1}{2}\right)$. The corresponding value of a is $b^2 + \frac{1}{2} = \left(\frac{1}{2}\right)^2 + \frac{1}{2} = \frac{3}{4}$. Thus, the two nonhorizontal normals are perpendicular when $a = \frac{3}{4}$.

51. (a)
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$b^2x^2 + a^2y^2 = a^2b^2$$
$$\frac{d}{dx}(b^2x^2) + \frac{d}{dx}(a^2y^2) = \frac{d}{dx}(a^2b^2)$$
$$2b^2x + 2a^2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{2b^2x}{2a^2y} = -\frac{b^2x}{a^2y}$$

The slope at (x_1, y_1) is $-\frac{b^2x_1}{a^2y_1}$.

The tangent line is $y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$. This gives:

$$a^2y_1y - a^2{y_1}^2 = -b^2x_1x + b^2{x_1}^2$$

$$a^2y_1y + b^2x_1x = a^2{y_1}^2 + b^2{x_1}^2.$$

But $a^2{y_1}^2 + b^2{x_1}^2 = a^2b^2$ since (x_1, y_1) is on the ellipse. Therefore, $a^2y_1y + b^2x_1x = a^2b^2$, and dividing by a^2b^2 gives $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$.

(b)
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$b^2x^2 - a^2y^2 = a^2b^2$$
$$\frac{d}{dx}(b^2x^2) - \frac{d}{dx}(a^2y^2) = \frac{d}{dx}(a^2b^2)$$
$$2b^2x - 2a^2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-2b^2x}{-2a^2y} = \frac{b^2x}{a^2y}$$

The slope at (x_1, y_1) is $\frac{b^2x_1}{a^2y_1}$.

The tangent line is $y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$.

This gives:

$$a^2y_1y - a^2{y_1}^2 = b^2x_1x - b^2{x_1}^2$$

$$b^2{x_1}^2 - a^2{y_1}^2 = b^2x_1x - a^2y_1y$$

But $b^2{x_1}^2 - a^2{y_1}^2 = a^2b^2$ since (x_1, y_1) is on the hyperbola. Therefore, $b^2x_1x - a^2y_1y = a^2b^2$, and dividing by a^2b^2 gives $\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1$.

52. (a) Solve for y:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$-\frac{y^2}{b^2} = -\frac{x^2}{a^2} + 1$$
$$y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$
$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$$

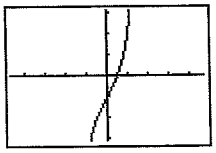
(b) $$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{\frac{b}{a}\sqrt{x^2-a^2}}{\frac{b}{a}|x|} = \lim_{x\to\infty}\frac{\sqrt{x^2-a^2}}{\sqrt{x^2}} = \lim_{x\to\infty}\sqrt{1-\frac{a^2}{x^2}} = 1$$

(c) $$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{-\frac{b}{a}\sqrt{x^2-a^2}}{-\frac{b}{a}|x|} = \lim_{x\to\infty}\frac{\sqrt{x^2-a^2}}{\sqrt{x^2}} = \lim_{x\to\infty}\sqrt{1-\frac{a^2}{x^2}} = 1$$

■ Section 3.8 Derivatives of Inverse Trigonometric Functions (pp. 157–163)

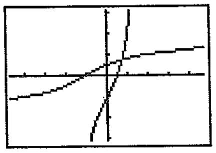
Exploration 1 Finding a Derivative on an Inverse Graph Geometrically

1. The graph is shown at the right. It appears to be a one-to-one function



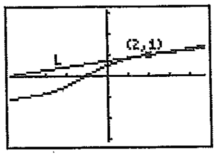
[−4.7, 4.7] by [−3.1, 3.1]

2. $f'(x) = 5x^4 + 2$. The fact that this function is always positive enables us to conclude that f is everywhere increasing, and hence one-to-one.
3. The graph of f^{-1} is shown to the right, along with the graph of f. The graph of f^{-1} is obtained from the graph of f by reflecting it in the line $y = x$.



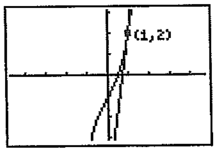
[−4.7, 4.7] by [−3.1, 3.1]

4. The line L is tangent to the graph of f^{-1} at the point (2, 1).



[−4.7, 4.7] by [−3.1, 3.1]

5. The reflection of line L is tangent to the graph of f at the point (1, 2).



[−4.7, 4.7] by [−3.1, 3.1]

6. The reflection of line L is the tangent line to the graph of $y = x^5 + 2x - 1$ at the point (1, 2). The slope is $\frac{dy}{dx}$ at $x = 1$, which is 7.
7. The slope of L is the reciprocal of the slope of its reflection $\left(\text{since } \frac{\Delta y}{\Delta x} \text{ gets reflected to become } \frac{\Delta x}{\Delta y}\right)$. It is $\frac{1}{7}$.
8. $\frac{1}{7}$

Quick Review 3.8

1. Domain: $[-1, 1]$
   Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
   At 1: $\frac{\pi}{2}$
2. Domain: $[-1, 1]$
   Range: $[0, \pi]$
   At 1: 0
3. Domain: all reals
   Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
   At 1: $\frac{\pi}{4}$
4. Domain: $(-\infty, -1] \cup [1, \infty)$
   Range: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
   At 1: 0
5. Domain: all reals
   Range: all reals
   At 1: 1
6. $$f(x) = y = 3x - 8$$
   $$y + 8 = 3x$$
   $$x = \frac{y+8}{3}$$
   Interchange x and y:
   $$y = \frac{x+8}{3}$$
   $$f^{-1}(x) = \frac{x+8}{3}$$
7. $$f(x) = y = \sqrt[3]{x+5}$$
   $$y^3 = x + 5$$
   $$x = y^3 - 5$$
   Interchange x and y:
   $$y = x^3 - 5$$
   $$f^{-1}(x) = x^3 - 5$$

8. $f(x) = y = \frac{8}{x}$

$$x = \frac{8}{y}$$

Interchange x and y:

$$y = \frac{8}{x}$$

$$f^{-1}(x) = \frac{8}{x}$$

9. $f(x) = y = \frac{3x-2}{x}$

$$xy = 3x - 2$$

$$(y-3)x = -2$$

$$x = \frac{-2}{y-3} = \frac{2}{3-y}$$

Interchange x and y:

$$y = \frac{2}{3-x}$$

$$f^{-1}(x) = \frac{2}{3-x}$$

10. $f(x) = y = \arctan \frac{x}{3}$

$$\tan y = \frac{x}{3},\ -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$x = 3\tan y,\ -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Interchange x and y:

$$y = 3\tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$f^{-1}(x) = 3\tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Section 3.8 Exercises

1. $\frac{dy}{dx} = \frac{d}{dx}\cos^{-1}(x^2) = -\frac{1}{\sqrt{1-(x^2)^2}}\frac{d}{dx}(x^2)$

$$= -\frac{1}{\sqrt{1-x^4}}(2x) = -\frac{2x}{\sqrt{1-x^4}}$$

2. $\frac{dy}{dx} = \frac{d}{dx}\cos^{-1}\left(\frac{1}{x}\right) = -\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\frac{d}{dx}\left(\frac{1}{x}\right)$

$$= -\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\left(-\frac{1}{x^2}\right) = \frac{1}{|x|\sqrt{x^2-1}}$$

[redacted]

3. $\frac{dy}{dt} = \frac{d}{dt}\sin^{-1}\sqrt{2}t = \frac{1}{\sqrt{1-(\sqrt{2}\,t)^2}}\frac{d}{dt}(\sqrt{2}\,t) = \frac{\sqrt{2}}{\sqrt{1-2t^2}}$

4. $\frac{dy}{dt} = \frac{d}{dt}\sin^{-1}(1-t) = \frac{1}{\sqrt{1-(1-t)^2}}\frac{d}{dt}(1-t)$

$$= -\frac{1}{\sqrt{2t-t^2}}$$

5. $\frac{dy}{ds} = \frac{d}{ds}\sec^{-1}(2s+1)$

$$= \frac{1}{|2s+1|\sqrt{(2s+1)^2-1}}\frac{d}{ds}(2s+1)$$

$$= \frac{1}{|2s+1|\sqrt{4s^2+4s}}(2) = \frac{1}{|2s+1|\sqrt{s^2+s}}$$

6. $\frac{dy}{ds} = \frac{d}{ds}\sec^{-1}5s = \frac{1}{|5s|\sqrt{(5s)^2-1}}\frac{d}{ds}(5s) = \frac{1}{|s|\sqrt{25s^2-1}}$

7. $\frac{dy}{dx} = \frac{d}{dx}\csc^{-1}(x^2+1)$

$$= -\frac{1}{|x^2+1|\sqrt{(x^2+1)^2-1}}\frac{d}{dx}(x^2+1)$$

$$= -\frac{2x}{(x^2+1)\sqrt{x^4+2x^2}} = -\frac{2}{(x^2+1)\sqrt{x^2+2}}$$

Note that the condition $x > 0$ is required in the last step.

8. $\frac{dy}{dx} = \frac{d}{dx}\csc^{-1}\left(\frac{x}{2}\right) = -\frac{1}{\left|\frac{x}{2}\right|\sqrt{\left(\frac{x}{2}\right)^2-1}}\frac{d}{dx}\left(\frac{x}{2}\right)$

$$= -\frac{2}{|x|\sqrt{x^2-4}}$$

9. $\frac{dy}{dt} = \frac{d}{dt}\sec^{-1}\left(\frac{1}{t}\right) = \frac{1}{\left|\frac{1}{t}\right|\sqrt{\left(\frac{1}{t}\right)^2-1}}\frac{d}{dt}\left(\frac{1}{t}\right)$

$$= \frac{1}{\left|\frac{1}{t}\right|\sqrt{\left(\frac{1}{t}\right)^2-1}}\left(-\frac{1}{t^2}\right) = -\frac{1}{\sqrt{1-t^2}}$$

Note that the condition $t > 0$ is required in the last step.

10. $\frac{dy}{dt} = \frac{d}{dt}\sin^{-1}\left(\frac{3}{t^2}\right) = \frac{1}{\sqrt{1-\left(\frac{3}{t^2}\right)^2}}\frac{d}{dt}\left(\frac{3}{t^2}\right)$

$$= \frac{1}{\sqrt{1-\frac{9}{t^4}}}\left(-\frac{6}{t^3}\right) = -\frac{6}{t\sqrt{t^4-9}}$$

11. $\frac{dy}{dt} = \frac{d}{dt}\cot^{-1}\sqrt{t} = -\frac{1}{1+(\sqrt{t})^2}\frac{d}{dt}\sqrt{t}$

$$= -\frac{1}{2\sqrt{t}(t+1)}$$

12. $\frac{dy}{dt} = \frac{d}{dt}\cot^{-1}\sqrt{t-1} = -\frac{1}{1+(\sqrt{t-1})^2}\frac{d}{dt}\sqrt{t-1}$

$$= -\left(\frac{1}{1+t-1}\right)\left(\frac{1}{2\sqrt{t-1}}\right) = -\frac{1}{2t\sqrt{t-1}}$$

13. $\frac{dy}{ds} = \frac{d}{ds}(s\sqrt{1-s^2}) + \frac{d}{ds}(\cos^{-1}s)$

$= (s)\left(\frac{1}{2\sqrt{1-s^2}}\right)(-2s) + (\sqrt{1-s^2})(1) - \frac{1}{\sqrt{1-s^2}}$

$= -\frac{s^2}{\sqrt{1-s^2}} + \sqrt{1-s^2} - \frac{1}{\sqrt{1-s^2}}$

$= \frac{-s^2 + (1-s^2) - 1}{\sqrt{1-s^2}}$

$= -\frac{2s^2}{\sqrt{1-s^2}}$

14. $\frac{dy}{ds} = \frac{d}{ds}\sqrt{s^2-1} - \frac{d}{ds}\sec^{-1}s$

$= \frac{1}{2\sqrt{s^2-1}}(2s) - \frac{1}{|s|\sqrt{s^2-1}}$

$= \frac{s|s|-1}{|s|\sqrt{s^2-1}}$

15. $\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}\sqrt{x^2-1}) + \frac{d}{dx}(\csc^{-1}x)$

$= \frac{1}{1+(\sqrt{x^2-1})^2}\frac{d}{dx}(\sqrt{x^2-1}) - \frac{1}{|x|\sqrt{x^2-1}}$

$= \frac{1}{x^2}\frac{1}{2\sqrt{x^2-1}}(2x) - \frac{1}{|x|\sqrt{x^2-1}}$

$= \frac{1}{x\sqrt{x^2-1}} - \frac{1}{|x|\sqrt{x^2-1}}$

$= 0$

Note that the condition $x > 1$ is required in the last step.

