CHAPTER 6
Differential Equations

Section 6.1 Slope Fields and Euler's Method

1. Differential equation: $y' = 4y$

Solution: $y = Ce^{4x}$

Check: $y' = 4Ce^{4x} = 4y$

2. Differential Equation: $3y' + 5y = -e^{-2x}$

Solution: $y = e^{-2x}$

$y' = -2e^{-2x}$

Check: $3(-2e^{-2x}) + 5(e^{-2x}) = -e^{-2x}$

3. Differential equation: $y' = \dfrac{2xy}{x^2 - y^2}$

Solution: $x^2 + y^2 = Cy$

Check: $2x + 2yy' = Cy'$

$$y' = \frac{-2x}{(2y - C)}$$

$$y' = \frac{-2xy}{2y^2 - Cy}$$

$$= \frac{-2xy}{2y^2 - (x^2 + y^2)}$$

$$= \frac{-2xy}{y^2 - x^2}$$

$$= \frac{2xy}{x^2 - y^2}$$

4. Differential Equation: $\dfrac{dy}{dx} = \dfrac{xy}{y^2 - 1}$

Solution: $y^2 - 2\ln y = x^2$

Check: $2yy' - \dfrac{2}{y}y' = 2x$

$$\left(y - \frac{1}{y}\right)y' = x$$

$$y' = \frac{x}{y - \frac{1}{y}}$$

$$y' = \frac{xy}{y^2 - 1}$$

5. Differential Equation: $y'' + y = 0$

Solution: $y = C_1 \sin x - C_2 \cos x$

$y' = C_1 \cos x + C_2 \sin x$

$y'' = -C_1 \sin x + C_2 \cos x$

Check: $y'' + y = (-C_1 \sin x + C_2 \cos x) + (C_1 \sin x - C_2 \cos x) = 0$

6. Differential equation: $y'' + 2y' + 2y = 0$

Solution: $y = C_1 e^{-x} \cos x + C_2 e^{-x} \sin x$

Check:

$$y' = -(C_1 + C_2)e^{-x} \sin x + (-C_1 + C_2)e^{-x} \cos x$$

$$y'' = 2C_1 e^{-x} \sin x - 2C_2 e^{-x} \cos x$$

$$y'' + 2y' + 2y = 2C_1 e^{-x} \sin x - 2C_2 e^{-x} \cos x +$$

$$2\left(-(C_1 + C_2)e^{-x} \sin x + (-C_1 + C_2)e^{-x} \cos x\right) + 2\left(C_1 e^{-x} \cos x + C_2 e^{-x} \sin x\right)$$

$$= (2C_1 - 2C_1 - 2C_2 + 2C_2)e^{-x} \sin x + (-2C_2 - 2C_1 + 2C_2 + 2C_1)e^{-x} \cos x = 0$$

7. Differential Equation: $y'' + y = \tan x$

Solution: $y = -\cos x \ln|\sec x + \tan x|$

$$y' = (-\cos x)\frac{1}{\sec x + \tan x}(\sec x \cdot \tan x + \sec^2 x) + \sin x \ln|\sec x + \tan x|$$

$$= \frac{(-\cos x)}{\sec x + \tan x}(\sec x)(\tan x + \sec x) + \sin x \ln|\sec x + \tan x|$$

$$= -1 + \sin x \ln|\sec x + \tan x|$$

$$y'' = (\sin x)\frac{1}{\sec x + \tan x}(\sec x \cdot \tan x + \sec^2 x) + \cos x \ln|\sec x + \tan x|$$

$$= (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x|$$

Check: $y'' + y = (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x| - \cos x \ln|\sec x + \tan x| = \tan x.$

8. Differential Equation: $y'' + 4y' = 2e^x$

Solution: $y = \frac{2}{5}(e^{-4x} + e^x)$

$$y' = \frac{2}{5}(-4e^{-4x} + e^x) = -\frac{8}{5}e^{-4x} + \frac{2}{5}e^x$$

$$y'' = \frac{32}{5}e^{-4x} + \frac{2}{5}e^x$$

Check: $y'' + 4y' = \left(\frac{32}{5}e^{-4x} + \frac{2}{5}e^x\right) + 4\left(-\frac{8}{5}e^{-4x} + \frac{2}{5}e^x\right) = \left(\frac{2}{5} + \frac{8}{5}\right)e^x = 2e^x$