16. $\frac{dy}{dx} = \frac{d}{dx}\left(\cot^{-1}\frac{1}{x}\right) - \frac{d}{dx}(\tan^{-1}x)$

$= -\frac{1}{1+\left(\frac{1}{x^2}\right)}\frac{d}{dx}\left(\frac{1}{x}\right) - \frac{1}{1+x^2}$

$= \left(-\frac{1}{1+\frac{1}{x^2}}\right)\left(-\frac{1}{x^2}\right) - \frac{1}{1+x^2}$

$= \frac{1}{x^2+1} - \frac{1}{1+x^2}$

$= 0, x \neq 0$

The condition $x \neq 0$ is required because the original function was undefined when $x \neq 0$.

17. $\frac{dy}{dx} = \frac{d}{dx}(x\sin^{-1}x) + \frac{d}{dx}(\sqrt{1-x^2})$

$= (x)\left(\frac{1}{\sqrt{1-x^2}}\right) + (\sin^{-1}x)(1) + \frac{1}{2\sqrt{1-x^2}}(-2x)$

$= \sin^{-1}x$

18. $\frac{dy}{dx} = \frac{d}{dx}[\sin^{-1}(2x)]^{-1}$

$= -[\sin^{-1}(2x)]^{-2}\frac{d}{dx}\sin^{-1}(2x)$

$= -[\sin^{-1}(2x)]^{-2}\frac{1}{\sqrt{1-4x^2}}(2)$

$= -\frac{2}{[\sin^{-1}(2x)]^2\sqrt{1-4x^2}}$

19. **(a)** Since $\frac{dy}{dx} = \sec^2 x$, the slope at $\left(\frac{\pi}{4}, 1\right)$ is $\sec^2\left(\frac{\pi}{4}\right) = 2$. The tangent line is given by $y = 2\left(x - \frac{\pi}{4}\right) + 1$, or

$y = 2x - \frac{\pi}{2} + 1$.

(b) Since $\frac{dy}{dx} = \frac{1}{1+x^2}$, the slope at $\left(1, \frac{\pi}{4}\right)$ is $\frac{1}{1+1^2} = \frac{1}{2}$. The tangent line is given by $y = \frac{1}{2}(x-1) + \frac{\pi}{4}$, or

$y = \frac{1}{2}x - \frac{1}{2} + \frac{\pi}{4}$.

20. **(a)** Note that $f'(x) = 5x^4 + 6x^2 + 1$. Thus $f(1) = 3$ and $f'(1) = 12$.

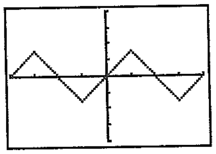
(b) Since the graph of $y = f(x)$ includes the point (1, 3) and the slope of the graph is 12 at this point, the graph of $y = f^{-1}(x)$ will include (3, 1) and the slope will be $\frac{1}{12}$. Thus, $f^{-1}(3) = 1$ and $(f^{-1})'(3) = \frac{1}{12}$. (We have assumed that $f^{-1}(x)$ is defined and differentiable at $x = 3$. This is true by Theorem 3, because $f'(x) = 5x^4 + 6x^2 + 1$, which is never zero.)

21. **(a)** Note that $f'(x) = -\sin x + 3$, which is always between 2 and 4. Thus f is differentiable at every point on the interval$(-\infty, \infty)$ and $f'(x)$ is never zero on this interval, so f has a differentiable inverse by Theorem 3.

(b) $f(0) = \cos 0 + 3(0) = 1$;
$f'(0) = -\sin 0 + 3 = 3$

(c) Since the graph of $y = f(x)$ includes the point (0, 1) and the slope of the graph is 3 at this point, the graph of $y = f^{-1}(x)$ will include (1, 0) and the slope will be $\frac{1}{3}$. Thus, $f^{-1}(1) = 0$ and $(f^{-1})'(1) = \frac{1}{3}$.

22.



$[-2\pi, 2\pi]$ by $[-4, 4]$

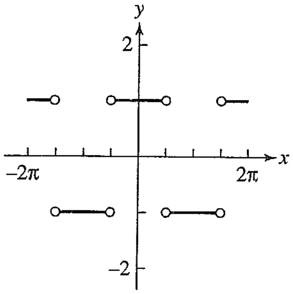
(a) All reals

(b) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

22. continued

(c) At the points $x = k\frac{\pi}{2}$, where k is an odd integer.

(d)



(e) $$f'(x) = \frac{d}{dx}\sin^{-1}(\sin x)$$
$$= \frac{1}{\sqrt{1-\sin^2 x}}\frac{d}{dx}\sin x$$
$$= \frac{\cos x}{\sqrt{1-\sin^2 x}}$$

which is ±1 depending on whether

$\cos x$ is positive or negative.

23. (a) $v(t) = \frac{dx}{dt} = \frac{1}{1+t^2}$ which is always positive.

(b) $a(t) = \frac{dv}{dt} = -\frac{2t}{(1+t^2)^2}$ which is always negative.

(c) $\frac{\pi}{2}$

24. $$\frac{d}{dx}\cos^{-1}(x) = \frac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1}x\right)$$
$$= 0 - \frac{d}{dx}\sin^{-1}(x)$$
$$= -\frac{1}{\sqrt{1-x^2}}$$

25. $$\frac{d}{dx}\cot^{-1}x = \frac{d}{dx}\left(\frac{\pi}{2} - \tan^{-1}(x)\right)$$
$$= 0 - \frac{d}{dx}\tan^{-1}(x)$$
$$= -\frac{1}{1+x^2}$$

26. $$\frac{d}{dx}\csc^{-1}(x) = \frac{d}{dx}\left(\frac{\pi}{2} - \sec^{-1}(x)\right)$$
$$= 0 - \frac{d}{dx}\sec^{-1}(x)$$
$$= -\frac{1}{|x|\sqrt{x^2-1}}$$

27. (a) $y = \frac{\pi}{2}$

(b) $y = -\frac{\pi}{2}$

(c) None, since $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \neq 0$.

28. (a) $y = 0$

(b) $y = \pi$

(c) None, since $\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2} \neq 0$.

29. (a) $y = \frac{\pi}{2}$

(b) $y = \frac{\pi}{2}$

(c) None, since $\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}} \neq 0$.

30. (a) $y = 0$

(b) $y = 0$

(c) None, since $\frac{d}{dx}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}} \neq 0$.

31. (a) None, since $\sin^{-1}x$ is undefined for $x > 1$.

(b) None, since $\sin^{-1}x$ is undefined for $x < -1$.

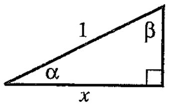
(c) None, since $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \neq 0$.

32. (a) None, since $\cos^{-1}x$ is undefined for $x > 1$.

(b) None, since $\cos^{-1}x$ is undefined for $x < -1$.

(c) None, since $\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}} \neq 0$.

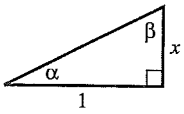
33. (a)



$\alpha = \cos^{-1}x,\ \beta = \sin^{-1}x$

So $\cos^{-1}x + \sin^{-1}x = \alpha + \beta = \frac{\pi}{2}$.

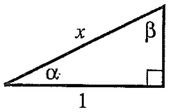
(b)



$\alpha = \tan^{-1}x,\ \beta = \cot^{-1}x$

So $\tan^{-1}x + \cot^{-1}x = \alpha + \beta = \frac{\pi}{2}$.

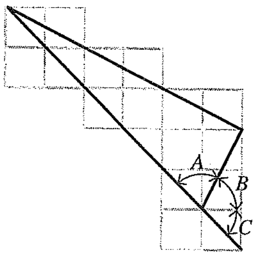
(c)



$\alpha = \sec^{-1}x,\ \beta = \csc^{-1}x$

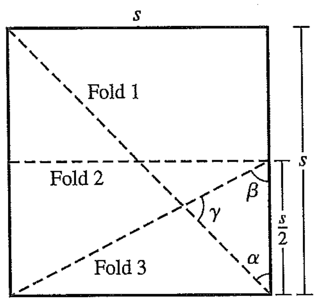
So $\sec^{-1}x + \csc^{-1}x = \alpha + \beta = \frac{\pi}{2}$.

34.



The "straight angle" with the arrows in it is the sum of the three angles A, B, and C.
A is equal to $\tan^{-1}3$ since the opposite side is 3 times as long as the adjacent side.
B is equal to $\tan^{-1}2$ since the side opposite it is 2 units and the adjacent side is one unit.
C is equal to $\tan^{-1}1$ since both the opposite and adjacent sides are one unit long.
But the sum of these three angles is the "straight angle," which has measure π radians.

35.



If s is the length of a side of the square, then

$\tan \alpha = \frac{s}{s} = 1$, so $\alpha = \tan^{-1} 1$ and

$\tan \beta = \frac{s}{\frac{s}{2}} = 2$, so $\beta = \tan^{-1} 2$.

From Exercise 34, we have

$\gamma = \pi - \alpha - \beta = \pi - \tan^{-1} 1 - \tan^{-1} 2 = \tan^{-1} 3$.

Section 3.9 Derivatives of Exponential and Logarithmic Functions (pp. 163–171)

Exploration 1 Leaving Milk on the Counter

1. The temperature of the refrigerator is 42°F, the temperature of the milk at time $t = 0$.
2. The temperature of the room is 72°F, the limit to which y tends as t increases.
3. The milk is warming up the fastest at $t = 0$. The second derivative $y'' = -30(\ln(0.98))^2(0.98)^t$ is negative, so y' (the rate at which the milk is warming) is maximized at the lowest value of t.
4. We set $y = 55$ and solve;

$$72 - 30(0.98)^t = 55$$
$$(0.98)^t = \frac{17}{30}$$
$$t \ln(0.98) = \ln\left(\frac{17}{30}\right)$$
$$t = \frac{\ln\left(\frac{17}{30}\right)}{\ln(0.98)} \approx 28.114$$

The milk reaches a temperature of 55°F after about 28 minutes.

5. $\frac{dy}{dt} = -30 \ln(0.98) \cdot (0.98)^t$. At $t = \frac{\ln\left(\frac{17}{30}\right)}{\ln(0.98)}$, $\frac{dy}{dt} \approx 0.343$ degrees/minute.