9. $y = \sin x \cos x - \cos^2 x$

$y' = -\sin^2 x + \cos^2 x + 2 \cos x \sin x$

$= -1 + 2\cos^2 x + \sin 2x$

Differential Equation:

$$2y + y' = 2(\sin x \cos x - \cos^2 x) + (-1 + 2\cos^2 x + \sin 2x)$$

$$= 2 \sin x \cos x - 1 + \sin 2x$$

$$= 2 \sin 2x - 1$$

Initial condition $\left(\frac{\pi}{4}, 0\right)$:

$$\sin\frac{\pi}{4}\cos\frac{\pi}{4} - \cos^2\frac{\pi}{4} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} - \left(\frac{\sqrt{2}}{2}\right)^2 = 0$$

10. $y = \frac{1}{2}x^2 - 2\cos x - 3$

$y' = x + 2\sin x$

Differential equation: $y' = x + 2\sin x$

Initial condition $(0, -5)$: $0 - 2\cos 0 - 3 = -5$

11. $y = 4e^{-6x^2}$

$y' = 4e^{-6x^2}(-12x) = -48xe^{-6x^2}$

Differential equation:

$$y' = -12xy = -12x\left(4e^{-6x^2}\right) = -48xe^{-6x^2}$$

Initial condition $(0, 4)$: $4e^0 = 4$

12. $y = e^{-\cos x}$

$y' = e^{-\cos x}(\sin x) = \sin x \cdot e^{-\cos x}$

Differential Equation:

$$y' = \sin x \cdot e^{-\cos x} = \sin x(y) = y \sin x$$

Initial condition $\left(\frac{\pi}{2}, 1\right)$: $e^{-\cos(\pi/2)} = e^0 = 1$

In Exercises 13–20, the differential equation is $y^{(4)} - 16y = 0$.

13.
$$y = 3\cos x$$
$$y^{(4)} = 3\cos x$$
$$y^{(4)} - 16y = -45\cos x \neq 0,$$
No

14.
$$y = 2\sin x$$
$$y^{(4)} = 2\sin x$$
$$y^{(4)} - 16y = 2\sin x - 16(2\sin x) \neq 0$$
No

15.
$$y = 3\cos 2x$$
$$y^{(4)} = 48\cos 2x$$
$$y^{(4)} - 16y = 48\cos 2x - 48\cos 2x = 0,$$
Yes

16.
$$y = 3\sin 2x$$
$$y^{(4)} = 48\sin 2x$$
$$y^{(4)} - 16y = 48\sin 2x - 16(3\sin 2x) = 0$$
Yes

17.
$$y = e^{-2x}$$
$$y^{(4)} = 16e^{-2x}$$
$$y^{(4)} - 16y = 16e^{-2x} - 16e^{-2x} = 0,$$
Yes

18.
$$y = 5\ln x$$
$$y^{(4)} = -\frac{30}{x^4}$$
$$y^{(4)} - 16y = -\frac{30}{x^4} - 80\ln x \neq 0,$$
No

19.
$$y = C_1e^{2x} + C_2e^{-2x} + C_3\sin 2x + C_4\cos 2x$$
$$y^{(4)} = 16C_1e^{2x} + 16C_2e^{-2x} + 16C_3\sin 2x + 16C_4\cos 2x$$
$$y^{(4)} - 16y = 0,$$
Yes

20.
$$y = 3e^{2x} - 4\sin 2x$$
$$y^{(4)} = 48e^{2x} - 64\sin 2x$$
$$y^{(4)} - 16y = \left(48e^{2x} - 64\sin 2x\right) - 16\left(3e^{2x} - 4\sin 2x\right) = 0,$$
Yes

In Exercises 21–28, the differential equation is $xy' - 2y = x^3e^x$.