Quick Review 3.9

1. $\log_5 8 = \frac{\ln 8}{\ln 5}$
2. $7^x = e^{\ln 7^x} = e^{x \ln 7}$
3. $\ln(e^{\tan x}) = \tan x$
4. $\ln(x^2 - 4) - \ln(x + 2) = \ln \frac{x^2 - 4}{x + 2} = \ln \frac{(x + 2)(x - 2)}{x + 2}$
   $= \ln(x - 2)$
5. $\log_2(8^{x-5}) = \log_2(2^3)^{x-5} = \log_2 2^{3x-15} = 3x - 15$
6. $\frac{\log_4 x^{15}}{\log_4 x^{12}} = \frac{15 \log_4 x}{12 \log_4 x} = \frac{15}{12} = \frac{5}{4}, x > 0$
7. $3 \ln x - \ln 3x + \ln(12x^2) = \ln x^3 - \ln 3x + \ln(12x^2)$
   $= \ln \frac{(x^3)(12x^2)}{3x} = \ln(4x^4)$
8. $$\begin{aligned} 3^x &= 19 \\ \ln 3^x &= \ln 19 \\ x \ln 3 &= \ln 19 \\ x &= \frac{\ln 19}{\ln 3} \approx 2.68 \end{aligned}$$
9. $$\begin{aligned} 5^t \ln 5 &= 18 \\ 5^t &= \frac{18}{\ln 5} \\ \ln 5^t &= \ln \frac{18}{\ln 5} \\ t \ln 5 &= \ln 18 - \ln(\ln 5) \\ t &= \frac{\ln 18 - \ln(\ln 5)}{\ln 5} \approx 1.50 \end{aligned}$$
10. $$\begin{aligned} 3^{x+1} &= 2^x \\ \ln 3^{x+1} &= \ln 2^x \\ (x + 1) \ln 3 &= x \ln 2 \\ x(\ln 3 - \ln 2) &= -\ln 3 \\ x &= \frac{\ln 3}{\ln 2 - \ln 3} \approx -2.71 \end{aligned}$$

Section 3.9 Exercises

1. $\frac{dy}{dx} = \frac{d}{dx}(2e^x) = 2e^x$
2. $\frac{dy}{dx} = \frac{d}{dx}(e^{2x}) = e^{2x} \frac{d}{dx}(2x) = 2e^{2x}$
3. $\frac{dy}{dx} = \frac{d}{dx}e^{-x} = e^{-x} \frac{d}{dx}(-x) = -e^{-x}$
4. $\frac{dy}{dx} = \frac{d}{dx}e^{-5x} = e^{-5x} \frac{d}{dx}(-5x) = -5e^{-5x}$
5. $\frac{dy}{dx} = \frac{d}{dx}e^{2x/3} = e^{2x/3} \frac{d}{dx}\left(\frac{2x}{3}\right) = \frac{2}{3}e^{2x/3}$
6. $\frac{dy}{dx} = \frac{d}{dx}e^{-x/4} = e^{-x/4} \frac{d}{dx}\left(-\frac{x}{4}\right) = -\frac{1}{4}e^{-x/4}$
7. $\frac{dy}{dx} = \frac{d}{dx}(xe^2) - \frac{d}{dx}(e^x) = e^2 - e^x$
8. $\frac{dy}{dx} = \frac{d}{dx}(x^2e^x) - \frac{d}{dx}(xe^x)$
   $= (x^2)(e^x) + (e^x)(2x) - [(x)(e^x) + (e^x)(1)]$
   $= x^2e^x + xe^x - e^x$
9. $\frac{dy}{dx} = \frac{d}{dx} e^{\sqrt{x}} = e^{\sqrt{x}} \frac{d}{dx}(\sqrt{x}) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$
10. $\frac{dy}{dx} = \frac{d}{dx} e^{(x^2)} = e^{(x^2)} \frac{d}{dx}(x^2) = 2xe^{(x^2)}$

11. $\frac{dy}{dx} = \frac{d}{dx}(x^{\pi}) = \pi x^{\pi - 1}$

12. $\frac{dy}{dx} = \frac{d}{dx}(x^{1+\sqrt{2}}) = (1 + \sqrt{2})x^{1+\sqrt{2}-1} = (1 + \sqrt{2})x^{\sqrt{2}}$

13. $\frac{dy}{dx} = \frac{d}{dx}x^{-\sqrt{2}} = -\sqrt{2}x^{-\sqrt{2}-1}$

14. $\frac{dy}{dx} = \frac{d}{dx}x^{1-e} = (1 - e)x^{1-e-1} = (1 - e)x^{-e}$

15. $\frac{dy}{dx} = \frac{d}{dx}8^x = 8^x \ln 8$

16. $\frac{dy}{dx} = \frac{d}{dx}9^{-x} = 9^{-x}(\ln 9)\frac{d}{dx}(-x) = -9^{-x}\ln 9$

17.
$$\begin{aligned} \frac{dy}{dx} = \frac{d}{dx}3^{\csc x} &= 3^{\csc x}(\ln 3)\frac{d}{dx}(\csc x) \\ &= 3^{\csc x}(\ln 3)(-\csc x \cot x) \\ &= -3^{\csc x}(\ln 3)(\csc x \cot x) \end{aligned}$$

18.
$$\begin{aligned} \frac{dy}{dx} = \frac{d}{dx}3^{\cot x} &= 3^{\cot x}(\ln 3)\frac{d}{dx}(\cot x) \\ &= 3^{\cot x}(\ln 3)(-\csc^2 x) \\ &= -3^{\cot x}(\ln 3)(\csc^2 x) \end{aligned}$$

19. Use logarithmic differentiation.
$$\begin{aligned} y &= x^{\ln x} \\ \ln y &= \ln x^{\ln x} \\ \ln y &= \ln x \ln x \\ \frac{d}{dx}(\ln y) &= \frac{d}{dx}(\ln x)^2 \\ \frac{1}{y}\frac{dy}{dx} &= (2\ln x)\left(\frac{1}{x}\right) \\ \frac{dy}{dx} &= \frac{2y \ln x}{x} \\ \frac{dy}{dx} &= \frac{2x^{\ln x}\ln x}{x} \end{aligned}$$

20. Use logarithmic differentiation.
$$\begin{aligned} y &= x^{1/\ln x} \\ \ln y &= \ln x^{1/\ln x} \\ \ln y &= \frac{1}{\ln x}\ln x \\ \ln y &= 1 \\ y &= e \\ \frac{dy}{dx} &= \frac{d}{dx}(e) = 0, x > 0 \end{aligned}$$

21. $\frac{dy}{dx} = \frac{d}{dx}\ln(x^2) = \frac{1}{x^2}\frac{d}{dx}(x^2) = \frac{1}{x^2}(2x) = \frac{2}{x}$

22. $\frac{dy}{dx} = \frac{d}{dx}(\ln x)^2 = 2\ln x \frac{d}{dx}(\ln x) = \frac{2\ln x}{x}$

23. $\frac{dy}{dx} = \frac{d}{dx}\ln(x^{-1}) = \frac{d}{dx}(-\ln x) = -\frac{1}{x}, x > 0$

24. $\frac{dy}{dx} = \frac{d}{dx}\ln\frac{10}{x} = \frac{d}{dx}(\ln 10 - \ln x) = 0 - \frac{1}{x} = -\frac{1}{x}, x > 0$

25. $\frac{dy}{dx} = \frac{d}{dx}\ln(x + 2) = \frac{1}{x+2}\frac{d}{dx}(x + 2) = \frac{1}{x+2}, x > -2$

26.
$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\ln(2x + 2) = \frac{1}{2x+2}\frac{d}{dx}(2x + 2) = \frac{2}{2x+2} \\ &= \frac{1}{x+1}, x > -1 \end{aligned}$$

27.
$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\ln(2 - \cos x) = \frac{1}{2 - \cos x}\frac{d}{dx}(2 - \cos x) \\ &= \frac{\sin x}{2 - \cos x} \end{aligned}$$

28. $\frac{dy}{dx} = \frac{d}{dx}\ln(x^2 + 1) = \frac{1}{x^2+1}\frac{d}{dx}(x^2 + 1) = \frac{2x}{x^2+1}$

29. $\frac{d}{dx}\ln(\ln x) = \frac{1}{\ln x}\frac{d}{dx}\ln x = \frac{1}{\ln x} \cdot \frac{1}{x} = \frac{1}{x\ln x}$

30.
$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(x\ln x - x) = (x)\left(\frac{1}{x}\right) + (\ln x)(1) - 1 \\ &= 1 + \ln x - 1 = \ln x \end{aligned}$$

31.
$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(\log_4 x^2) = \frac{d}{dx}\frac{\ln x^2}{\ln 4} = \frac{d}{dx}\left[\left(\frac{2}{\ln 4}\right)(\ln x)\right] \\ &= \frac{2}{\ln 4} \cdot \frac{1}{x} = \frac{2}{x\ln 4} = \frac{1}{x\ln 2} \end{aligned}$$

32.
$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(\log_5 \sqrt{x}) = \frac{d}{dx}\frac{\ln x^{1/2}}{\ln 5} = \frac{d}{dx}\frac{\frac{1}{2}\ln x}{\ln 5} \\ &= \frac{1}{2\ln 5}\frac{d}{dx}(\ln x) = \frac{1}{2\ln 5} \cdot \frac{1}{x} = \frac{1}{2x\ln 5}, x > 0 \end{aligned}$$

33.
$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\log_2(3x + 1) = \frac{1}{(3x + 1)\ln 2}\frac{d}{dx}(3x + 1) \\ &= \frac{3}{(3x + 1)\ln 2}, x > -\frac{1}{3} \end{aligned}$$

34.
$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\log_{10}(x + 1)^{1/2} = \frac{1}{2}\frac{d}{dx}\log_{10}(x + 1) \\ &= \frac{1}{2}\frac{1}{(x + 1)\ln 10}\frac{d}{dx}(x + 1) = \frac{1}{2(x + 1)\ln 10}, x > -1 \end{aligned}$$

35. $\frac{dy}{dx} = \frac{d}{dx}\log_2\left(\frac{1}{x}\right) = \frac{d}{dx}(-\log_2 x) = -\frac{1}{x\ln 2}, x > 0$

36.
$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\frac{1}{\log_2 x} = -\frac{1}{(\log_2 x)^2}\frac{d}{dx}(\log_2 x) \\ &= -\frac{1}{(\log_2 x)^2}\frac{1}{x\ln 2} = -\frac{1}{x(\ln 2)(\log_2 x)^2} \text{ or } -\frac{\ln 2}{x(\ln x)^2} \end{aligned}$$

37.
$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(\ln 2 \cdot \log_2 x) = (\ln 2)\frac{d}{dx}(\log_2 x) \\ &= (\ln 2)\left(\frac{1}{x\ln 2}\right) = \frac{1}{x}, x > 0 \end{aligned}$$

38.
$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\log_3(1 + x\ln 3) = \frac{1}{(1 + x\ln 3)\ln 3}\frac{d}{dx}(1 + x\ln 3) \\ &= \frac{\ln 3}{(1 + x\ln 3)\ln 3} = \frac{1}{1 + x\ln 3}, x > -\frac{1}{\ln 3} \end{aligned}$$

39.
$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(\log_{10} e^x) = \frac{d}{dx}(x\log_{10} e) = \log_{10} e = \frac{\ln e}{\ln 10} \\ &= \frac{1}{\ln 10} \end{aligned}$$

40. $\frac{dy}{dx} = \frac{d}{dx}\ln 10^x = \frac{d}{dx}(x\ln 10) = \ln 10$

41. The line passes through (a, e^a) for some value of a and has slope $m = e^a$. Since the line also passes through the origin, the slope is also given by $m = \frac{e^a - 0}{a - 0}$ and we have $e^a = \frac{e^a}{a}$, so $a = 1$. Hence, the slope is e and the equation is $y = ex$.

42. For $y = xe^x$, we have $y' = (x)(e^x) + (e^x)(1) = (x + 1)e^x$, so the normal line through the point (a, ae^a) has slope $m = -\frac{1}{(a+1)e^a}$ and its equation is $y = -\frac{1}{(a+1)e^a}(x - a) + ae^a$. The desired normal line includes the point $(0, 0)$, so we have:

$$\begin{aligned}
0 &= -\frac{1}{(a+1)e^a}(0 - a) + ae^a \\
0 &= \frac{a}{(a+1)e^a} + ae^a \\
0 &= a\left(\frac{1}{(a+1)e^a} + e^a\right) \\
a &= 0 \text{ or } \frac{1}{(a+1)e^a} + e^a = 0
\end{aligned}$$

The equation $\frac{1}{(a+1)e^a} + e^a = 0$ has no solution, so we need to use $a = 0$. The equation of the normal line is $y = -\frac{1}{(0+1)e^0}(x - 0) + 0e^0$, or $y = -x$.

43.

$$\begin{aligned}
y &= (\sin x)^x \\
\ln y &= \ln (\sin x)^x \\
\ln y &= x \ln (\sin x) \\
\frac{d}{dx} \ln y &= \frac{d}{dx}[x \ln (\sin x)] \\
\frac{1}{y}\frac{dy}{dx} &= (x)\left(\frac{1}{\sin x}\right)(\cos x) + \ln (\sin x)(1) \\
\frac{dy}{dx} &= y[x \cot x + \ln (\sin x)] \\
\frac{dy}{dx} &= (\sin x)^x[x \cot x + \ln (\sin x)]
\end{aligned}$$

44.

$$\begin{aligned}
y &= x^{\tan x} \\
\ln y &= \ln (x^{\tan x}) \\
\ln y &= (\tan x)(\ln x) \\
\frac{d}{dx} \ln y &= \frac{d}{dx}[(\tan x)(\ln x)] \\
\frac{1}{y}\frac{dy}{dx} &= (\tan x)\left(\frac{1}{x}\right) + (\ln x)(\sec^2 x) \\
\frac{dy}{dx} &= y\left[\frac{\tan x}{x} + (\ln x)(\sec^2 x)\right] \\
\frac{dy}{dx} &= x^{\tan x}\left[\frac{\tan x}{x} + (\ln x)(\sec^2 x)\right]
\end{aligned}$$

45.

$$\begin{aligned}
y &= \sqrt[5]{\frac{(x-3)^4(x^2+1)}{(2x+5)^3}} = \left(\frac{(x-3)^4(x^2+1)}{(2x+5)^3}\right)^{1/5} \\
\ln y &= \ln \left(\frac{(x-3)^4(x^2+1)}{(2x+5)^3}\right)^{1/5} \\
\ln y &= \frac{1}{5} \ln \frac{(x-3)^4(x^2+1)}{(2x+5)^3} \\
\ln y &= \frac{1}{5}[4 \ln (x-3) + \ln (x^2+1) - 3 \ln (2x+5)] \\
\frac{d}{dx}(\ln y) &= \frac{4}{5}\frac{d}{dx} \ln (x-3) + \\
&\quad \frac{1}{5}\frac{d}{dx} \ln (x^2+1) - \frac{3}{5}\frac{d}{dx} \ln (2x+5) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{4}{5}\frac{1}{x-3} + \frac{1}{5}\frac{1}{x^2+1}(2x) - \frac{3}{5}\frac{1}{2x+5}(2) \\
\frac{dy}{dx} &= y\left(\frac{4}{5(x-3)} + \frac{2x}{5(x^2+1)} - \frac{6}{5(2x+5)}\right) \\
\frac{dy}{dx} &= \left(\frac{(x-3)^4(x^2+1)}{(2x+5)^3}\right)^{1/5} \cdot \\
&\quad \left(\frac{4}{5(x-3)} + \frac{2x}{5(x^2+1)} - \frac{6}{5(2x+5)}\right)
\end{aligned}$$

46.

$$\begin{aligned}
y &= \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}} = \frac{x(x^2+1)^{1/2}}{(x+1)^{2/3}} \\
\ln y &= \ln \frac{x(x^2+1)^{1/2}}{(x+1)^{2/3}} \\
\ln y &= \ln x + \frac{1}{2} \ln (x^2+1) - \frac{2}{3} \ln (x+1) \\
\frac{d}{dx} \ln y &= \frac{d}{dx} \ln x + \frac{1}{2}\frac{d}{dx} \ln (x^2+1) - \frac{2}{3}\frac{d}{dx} \ln (x+1) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{x} + \frac{1}{2}\frac{1}{x^2+1}(2x) - \frac{2}{3}\frac{1}{x+1}(1) \\
\frac{dy}{dx} &= y\left(\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}\right) \\
\frac{dy}{dx} &= \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\left(\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}\right)
\end{aligned}$$

47.

$$\begin{aligned}
\frac{dA}{dt} &= 20\frac{d}{dt}\left(\frac{1}{2}\right)^{t/140} \\
&= 20\frac{d}{dt} 2^{-t/140} \\
&= 20\,(2^{-t/140})(\ln 2)\frac{d}{dt}\left(-\frac{t}{140}\right) \\
&= 20(2^{-t/140})(\ln 2)\left(-\frac{1}{140}\right) \\
&= -\frac{(2^{-t/140})(\ln 2)}{7}
\end{aligned}$$

At $t = 2$ days, we have

$$\frac{dA}{dt} = -\frac{(2^{-1/70})(\ln 2)}{7} \approx -0.098 \text{ grams/day.}$$

This means that the rate of *decay* is the positive rate of approximately 0.098 grams/day.

48. (a) $\frac{d}{dx} \ln (kx) = \frac{1}{kx}\frac{d}{dx} kx = \frac{k}{kx} = \frac{1}{x}$

(b)

$$\begin{aligned}
\frac{d}{dx} \ln (kx) &= \frac{d}{dx} (\ln k + \ln x) \\
&= 0 + \frac{d}{dx} \ln x = \frac{1}{x}
\end{aligned}$$

49. **(a)** Since $f'(x) = 2^x \ln 2$, $f'(0) = 2^0 \ln 2 = \ln 2$.

(b) $f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{2^h - 2^0}{h} = \lim_{h\to 0} \frac{2^h - 1}{h}$

(c) Since quantities in parts (a) and (b) are equal,

$\lim_{h\to 0} \frac{2^h - 1}{h} = \ln 2$.

(d) By following the same procedure as above using

$g(x) = 7^x$, we may see that $\lim_{h\to 0} \frac{7^h - 1}{h} = \ln 7$.

50. Recall that a point (a, b) is on the graph of $y = e^x$ if and only if the point (b, a) is on the graph of $y = \ln x$. Since there are points (x, e^x) on the graph of $y = e^x$ with arbitrarily large x-coordinates, there will be points $(x, \ln x)$ on the graph of $y = \ln x$ with arbitrarily large y-coordinates.

51. **(a)** The graph y_4 is a horizontal line at $y = a$.

(b) The graph of y_3 is always a horizontal line.

a	2	3	4	5
y_3	0.693147	1.098613	1.386295	1.609439
$\ln a$	0.693147	1.098612	1.386294	1.609438

We conclude that the graph of y_3 is a horizontal line at $y = \ln a$.

(c) $\frac{d}{dx} a^x = a^x$ if and only if $y_3 = \frac{y_2}{y_1} = 1$.

So if $y_3 = \ln a$, then $\frac{d}{dx} a^x$ will equal a^x if and only if

$\ln a = 1$, or $a = e$.

(d) $y_2 = \frac{d}{dx} a^x = a^x \ln a$. This will equal $y_1 = a^x$ if and only if $\ln a = 1$, or $a = e$.

52. $\frac{d}{dx}\left(-\frac{1}{2}x^2 + k\right) = -x$ and $\frac{d}{dx}(\ln x + c) = \frac{1}{x}$.

Therefore, at any given value of x, these two curves will have perpendicular tangent lines.

53. **(a)** Since the line passes through the origin and has slope $\frac{1}{e}$, its equation is $y = \frac{x}{e}$.

(b) The graph of $y = \ln x$ lies below the graph of the line $y = \frac{x}{e}$ for all positive $x \neq e$. Therefore, $\ln x < \frac{x}{e}$ for all positive $x \neq e$.

(c) Multiplying by e, $e \ln x < x$ or $\ln x^e < x$.

(d) Exponentiating both sides of $\ln x^e < x$, we have $e^{\ln x^e} < e^x$, or $x^e < e^x$ for all positive $x \neq e$.

(e) Let $x = \pi$ to see that $\pi^e < e^\pi$. Therefore, e^π is bigger.

Chapter 3 Review Exercises

(pp. 172–175)

1. $\frac{dy}{dx} = \frac{d}{dx}\left(x^5 - \frac{1}{8}x^2 + \frac{1}{4}x\right) = 5x^4 - \frac{1}{4}x + \frac{1}{4}$

2. $\frac{dy}{dx} = \frac{d}{dx}(3 - 7x^3 + 3x^7) = -21x^2 + 21x^6$

3. $\frac{dy}{dx} = \frac{d}{dx}(2 \sin x \cos x)$

$= 2(\sin x)\frac{d}{dx}(\cos x) + 2(\cos x)\frac{d}{dx}(\sin x)$

$= -2 \sin^2 x + 2 \cos^2 x$

Alternate solution:

$\frac{dy}{dx} = \frac{d}{dx}(2 \sin x \cos x) = \frac{d}{dx} \sin 2x = (\cos 2x)(2)$

$= 2 \cos 2x$

4. $\frac{dy}{dx} = \frac{d}{dx} \frac{2x + 1}{2x - 1} = \frac{(2x - 1)(2) - (2x + 1)(2)}{(2x - 1)^2} = -\frac{4}{(2x - 1)^2}$

5. $\frac{ds}{dt} = \frac{d}{dt} \cos (1 - 2t) = -\sin (1 - 2t)(-2) = 2 \sin (1 - 2t)$

6. $\frac{ds}{dt} = \frac{d}{dt} \cot \left(\frac{2}{t}\right) = -\csc^2 \left(\frac{2}{t}\right)\frac{d}{dt}\left(\frac{2}{t}\right) = -\csc^2 \left(\frac{2}{t}\right)\left(-\frac{2}{t^2}\right)$

$= \frac{2}{t^2} \csc^2 \left(\frac{2}{t}\right)$

7. $\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x} + 1 + \frac{1}{\sqrt{x}}\right) = \frac{d}{dx}(x^{1/2} + 1 + x^{-1/2})$

$= \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} = \frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}}$

8. $\frac{dy}{dx} = \frac{d}{dx}(x\sqrt{2x + 1}) = (x)\left(\frac{1}{2\sqrt{2x + 1}}\right)(2) + (\sqrt{2x + 1})(1)$