21. $y = x^2$, $y' = 2x$
$$xy' - 2y = x(2x) - 2(x^2) = 0 \neq x^3e^x,$$
No

22. $y = x^3$, $y' = 3x^2$
$$xy' - 2y = x(3x^2) - 2x^3 = x^3 \neq x^3e^x$$
No

23. $y = x^2e^x$, $y' = x^2e^x + 2xe^x = e^x(x^2 + 2x)$
$$xy' - 2y = x\left(e^x(x^2 + 2x)\right) - 2(x^2e^x) = x^3e^x,$$
Yes

24. $y = x^2(2 + e^x)$, $y' = x^2(e^x) + 2x(2 + e^x)$
$$xy' - 2y = x\left[x^2e^x + 2xe^x + 4x\right] - 2\left[x^2e^x + 2x^2\right]$$
$$= x^3e^x,$$
Yes

25. $y = \sin x$, $y' = \cos x$
$$xy' - 2y = x(\cos x) - 2(\sin x) \neq x^3e^x,$$
No

26. $y = \cos x$, $y' = -\sin x$
$$xy' - 2y = x(-\sin x) - 2\cos x \neq x^3e^x$$
No

27. $y = \ln x$, $y' = \frac{1}{x}$
$$xy' - 2y = x\left(\frac{1}{x}\right) - 2\ln x \neq x^3e^x,$$
No

28. $y = x^2e^x - 5x^2$, $y' = x^2e^x + 2xe^x - 10x$
$$xy' - 2y = x\left[x^2e^x + 2xe^x - 10x\right] - 2\left[x^2e^x - 5x^2\right]$$
$$= x^3e^x,$$
Yes

29. $y = Ce^{-x/2}$ passes through $(0, 3)$.

$3 = Ce^0 = C \Rightarrow C = 3$

Particular solution: $y = 3e^{-x/2}$

30. $y(x^2 + y) = C$ passes through $(0, 2)$.

$2(0 + 2) = C \Rightarrow C = 4$

Particular solution: $y(x^2 + y) = 4$

31. $y^2 = Cx^3$ passes through $(4, 4)$.

$16 = C(64) \Rightarrow C = \frac{1}{4}$

Particular solution: $y^2 = \frac{1}{4}x^3$ or $4y^2 = x^3$

32. $2x^2 - y^2 = C$ passes through $(3, 4)$.

$2(9) - 16 = C \Rightarrow C = 2$

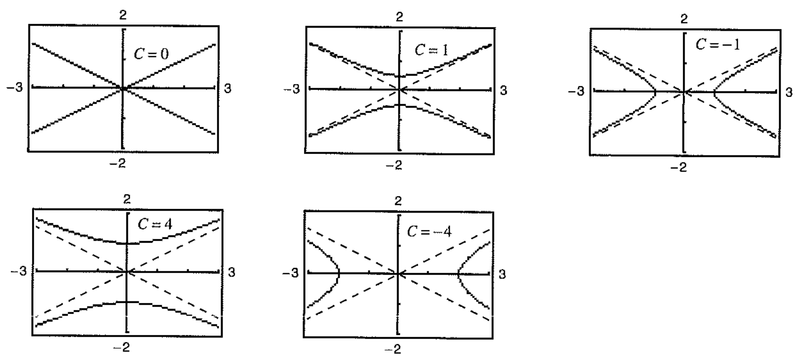
Particular solution: $2x^2 - y^2 = 2$

33. Differential equation: $4yy' - x = 0$

General solution: $4y^2 - x^2 = C$

Particular solutions: $C = 0$, Two intersecting lines

$C = \pm 1, C = \pm 4$, Hyperbolas

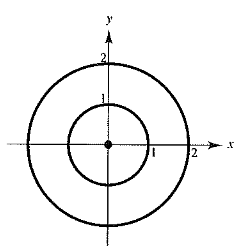


34. Differential equation: $yy' + x = 0$

General solution: $x^2 + y^2 = C$

Particular solutions: $C = 0$, Point

$C = 1, C = 4$, Circles



35. Differential equation: $y' + 2y = 0$

General solution: $y = Ce^{-2x}$

$y' + 2y = C(-2)e^{-2x} + 2(Ce^{-2x}) = 0$

Initial condition $(0, 3)$: $3 = Ce^0 = C$

Particular solution: $y = 3e^{-2x}$

36. Differential equation: $3x + 2yy' = 0$

General solution: $3x^2 + 2y^2 = C$

$6x + 4yy' = 0$

$2(3x + 2yy') = 0$

$3x + 2yy' = 0$

Initial condition $(1, 3)$:

$3(1)^2 + 2(3)^2 = 3 + 18 = 21 = C$

Particular solution: $3x^2 + 2y^2 = 21$

37. Differential equation: $y'' + 9y = 0$

General solution: $y = C_1 \sin 3x + C_2 \cos 3x$

$$y' = 3C_1 \cos 3x - 3C_2 \sin 3x,$$
$$y'' = -9C_1 \sin 3x - 9C_2 \cos 3x$$
$$y'' + 9y = (-9C_1 \sin 3x - 9C_2 \cos 3x) + 9(C_1 \sin 3x + C_2 \cos 3x) = 0$$

Initial conditions $\left(\frac{\pi}{6}, 2\right)$ and $y' = 1$ when $x = \frac{\pi}{6}$:

$$2 = C_1 \sin\left(\frac{\pi}{2}\right) + C_2 \cos\left(\frac{\pi}{2}\right) \Rightarrow C_1 = 2$$
$$y' = 3C_1 \cos 3x - 3C_2 \sin 3x$$
$$1 = 3C_1 \cos\left(\frac{\pi}{2}\right) - 3C_2 \sin\left(\frac{\pi}{2}\right) = -3C_2 \Rightarrow C_2 = -\frac{1}{3}$$

Particular solution: $y = 2 \sin 3x - \frac{1}{3} \cos 3x$

38. Differential equation: $xy'' + y' = 0$

General solution: $y = C_1 + C_2 \ln x$

$$y' = C_2\left(\frac{1}{x}\right), y'' = -C_2\left(\frac{1}{x^2}\right)$$
$$xy'' + y' = x\left(-C_2\frac{1}{x^2}\right) + C_2\frac{1}{x} = 0$$

Initial conditions $(2, 0)$ and $y' = \frac{1}{2}$ when $x = 2$:

$$0 = C_1 + C_2 \ln 2$$
$$y' = \frac{C_2}{x}$$
$$\frac{1}{2} = \frac{C_2}{2} \Rightarrow C_2 = 1, C_1 = -\ln 2$$

Particular solution: $y = -\ln 2 + \ln x = \ln \frac{x}{2}$

39. Differential equation: $x^2y'' - 3xy' + 3y = 0$

General solution: $y = C_1x + C_2x^3$

$$y' = C_1 + 3C_2x^2, y'' = 6C_2x$$
$$x^2y'' - 3xy' + 3y = x^2(6C_2x) - 3x(C_1 + 3C_2x^2) + 3(C_1x + C_2x^3) = 0$$

Initial conditions $(2, 0)$ and $y' = 4$ when $x = 2$:

$$0 = 2C_1 + 8C_2$$
$$y' = C_1 + 3C_2x^2$$
$$4 = C_1 + 12C_2$$
$$\left.\begin{array}{l} C_1 + 4C_2 = 0 \\ C_1 + 12C_2 = 4 \end{array}\right\} C_2 = \tfrac{1}{2}, C_1 = -2$$

Particular solution: $y = -2x + \frac{1}{2}x^3$

40. Differential equation: $9y'' - 12y' + 4y = 0$

General solution: $y = e^{2x/3}(C_1 + C_2x)$

$$y' = \frac{2}{3}e^{2x/3}(C_1 + C_2x) + C_2e^{2x/3} = e^{2x/3}\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right)$$

$$y'' = \frac{2}{3}e^{2x/3}\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right) + e^{2x/3}\frac{2}{3}C_2 = \frac{2}{3}e^{2x/3}\left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2x\right)$$

$$9y'' - 12y' + 4y = 9\left(\frac{2}{3}e^{2x/3}\right)\left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2x\right) - 12\left(e^{2x/3}\right)\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right) + 4\left(e^{2x/3}\right)(C_1 + C_2x) = 0$$