$= \frac{x + (2x + 1)}{\sqrt{2x + 1}} = \frac{3x + 1}{\sqrt{2x + 1}}$

9. $\frac{dr}{d\theta} = \frac{d}{d\theta} \sec (1 + 3\theta) = \sec (1 + 3\theta) \tan (1 + 3\theta)(3)$

$= 3 \sec (1 + 3\theta) \tan (1 + 3\theta)$

10. $\frac{dr}{d\theta} = \frac{d}{d\theta} \tan^2 (3 - \theta^2)$

$= 2 \tan (3 - \theta^2)\frac{d}{d\theta} \tan (3 - \theta^2)$

$= 2 \tan (3 - \theta^2) \sec^2 (3 - \theta^2)(-2\theta)$

$= -4\theta \tan (3 - \theta^2) \sec^2 (3 - \theta^2)$

11. $\frac{dy}{dx} = \frac{d}{dx}(x^2 \csc 5x)$

$= (x^2)(-\csc 5x \cot 5x)(5) + (\csc 5x)(2x)$

$= -5x^2 \csc 5x \cot 5x + 2x \csc 5x$

12. $\frac{dy}{dx} = \frac{d}{dx} \ln \sqrt{x} = \frac{1}{\sqrt{x}} \frac{d}{dx} \sqrt{x} = \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x}, x > 0$

13. $\frac{dy}{dx} = \frac{d}{dx} \ln (1 + e^x) = \frac{1}{1 + e^x} \frac{d}{dx}(1 + e^x) = \frac{e^x}{1 + e^x}$

14. $\frac{dy}{dx} = \frac{d}{dx}(xe^{-x}) = (x)(e^{-x})(-1) + (e^{-x})(1) = -xe^{-x} + e^{-x}$

15. $\frac{dy}{dx} = \frac{d}{dx}(e^{1+\ln x}) = \frac{d}{dx}(e^1 e^{\ln x}) = \frac{d}{dx}(ex) = e$

16. $\frac{dy}{dx} = \frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{\cos x}{\sin x} = \cot x$, for values of x in the intervals $(k\pi, (k+1)\pi)$, where k is even.

17. $\frac{dr}{dx} = \frac{d}{dx}\ln(\cos^{-1} x) = \frac{1}{\cos^{-1} x}\frac{d}{dx}\cos^{-1} x$

$$= \frac{1}{\cos^{-1} x}\left(-\frac{1}{\sqrt{1-x^2}}\right) = -\frac{1}{\cos^{-1} x\sqrt{1-x^2}}$$

18. $\frac{dr}{d\theta} = \frac{d}{d\theta}\log_2(\theta^2) = \frac{1}{\theta^2 \ln 2}\frac{d}{d\theta}(\theta^2) = \frac{2\theta}{\theta^2 \ln 2} = \frac{2}{\theta \ln 2}$

19. $\frac{ds}{dt} = \frac{d}{dt}\log_5(t-7) = \frac{1}{(t-7)\ln 5}\frac{d}{dt}(t-7) = \frac{1}{(t-7)\ln 5}$, $t > 7$

20. $\frac{ds}{dt} = \frac{d}{dt}(8^{-t}) = 8^{-t}(\ln 8)\frac{d}{dt}(-t) = -8^{-t}\ln 8$

21. Use logarithmic differentiation.

$$y = x^{\ln x}$$

$$\ln y = \ln(x^{\ln x})$$

$$\ln y = (\ln x)(\ln x)$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}(\ln x)^2$$

$$\frac{1}{y}\frac{dy}{dx} = 2\ln x \frac{d}{dx}\ln x$$

$$\frac{dy}{dx} = \frac{2y\ln x}{x}$$

$$\frac{dy}{dx} = \frac{2x^{\ln x}\ln x}{x}$$

22. $\frac{dy}{dx} = \frac{d}{dx}\frac{(2x)2^x}{\sqrt{x^2+1}}$

$$= \frac{\sqrt{x^2+1}\frac{d}{dx}[(2x)2^x] - (2x)(2^x)\frac{d}{dx}\sqrt{x^2+1}}{x^2+1}$$

$$= \frac{\sqrt{x^2+1}[(2x)(2^x)(\ln 2) + (2^x)(2)] - (2x)(2^x)\frac{1}{2\sqrt{x^2+1}}(2x)}{x^2+1}$$

$$= \frac{(x^2+1)(2^x)(2x\ln 2 + 2) - 2x^2(2^x)}{(x^2+1)^{3/2}}$$

$$= \frac{(2 \cdot 2^x)[(x^2+1)(x\ln 2 + 1) - x^2]}{(x^2+1)^{3/2}}$$

$$= \frac{(2 \cdot 2^x)(x^3\ln 2 + x^2 + x\ln 2 + 1 - x^2)}{(x^2+1)^{3/2}}$$

$$= \frac{(2 \cdot 2^x)(x^3\ln 2 + x\ln 2 + 1)}{(x^2+1)^{3/2}}$$

Alternate solution, using logarithmic differentiation:

$$y = \frac{(2x)2^x}{\sqrt{x^2+1}}$$

$$\ln y = \ln(2x) + \ln(2^x) - \ln\sqrt{x^2+1}$$

$$\ln y = \ln 2 + \ln x + x\ln 2 - \frac{1}{2}\ln(x^2+1)$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}[\ln 2 + \ln x + x\ln 2 - \frac{1}{2}\ln(x^2+1)]$$

$$\frac{1}{y}\frac{dy}{dx} = 0 + \frac{1}{x} + \ln 2 - \frac{1}{2}\frac{1}{x^2+1}(2x)$$

$$\frac{dy}{dx} = y\left(\frac{1}{x} + \ln 2 - \frac{x}{x^2+1}\right)$$

$$\frac{dy}{dx} = \frac{(2x)2^x}{\sqrt{x^2+1}}\left(\frac{1}{x} + \ln 2 - \frac{x}{x^2+1}\right)$$

23. $\frac{dy}{dx} = \frac{d}{dx}e^{\tan^{-1}x} = e^{\tan^{-1}x}\frac{d}{dx}\tan^{-1}x = \frac{e^{\tan^{-1}x}}{1+x^2}$

24. $\frac{dy}{du} = \frac{d}{du}\sin^{-1}\sqrt{1-u^2}$

$$= \frac{1}{\sqrt{1-(\sqrt{1-u^2})^2}}\frac{d}{du}\sqrt{1-u^2}$$

$$= \frac{1}{\sqrt{u^2}}\frac{1}{2\sqrt{1-u^2}}(-2u) = -\frac{u}{|u|\sqrt{1-u^2}}$$

25. $\frac{dy}{dt} = \frac{d}{dt}(t\sec^{-1} t - \frac{1}{2}\ln t)$

$$= (t)\left(\frac{1}{|t|\sqrt{t^2-1}}\right) + (\sec^{-1} t)(1) - \frac{1}{2t}$$

$$= \frac{t}{|t|\sqrt{t^2-1}} + \sec^{-1} t - \frac{1}{2t}$$

26. $\frac{dy}{dt} = \frac{d}{dt}[(1+t^2)\cot^{-1} 2t]$

$$= (1+t^2)\left(-\frac{1}{1+(2t)^2}\right)(2) + (\cot^{-1} 2t)(2t)$$

$$= -\frac{2+2t^2}{1+4t^2} + 2t\cot^{-1} 2t$$

27. $\frac{dy}{dz} = \frac{d}{dz}(z\cos^{-1} z - \sqrt{1-z^2})$

$$= (z)\left(-\frac{1}{\sqrt{1-z^2}}\right) + (\cos^{-1} z)(1) - \frac{1}{2\sqrt{1-z^2}}(-2z)$$

$$= -\frac{z}{\sqrt{1-z^2}} + \cos^{-1} z + \frac{z}{\sqrt{1-z^2}}$$

$$= \cos^{-1} z$$

28. $\frac{dy}{dx} = \frac{d}{dx}(2\sqrt{x-1}\csc^{-1}\sqrt{x})$

$$= (2\sqrt{x-1})\left(-\frac{1}{|\sqrt{x}|\sqrt{(\sqrt{x})^2-1}}\right)\left(\frac{1}{2\sqrt{x}}\right)$$

$$+ (2\csc^{-1}\sqrt{x})\left(\frac{1}{2\sqrt{x-1}}\right)$$

$$= -\frac{\sqrt{x-1}}{(\sqrt{x})^2\sqrt{x-1}} + \frac{\csc^{-1}\sqrt{x}}{\sqrt{x-1}}$$

$$= -\frac{1}{x} + \frac{\csc^{-1}\sqrt{x}}{\sqrt{x-1}}$$

29. $\frac{dy}{dx} = \frac{d}{dx}\csc^{-1}(\sec x)$

$$= \left(-\frac{1}{|\sec x|\sqrt{\sec^2 x - 1}}\right)\frac{d}{dx}(\sec x)$$

$$= -\frac{1}{|\sec x|\sqrt{\tan^2 x}}\sec x \tan x$$

$$= -\frac{\sec x \tan x}{|\sec x \tan x|}$$

$$= -\frac{\frac{1}{\cos x}\frac{\sin x}{\cos x}}{\left|\frac{1}{\cos x}\frac{\sin x}{\cos x}\right|} = -\frac{\sin x}{|\sin x|}$$

$$= \begin{cases} -1, & 0 \le x < \pi, \quad x \ne \frac{\pi}{2} \\ 1, & \pi < x \le 2\pi, \quad x \ne \frac{3\pi}{2} \end{cases}$$

Alternate method:

On the domain $0 \le x \le 2\pi$, $x \ne \frac{\pi}{2}$, $x \ne \frac{3\pi}{2}$, we may rewrite the function as follows:

$$y = \csc^{-1}(\sec x)$$

$$= \frac{\pi}{2} - \sec^{-1}(\sec x)$$

$$= \frac{\pi}{2} - \cos^{-1}(\cos x)$$

$$= \begin{cases} \frac{\pi}{2} - x, & 0 \le x \le \pi, \quad x \ne \frac{\pi}{2} \\ \frac{\pi}{2} - (\pi - x), & \pi < x \le 2\pi, \quad x \ne \frac{3\pi}{2} \end{cases}$$

$$= \begin{cases} \frac{\pi}{2} - x, & 0 \le x \le \pi, \quad x \ne \frac{\pi}{2} \\ -\frac{\pi}{2} + x, & \pi < x \le 2\pi, \quad x \ne \frac{3\pi}{2} \end{cases}$$

Therefore, $\frac{dy}{dx} = \begin{cases} -1, & 0 \le x < \pi, \quad x \ne \frac{\pi}{2} \\ 1, & \pi < x \le 2\pi, \quad x \ne \frac{3\pi}{2} \end{cases}$

Note that the derivative exists at 0 and 2π only because these are the endpoints of the given domain; the two-sided derivative of $y = \csc^{-1}(\sec x)$ does not exist at these points.

30. $\frac{dr}{d\theta} = \frac{d}{d\theta}\left(\frac{1 + \sin\theta}{1 - \cos\theta}\right)^2$

$$= 2\left(\frac{1 + \sin\theta}{1 - \cos\theta}\right)\left(\frac{(1 - \cos\theta)(\cos\theta) - (1 + \sin\theta)(\sin\theta)}{(1 - \cos\theta)^2}\right)$$

$$= 2\left(\frac{1 + \sin\theta}{1 - \cos\theta}\right)\left(\frac{\cos\theta - \cos^2\theta - \sin\theta - \sin^2\theta}{(1 - \cos\theta)^2}\right)$$

$$= 2\left(\frac{1 + \sin\theta}{1 - \cos\theta}\right)\left(\frac{\cos\theta - \sin\theta - 1}{(1 - \cos\theta)^2}\right)$$

31. Since $y = \ln x^2$ is defined for all $x \ne 0$ and $\frac{dy}{dx} = \frac{1}{x^2}\frac{d}{dx}(x^2) = \frac{2x}{x^2} = \frac{2}{x}$, the function is differentiable for all $x \ne 0$.

32. Since $y = \sin x - x\cos x$ is defined for all real x and $\frac{dy}{dx} = \cos x - (x)(-\sin x) - (\cos x)(1) = x\sin x$, the function is differentiable for all real x.

33. Since $y = \sqrt{\frac{1 - x}{1 + x^2}}$ is defined for all $x < 1$ and

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\frac{1 - x}{1 + x^2}}}\,\frac{(1 + x^2)(-1) - (1 - x)(2x)}{(1 + x^2)^2}$$

$$= \frac{x^2 - 2x - 1}{2\sqrt{1 - x}\,(1 + x^2)^{3/2}},$$ which is defined only for $x < 1$,

the function is differentiable for all $x < 1$.

34. Since $y = (2x - 7)^{-1}(x + 5) = \frac{x + 5}{2x - 7}$ is defined for all $x \ne \frac{7}{2}$ and $\frac{dy}{dx} = \frac{(2x - 7)(1) - (x + 5)(2)}{(2x - 7)^2} = -\frac{17}{(2x - 7)^2}$, the function is differentiable for all $x \ne \frac{7}{2}$.

35. Use implicit differentiation.

$$xy + 2x + 3y = 1$$

$$\frac{d}{dx}(xy) + \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}(1)$$

$$x\frac{dy}{dx} + (y)(1) + 2 + 3\frac{dy}{dx} = 0$$

$$(x + 3)\frac{dy}{dx} = -(y + 2)$$

$$\frac{dy}{dx} = -\frac{y + 2}{x + 3}$$

36. Use implicit differentiation.

$$5x^{4/5} + 10y^{6/5} = 15$$

$$\frac{d}{dx}(5x^{4/5}) + \frac{d}{dx}(10y^{6/5}) = \frac{d}{dx}(15)$$

$$4x^{-1/5} + 12y^{1/5}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{4x^{-1/5}}{12y^{1/5}} = -\frac{1}{3(xy)^{1/5}}$$

37. Use implicit differentiation.

$$\sqrt{xy} = 1$$

$$\frac{d}{dx}\sqrt{xy} = \frac{d}{dx}(1)$$

$$\frac{1}{2\sqrt{xy}}\left[x\frac{dy}{dx} + (y)(1)\right] = 0$$

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

Alternate method:

38. Use implicit differentiation.

$$
\begin{aligned}
y^2 &= \frac{x}{x+1} \\
\frac{d}{dx}y^2 &= \frac{d}{dx}\frac{x}{x+1} \\
2y\frac{dy}{dx} &= \frac{(x+1)(1)-(x)(1)}{(x+1)^2} \\
\frac{dy}{dx} &= \frac{1}{2y(x+1)^2}
\end{aligned}
$$

39.

$$
\begin{aligned}
x^3+y^3 &= 1 \\
\frac{d}{dx}(x^3)+\frac{d}{dx}(y^3) &= \frac{d}{dx}(1) \\
3x^2+3y^2y' &= 0 \\
y' &= -\frac{x^2}{y^2} \\
y'' &= \frac{d}{dx}\left(-\frac{x^2}{y^2}\right) \\
&= -\frac{(y^2)(2x)-(x^2)(2y)(y')}{y^4} \\
&= -\frac{(y^2)(2x)-(x^2)(2y)\left(-\frac{x^2}{y^2}\right)}{y^4} \\
&= -\frac{2xy^3+2x^4}{y^5} \\
&= -\frac{2x(x^3+y^3)}{y^5} \\
&= -\frac{2x}{y^5}
\end{aligned}
$$

since $x^3+y^3=1$

40.

$$
\begin{aligned}
y^2 &= 1-\frac{2}{x} \\
\frac{d}{dx}(y^2) &= \frac{d}{dx}(1)-\frac{d}{dx}\left(\frac{2}{x}\right) \\
2yy' &= \frac{2}{x^2} \\
y' &= \frac{2}{x^2(2y)} = \frac{1}{x^2y} \\
y'' &= \frac{d}{dx}\left(\frac{1}{x^2y}\right) \\
&= -\frac{1}{(x^2y)^2}\frac{d}{dx}(x^2y) \\
&= -\frac{1}{(x^2y)^2}[(x^2)(y')+(y)(2x)] \\
&= -\frac{1}{(x^2y)^2}\left[(x^2)\left(\frac{1}{x^2y}\right)+2xy\right] \\
&= -\frac{1}{x^4y^2}\left(\frac{1}{y}+2xy\right) \\
&= -\frac{1+2xy^2}{x^4y^3}
\end{aligned}
$$

41.

$$
\begin{aligned}
y^3+y &= 2\cos x \\
\frac{d}{dx}(y^3)+\frac{d}{dx}(y) &= \frac{d}{dx}(2\cos x) \\
3y^2y'+y' &= -2\sin x \\
(3y^2+1)y' &= -2\sin x \\
y' &= -\frac{2\sin x}{3y^2+1} \\
y'' &= \frac{d}{dx}\left(-\frac{2\sin x}{3y^2+1}\right) \\
&= -\frac{(3y^2+1)(2\cos x)-(2\sin x)(6yy')}{(3y^2+1)^2} \\
&= -\frac{(3y^2+1)(2\cos x)-(12y\sin x)\left(-\frac{2\sin x}{3y^2+1}\right)}{(3y^2+1)^2} \\
&= -2\frac{(3y^2+1)^2\cos x+12y\sin^2 x}{(3y^2+1)^3}
\end{aligned}
$$

42.

$$
\begin{aligned}
x^{1/3}+y^{1/3} &= 4 \\
\frac{d}{dx}(x^{1/3})+\frac{d}{dx}(y^{1/3}) &= \frac{d}{dx}(4) \\
\frac{1}{3}x^{-2/3}+\frac{1}{3}y^{-2/3}y' &= 0 \\
y' &= -\frac{x^{-2/3}}{y^{-2/3}} = -\left(\frac{y}{x}\right)^{2/3} \\
y'' &= \frac{d}{dx}\left[-\left(\frac{y}{x}\right)^{2/3}\right] \\
&= -\frac{2}{3}\left(\frac{y}{x}\right)^{-1/3}\left(\frac{xy'-(y)(1)}{x^2}\right) \\
&= -\frac{2}{3}\left(\frac{y}{x}\right)^{-1/3}\left(\frac{(x)\left[-\left(\frac{y}{x}\right)^{2/3}\right]-y}{x^2}\right) \\
&= -\frac{2}{3}x^{1/3}y^{-1/3}(-x^{-5/3}y^{2/3}-x^{-2}y) \\
&= \frac{2}{3}x^{-4/3}y^{1/3}+\frac{2}{3}x^{-5/3}y^{2/3}
\end{aligned}
$$

43. $y' = 2x^3-3x-1$,
$y'' = 6x^2-3$,
$y''' = 12x$,
$y^{(4)} = 12$, and the rest are all zero.

44. $y' = \frac{x^4}{24}$,
$y'' = \frac{x^3}{6}$,
$y''' = \frac{x^2}{2}$,
$y^{(4)} = x$,
$y^{(5)} = 1$, and the rest are all zero.

45. $\frac{dy}{dx} = \frac{d}{dx}\sqrt{x^2 - 2x} = \frac{1}{2\sqrt{x^2 - 2x}}(2x - 2) = \frac{x - 1}{\sqrt{x^2 - 2x}}$

At $x = 3$, we have $y = \sqrt{3^2 - 2(3)} = \sqrt{3}$

and $\frac{dy}{dx} = \frac{3 - 1}{\sqrt{3^2 - 2(3)}} = \frac{2}{\sqrt{3}}$.

(a) Tangent: $y = \frac{2}{\sqrt{3}}(x - 3) + \sqrt{3}$ or $y = \frac{2}{\sqrt{3}}x - \sqrt{3}$

(b) Normal: $y = -\frac{\sqrt{3}}{2}(x - 3) + \sqrt{3}$

or $y = -\frac{\sqrt{3}}{2}x + \frac{5\sqrt{3}}{2}$

46. $\frac{dy}{dx} = \frac{d}{dx}(4 + \cot x - 2\csc x)$

$= -\csc^2 x + 2\csc x \cot x$

x_1

At $x = \frac{\pi}{2}$, we have

y_1

$y = 4 + \cot\frac{\pi}{2} - 2\csc\frac{\pi}{2} = 4 + 0 - 2 = 2$ and

m

$\frac{dy}{dx} = -\csc^2\frac{\pi}{2} + 2\csc\frac{\pi}{2}\cot\frac{\pi}{2} = -1 + 2(1)(0) = -1.$

(a) Tangent: $y = -1\left(x - \frac{\pi}{2}\right) + 2$ or $y = -x + \frac{\pi}{2} + 2$

(b) Normal: $y = 1\left(x - \frac{\pi}{2}\right) + 2$ or $y = x - \frac{\pi}{2} + 2$

47. Use implicit differentiation.

$$x^2 + 2y^2 = 9$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(2y^2) = \frac{d}{dx}(9)$$

$$2x + 4y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{4y} = -\frac{x}{2y}$$

Slope at (1, 2): $-\frac{1}{2(2)} = -\frac{1}{4}$

(a) Tangent: $y = -\frac{1}{4}(x - 1) + 2$ or $y = -\frac{1}{4}x + \frac{9}{4}$

(b) Normal: $y = 4(x - 1) + 2$ or $y = 4x - 2$

48. Use implicit differentiation.

$$x + \sqrt{xy} = 6$$

$$\frac{d}{dx}(x) + \frac{d}{dx}(\sqrt{xy}) = \frac{d}{dx}(6)$$

$$1 + \frac{1}{2\sqrt{xy}}\left[(x)\left(\frac{dy}{dx}\right) + (y)(1)\right] = 0$$

$$\frac{x}{2\sqrt{xy}}\frac{dy}{dx} = -1 - \frac{y}{2\sqrt{xy}}$$

$$\frac{dy}{dx} = \frac{2\sqrt{xy}}{x}\left(-1 - \frac{y}{2\sqrt{xy}}\right)$$

$$= -2\sqrt{\frac{y}{x}} - \frac{y}{x}$$

Slope at (4, 1): $-2\sqrt{\frac{1}{4}} - \frac{1}{4} = -\frac{2}{2} - \frac{1}{4} = -\frac{5}{4}$

(a) Tangent: $y = -\frac{5}{4}(x - 4) + 1$ or $y = -\frac{5}{4}x + 6$

(b) Normal: $y = \frac{4}{5}(x - 4) + 1$ or $y = \frac{4}{5}x - \frac{11}{5}$

49. $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\sin t}{2\cos t} = -\tan t$

At $t = \frac{3\pi}{4}$, we have $x = 2\sin\frac{3\pi}{4} = \sqrt{2}$,

$y = 2\cos\frac{3\pi}{4} = -\sqrt{2}$, and $\frac{dy}{dx} = -\tan\frac{3\pi}{4} = 1.$

The equation of the tangent line is

$y = 1(x - \sqrt{2}) + (-\sqrt{2})$, or $y = x - 2\sqrt{2}$.

50. $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4\cos t}{-3\sin t} = -\frac{4}{3}\cot t$

At $t = \frac{3\pi}{4}$, we have $x = 3\cos\frac{3\pi}{4} = -\frac{3\sqrt{2}}{2}$,

$y = 4\sin\frac{3\pi}{4} = 2\sqrt{2}$, and $\frac{dy}{dx} = -\frac{4}{3}\cot\frac{3\pi}{4} = \frac{4}{3}$.

The equation of the tangent line is

$y = \frac{4}{3}\left(x + \frac{3\sqrt{2}}{2}\right) + 2\sqrt{2}$, or $y = \frac{4}{3}x + 4\sqrt{2}$.

51. $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{5\sec^2 t}{3\sec t\tan t} = \frac{5\sec t}{3\tan t} = \frac{5}{3\sin t}$

At $t = \frac{\pi}{6}$, we have $x = 3\sec\frac{\pi}{6} = 2\sqrt{3}$,

$y = 5\tan\frac{\pi}{6} = \frac{5\sqrt{3}}{3}$, and $\frac{dy}{dx} = \frac{5}{3\sin\left(\frac{\pi}{6}\right)} = \frac{10}{3}$.

The equation of the tangent line is

$y = \frac{10}{3}(x - 2\sqrt{3}) + \frac{5\sqrt{3}}{3}$, or $y = \frac{10}{3}x - 5\sqrt{3}$.

52. $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 + \cos t}{-\sin t}$

At $t = -\frac{\pi}{4}$, we have $x = \cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$,

$y = -\frac{\pi}{4} + \sin\left(-\frac{\pi}{4}\right) = -\frac{\pi}{4} - \frac{\sqrt{2}}{2}$, and

$$\frac{dy}{dx} = \frac{1 + \cos\left(-\frac{\pi}{4}\right)}{-\sin\left(-\frac{\pi}{4}\right)} = \frac{1 + \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \sqrt{2} + 1.$$

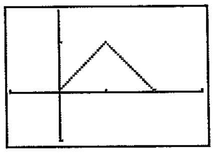
The equation of the tangent line is

$y = (\sqrt{2} + 1)\left(x - \frac{\sqrt{2}}{2}\right) - \frac{\pi}{4} - \frac{\sqrt{2}}{2}$, or

$y = (1 + \sqrt{2})x - \sqrt{2} - 1 - \frac{\pi}{4}$.

This is approximately $y = 2.414x - 3.200$.

53. (a)



$[-1, 3]$ by $\left[-1, \frac{5}{3}\right]$

(b) Yes, because both of the one-sided limits as $x \to 1$ are equal to $f(1) = 1$.

(c) No, because the left-hand derivative at $x = 1$ is $+1$ and the right-hand derivative at $x = 1$ is -1.

54. (a) The function is continuous for all values of m, because the right-hand limit as $x \to 0$ is equal to $f(0) = 0$ for any value of m.

(b) The left-hand derivative at $x = 0$ is $2\cos(2 \cdot 0) = 2$, and the right-hand derivative at $x = 0$ is m, so in order for the function to be differentiable at $x = 0$, m must be 2.

55. (a) For all $x \neq 0$ **(b)** At $x = 0$

(c) Nowhere

56. (a) For all x **(b)** Nowhere

(c) Nowhere

57. Note that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (2x - 3) = -3$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x - 3) = -3$. Since these values agree with $f(0)$, the function is continuous at $x = 0$. On the other hand,

$f'(x) = \begin{cases} 2, & -1 \le x < 0 \\ 1, & 0 < x \le 4 \end{cases}$, so the derivative is undefined at $x = 0$.

(a) $[-1, 0) \cup (0, 4]$ **(b)** At $x = 0$

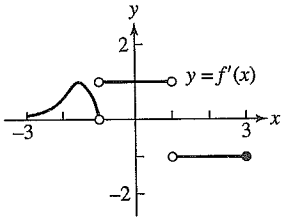
(c) Nowhere in its domain

58. Note that the function is undefined at $x = 0$.

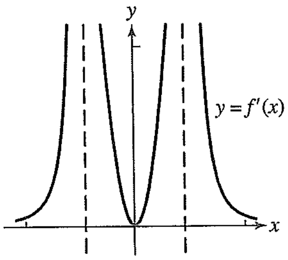
(a) $[-2, 0) \cup (0, 2]$ **(b)** Nowhere

(c) Nowhere in its domain

59.



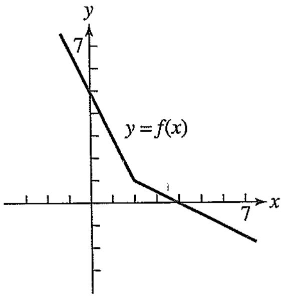
60.



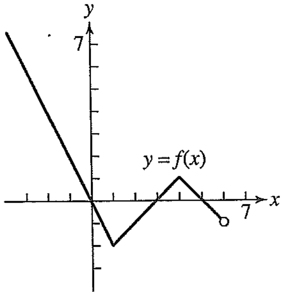
61. (a) iii **(b)** i

(c) ii

62. The graph passes through (0, 5) and has slope −2 for $x < 2$ and slope −0.5 for $x > 2$.



63. The graph passes through (−1, 2) and has slope −2 for $x < 1$, slope 1 for $1 < x < 4$, and slope −1 for $4 < x < 6$.



64. i. If $f(x) = \frac{9}{28}x^{7/3} + 9$, then $f'(x) = \frac{3}{4}x^{4/3}$ and $f''(x) = x^{1/3}$, which matches the given equation.

ii. If $f'(x) = \frac{9}{28}x^{7/3} - 2$, then $f''(x) = \frac{3}{4}x^{4/3}$, which contradicts the given equation $f''(x) = x^{1/3}$.

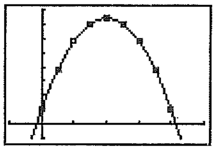
iii. If $f'(x) = \frac{3}{4}x^{4/3} + 6$, then $f''(x) = x^{1/3}$, which matches the given equation.

64. continued

iv. If $f(x) = \frac{3}{4}x^{4/3} - 4$, then $f'(x) = x^{1/3}$ and $f''(x) = \frac{1}{3}x^{-2/3}$, which contradicts the given equation $f''(x) = x^{1/3}$.

Answer is **D**: **i** and **iii** only could be true. Note, however that **i** and **iii** could not simultaneously be true.

65. **(a)**

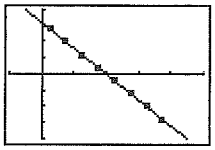


[−1, 5] by [−10, 80]

(b)

t interval	avg. vel.
[0, 0.5]	$\frac{38-10}{0.5-0} = 56$
[0.5, 1]	$\frac{58-38}{1-0.5} = 40$
[1, 1.5]	$\frac{70-58}{1.5-1} = 24$
[1.5, 2]	$\frac{74-70}{2-1.5} = 8$
[2, 2.5]	$\frac{70-74}{2.5-2} = -8$
[2.5, 3]	$\frac{58-70}{3-2.5} = -24$
[3, 3.5]	$\frac{38-58}{3.5-3} = -40$
[3.5, 4]	$\frac{10-38}{4-3.5} = -56$

(c)



[−1, 5] by [−80, 80]

(d) Average velocity is a good approximation to velocity.

66. **(a)** $\frac{d}{dx}[\sqrt{x}f(x)] = \sqrt{x}f'(x) + \frac{1}{2\sqrt{x}}f(x)$

At $x = 1$, the derivative is

$$\sqrt{1}f'(1) + \frac{1}{2\sqrt{1}}f(1) = 1\left(\frac{1}{5}\right) + \left(\frac{1}{2}\right)(-3) = -\frac{13}{10}.$$

(b) $\frac{d}{dx}\sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}}f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$

At $x = 0$, the derivative is $\frac{f'(0)}{2\sqrt{f(0)}} = -\frac{2}{2\sqrt{9}} = -\frac{1}{3}$.

(c) $\frac{d}{dx}f(\sqrt{x}) = f'(\sqrt{x})\frac{d}{dx}\sqrt{x} = \frac{f'(\sqrt{x})}{2\sqrt{x}}$

At $x = 1$, the derivative is $\frac{f'(\sqrt{1})}{2\sqrt{1}} = \frac{f'(1)}{2} = \frac{\frac{1}{5}}{2} = \frac{1}{10}$.

(d) $\frac{d}{dx}f(1 - 5\tan x) = f'(1 - 5\tan x)(-5\sec^2 x)$

At $x = 0$, the derivative is

$$f'(1 - 5\tan 0)(-5\sec^2 0) = f'(1)(-5) = \left(\frac{1}{5}\right)(-5)$$
$$= -1.$$

(e) $\frac{d}{dx}\frac{f(x)}{2 + \cos x} = \frac{(2 + \cos x)(f'(x)) - (f(x))(-\sin x)}{(2 + \cos x)^2}$

At $x = 0$, the derivative is

$$\frac{(2 + \cos 0)(f'(0)) - (f(0))(-\sin 0)}{(2 + \cos 0)^2} = \frac{3f'(0)}{3^2} = -\frac{2}{3}.$$

(f) $\frac{d}{dx}\left[10\sin\left(\frac{\pi x}{2}\right)f^2(x)\right]$

$$= 10\left(\sin\frac{\pi x}{2}\right)(2f(x)f'(x)) + 10f^2(x)\left(\cos\frac{\pi x}{2}\right)\left(\frac{\pi}{2}\right)$$
$$= 20f(x)f'(x)\sin\frac{\pi x}{2} + 5\pi f^2(x)\cos\frac{\pi x}{2}$$

At $x = 1$, the derivative is

$$20f(1)f'(1)\sin\frac{\pi}{2} + 5\pi f^2(1)\cos\frac{\pi}{2}$$
$$= 20(-3)\left(\frac{1}{5}\right)(1) + 5\pi(-3)^2(0)$$
$$= -12.$$

67. **(a)** $\frac{d}{dx}[3f(x) - g(x)] = 3f'(x) - g'(x)$

At $x = -1$, the derivative is

$$3f'(-1) - g'(-1) = 3(2) - 1 = 5.$$

(b) $\frac{d}{dx}[f^2(x)g^3(x)]$

$$= f^2(x)\cdot 3g^2(x)g'(x) + g^3(x)\cdot 2f(x)f'(x)$$
$$= f(x)g^2(x)[3f(x)g'(x) + 2g(x)f'(x)]$$

At $x = 0$, the derivative is

$$f(0)g^2(0)[3f(0)g'(0) + 2g(0)f'(0)]$$
$$= (-1)(-3)^2[3(-1)(4) + 2(-3)(-2)]$$
$$= -9[-12 + 12] = 0.$$

(c) $\frac{d}{dx}g(f(x)) = g'(f(x))f'(x)$

At $x = -1$, the derivative is

$$g'(f(-1))f'(-1) = g'(0)f'(-1) = (4)(2) = 8.$$

(d) $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

At $x = -1$, the derivative is

$$f'(g(-1))g'(-1) = f'(-1)g'(-1) = (2)(1) = 2.$$

(e) $\frac{d}{dx}\frac{f(x)}{g(x) + 2} = \frac{(g(x) + 2)f'(x) - f(x)g'(x)}{(g(x) + 2)^2}$

At $x = 0$, the derivative is

$$\frac{(g(0) + 2)f'(0) - f(0)g'(0)}{(g(0) + 2)^2} = \frac{(-3 + 2)(-2) - (-1)(4)}{(-3 + 2)^2}$$
$$= 6.$$

(f) $\frac{d}{dx}g(x+f(x)) = g'(x+f(x))\frac{d}{dx}(x+f(x))$

$= g'(x+f(x))(1+f'(x))$

At $x = 0$, the derivative is $g'(0+f(0))[1+f'(0)]$

$= g'(0-1)[1+(-2)] = (1)(-1) = -1$

68. $\frac{dw}{ds} = \frac{dw}{dr}\frac{dr}{ds} = \frac{d}{dr}[\sin(\sqrt{r}-2)]\frac{d}{ds}\left[8\sin\left(s+\frac{\pi}{6}\right)\right]$

$= \left[\cos(\sqrt{r}-2)\frac{1}{2\sqrt{r}}\right]\left[8\cos\left(s+\frac{\pi}{6}\right)\right]$

At $s = 0$, we have $r = 8\sin\left(0+\frac{\pi}{6}\right) = 4$ and so

$\frac{dw}{ds} = \left[\cos(\sqrt{4}-2)\frac{1}{2\sqrt{4}}\right]\left[8\cos\left(0+\frac{\pi}{6}\right)\right]$

$= \left(\frac{\cos 0}{4}\right)\left(8\cos\frac{\pi}{6}\right) = \left(\frac{1}{4}\right)\left(\frac{8\sqrt{3}}{2}\right) = \sqrt{3}$

69. Solving $\theta^2 t + \theta = 1$ for t, we have

$t = \frac{1-\theta}{\theta^2} = \theta^{-2} - \theta^{-1}$, and we may write:

$$\frac{dr}{d\theta} = \frac{dr}{dt}\frac{dt}{d\theta}$$

$$\frac{d}{d\theta}(\theta^2+7)^{1/3} = \frac{dr}{dt}\frac{d}{d\theta}(\theta^{-2}-\theta^{-1})$$

$$\frac{1}{3}(\theta^2+7)^{-2/3}(2\theta) = \left(\frac{dr}{dt}\right)(-2\theta^{-3}+\theta^{-2})$$

$$\frac{dr}{dt} = \frac{2\theta(\theta^2+7)^{-2/3}}{3(-2\theta^{-3}+\theta^{-2})} = \frac{2\theta^4(\theta^2+7)^{-2/3}}{3(\theta-2)}$$

At $t = 0$, we may solve $\theta^2 t + \theta = 1$ to obtain $\theta = 1$,

and so $\frac{dr}{dt} = \frac{2(1)^4(1^2+7)^{-2/3}}{3(1-2)} = \frac{2(8)^{-2/3}}{-3} = -\frac{1}{6}$.

70. (a) One possible answer:

$x(t) = 10\cos\left(t+\frac{\pi}{4}\right)$, $y(t) = 1$

(b) $s(0) = 10\cos\frac{\pi}{4} = 5\sqrt{2}$

(c) Farthest left:

When $\cos\left(t+\frac{\pi}{4}\right) = -1$, we have $s(t) = -10$.

Farthest right:

When $\cos\left(t+\frac{\pi}{4}\right) = 1$, we have $s(t) = 10$.

(d) Since $\cos\frac{\pi}{2} = 0$, the particle first reaches the origin at $t = \frac{\pi}{4}$. The velocity is given by $v(t) = -10\sin\left(t+\frac{\pi}{4}\right)$, so the velocity at $t = \frac{\pi}{4}$ is $-10\sin\frac{\pi}{2} = -10$, and the speed at $t = \frac{\pi}{4}$ is $|-10| = 10$. The acceleration is given by $a(t) = -10\cos\left(t+\frac{\pi}{4}\right)$, so the acceleration at $t = \frac{\pi}{4}$ is $-10\cos\frac{\pi}{2} = 0$.

71. (a) $\frac{ds}{dt} = \frac{d}{dt}(64t - 16t^2) = 64 - 32t$

$\frac{d^2s}{dt^2} = \frac{d}{dt}(64 - 32t) = -32$

(b) The maximum height is reached when $\frac{ds}{dt} = 0$, which occurs at $t = 2$ sec.

(c) When $t = 0$, $\frac{ds}{dt} = 64$, so the velocity is 64 ft/sec.

(d) Since $\frac{ds}{dt} = \frac{d}{dt}(64t - 2.6t^2) = 64 - 5.2t$, the maximum height is reached at $t = \frac{64}{5.2} \approx 12.3$ sec. The maximum height is $s\left(\frac{64}{5.2}\right) \approx 393.8$ ft.

72. (a) Solving $160 = 490t^2$, it takes $\frac{4}{7}$ sec. The average velocity is $\frac{160}{\frac{4}{7}} = 280$ cm/sec.

(b) Since $v(t) = \frac{ds}{dt} = 980t$, the velocity is $(980)\left(\frac{4}{7}\right) = 560$ cm/sec. Since $a(t) = \frac{dv}{dt} = 980$, the acceleration is 980 cm/sec^2.

73. $\frac{dV}{dx} = \frac{d}{dx}\left[\pi\left(10-\frac{x}{3}\right)x^2\right] = \frac{d}{dx}\left[\pi\left(10x^2 - \frac{1}{3}x^3\right)\right]$

$= \pi(20x - x^2)$

74. (a) $r(x) = \left(3 - \frac{x}{40}\right)^2 x = 9x - \frac{3}{20}x^2 + \frac{1}{1600}x^3$

(b) The marginal revenue is

$r'(x) = 9 - \frac{3}{10}x + \frac{3}{1600}x^2$

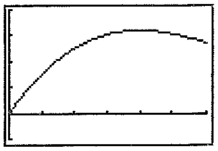
$= \frac{3}{1600}(x^2 - 160x + 4800)$

$= \frac{3}{1600}(x-40)(x-120)$,

which is zero when $x = 40$ or $x = 120$. Since the bus holds only 60 people, we require $0 \le x \le 60$. The marginal revenue is 0 when there are 40 people, and the corresponding fare is $p(40) = \left(3 - \frac{40}{40}\right)^2 = \$4.00$.

74. continued

(c) One possible answer:
If the current ridership is less than 40, then the proposed plan may be good. If the current ridership is greater than or equal to 40, then the plan is not a good idea. Look at the graph of $y = r(x)$.



$[0, 60]$ by $[-50, 200]$

75. (a) Since $x = \tan\theta$, we have
$\frac{dx}{dt} = (\sec^2\theta)\frac{d\theta}{dt} = -0.6\sec^2\theta$. At point A, we have $\theta = 0$ and $\frac{dx}{dt} = -0.6\sec^2 0 = -0.6$ km/sec.

(b) $0.6\frac{\text{rad}}{\text{sec}} \cdot \frac{1 \text{ revolution}}{2\pi \text{ rad}} \cdot \frac{60 \text{ sec}}{1 \text{ min}} = \frac{18}{\pi}$ revolutions per minute or approximately 5.73 revolutions per minute.

76. Let $f(x) = \sin(x - \sin x)$. Then
$f'(x) = \cos(x - \sin x)\frac{d}{dx}(x - \sin x)$
$= \cos(x - \sin x)(1 - \cos x)$. This derivative is zero when $\cos(x - \sin x) = 0$ (which we need not solve) or when $\cos x = 1$, which occurs at $x = 2k\pi$ for integers k. For each of these values, $f(x) = f(2k\pi) = \sin(2k\pi - \sin 2k\pi)$
$= \sin(2k\pi - 0) = 0$. Thus, $f(x) = f'(x) = 0$ for $x = 2k\pi$, which means that the graph has a horizontal tangent at each of these values of x.

77. $y'(r) = \frac{d}{dr}\left(\frac{1}{2rl}\sqrt{\frac{T}{\pi d}}\right) = \left(\frac{1}{2l}\sqrt{\frac{T}{\pi d}}\right)\frac{d}{dr}\left(\frac{1}{r}\right) = -\frac{1}{2r^2l}\sqrt{\frac{T}{\pi d}}$

$y'(l) = \frac{d}{dl}\left(\frac{1}{2rl}\sqrt{\frac{T}{\pi d}}\right) = \left(\frac{1}{2r}\sqrt{\frac{T}{\pi d}}\right)\frac{d}{dl}\left(\frac{1}{l}\right) = -\frac{1}{2rl^2}\sqrt{\frac{T}{\pi d}}$

$y'(d) = \frac{d}{dd}\left(\frac{1}{2rl}\sqrt{\frac{T}{\pi d}}\right) = \left(\frac{1}{2rl}\sqrt{\frac{T}{\pi}}\right)\frac{d}{dd}(d^{-1/2})$

$= \frac{1}{2rl}\sqrt{\frac{T}{\pi}}\left(-\frac{1}{2}d^{-3/2}\right) = -\frac{1}{4rl}\sqrt{\frac{T}{\pi d^3}}$

$y'(T) = \frac{d}{dT}\left(\frac{1}{2rl}\sqrt{\frac{T}{\pi d}}\right) = \left(\frac{1}{2rl}\sqrt{\frac{1}{\pi d}}\right)\frac{d}{dT}(\sqrt{T})$

$= \frac{1}{2rl}\sqrt{\frac{1}{\pi d}}\left(\frac{1}{2\sqrt{T}}\right) = \frac{1}{4rl\sqrt{\pi d T}}$

Since $y'(r) < 0$, $y'(l) < 0$, and $y'(d) < 0$, increasing r, l, or d would decrease the frequency. Since $y'(T) > 0$, increasing T would increase the frequency.

78. (a) $P(0) = \frac{200}{1 + e^5} \approx 1$ student

(b) $\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{200}{1 + e^{5-t}} = \frac{200}{1} = 200$ students

(c) $P'(t) = \frac{d}{dt}200(1 + e^{5-t})^{-1}$

$= -200(1 + e^{5-t})^{-2}(e^{5-t})(-1)$

$= \frac{200e^{5-t}}{(1 + e^{5-t})^2}$

$$P''(t) = \frac{(1 + e^{5-t})^2(200e^{5-t})(-1) - (200e^{5-t})(2)(1 + e^{5-t})(e^{5-t})(-1)}{(1 + e^{5-t})^4}$$

$$= \frac{(1 + e^{5-t})(-200e^{5-t}) + 400(e^{5-t})^2}{(1 + e^{5-t})^3}$$

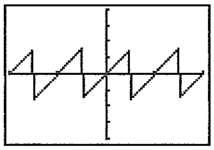
$$= \frac{(200e^{5-t})(e^{5-t} - 1)}{(1 + e^{5-t})^3}$$

Since $P'' = 0$ when $t = 5$, the critical point of $y = P'(t)$ occurs at $t = 5$. To confirm that this corresponds to the maximum value of $P'(t)$, note that $P''(t) > 0$ for $t < 5$ and $P''(t) < 0$ for $t > 5$. The maximum rate occurs at $t = 5$, and this rate is

$P'(5) = \frac{200e^0}{(1 + e^0)^2} = \frac{200}{2^2} = 50$ students per day.

Note: This problem can also be solved graphically.

79.



$[-\pi, \pi]$ by $[-4, 4]$

(a) $x \neq k\frac{\pi}{4}$, where k is an odd integer

(b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(c) Where it's not defined, at $x = k\frac{\pi}{4}$, k an odd integer

(d) It has period $\frac{\pi}{2}$ and continues to repeat the pattern seen in this window.