Initial conditions $(0, 4)$ and $(3, 0)$:

$$0 = e^2(C_1 + 3C_2)$$

$$4 = (1)(C_1 + 0) \Rightarrow C_1 = 4$$

$$0 = e^2(4 + 3C_2) \Rightarrow C_2 = -\frac{4}{3}$$

Particular solution: $y = e^{2x/3}\left(4 - \frac{4}{3}x\right)$

41. $\dfrac{dy}{dx} = 6x^2$

$$y = \int 6x^2\,dx = 2x^3 + C$$

42. $\dfrac{dy}{dx} = 2x^3 - 3x$

$$y = \int\left(2x^3 - 3x\right)dx = \frac{x^4}{2} - \frac{3}{2}x^2 + C$$

43. $\dfrac{dy}{dx} = \dfrac{x}{1 + x^2}$

$$y = \int\frac{x}{1 + x^2}\,dx = \frac{1}{2}\ln\left(1 + x^2\right) + C$$

$\left(u = 1 + x^2, du = 2x\,dx\right)$

44. $\dfrac{dy}{dx} = \dfrac{e^x}{4 + e^x}$

$$y = \int\frac{e^x}{4 + e^x}\,dx = \ln\left(4 + e^x\right) + C$$

45. $\dfrac{dy}{dx} = \dfrac{x - 2}{x} = 1 - \dfrac{2}{x}$

$$y = \int\left(1 - \frac{2}{x}\right)dx$$

$$= x - 2\ln|x| + C = x - \ln x^2 + C$$

46. $\dfrac{dy}{dx} = x\cos x^2$

$$y = \int x\cos\left(x^2\right)dx = \frac{1}{2}\sin\left(x^2\right) + C$$

$\left(u = x^2, du = 2x\,dx\right)$

47. $\dfrac{dy}{dx} = \sin 2x$

$$y = \int\sin 2x\,dx = -\frac{1}{2}\cos 2x + C$$

$(u = 2x, du = 2\,dx)$

48. $\dfrac{dy}{dx} = \tan^2 x = \sec^2 x - 1$

$$y = \int\left(\sec^2 x - 1\right)dx = \tan x - x + C$$

49. $\dfrac{dy}{dx} = x\sqrt{x - 6}$

Let $u = \sqrt{x - 6}$, then $x = u^2 + 6$ and $dx = 2u\,du$.

$$\begin{aligned}
y = \int x\sqrt{x - 6}\,dx &= \int\left(u^2 + 6\right)(u)(2u)\,du \\
&= 2\int\left(u^4 + 6u^2\right)du \\
&= 2\left(\frac{u^5}{5} + 2u^3\right) + C \\
&= \frac{2}{5}(x - 6)^{5/2} + 4(x - 6)^{3/2} + C \\
&= \frac{2}{5}(x - 6)^{3/2}(x - 6 + 10) + C \\
&= \frac{2}{5}(x - 6)^{3/2}(x + 4) + C
\end{aligned}$$

50. $\frac{dy}{dx} = 2x\sqrt{3 - x}$

Let $u = \sqrt{3 - x}$, then $x = 3 - u^2$ and $dx = -2u\,du$

$$y = \int 2x\sqrt{3 - x}\,dx = \int 2(3 - u^2)u(-2u\,du)$$
$$= \int (4u^4 - 12u^2)du$$
$$= \frac{4u^5}{5} - 4u^3 + C$$
$$= \frac{4}{5}(3 - x)^{5/2} - 4(3 - x)^{3/2} + C$$
$$= \frac{4}{5}(3 - x)^{3/2}(3 - x - 5) + C$$
$$= -\frac{4}{5}(3 - x)^{3/2}(x + 2) + C$$

51. $\frac{dy}{dx} = xe^{x^2}$

$$y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$$

$(u = x^2, du = 2x\,dx)$

52. $\frac{dy}{dx} = 5e^{-x/2}$

$$y = \int 5e^{-x/2}\,dx = 5(-2)\int e^{-x/2}\left(-\frac{1}{2}\right)dx = -10e^{-x/2} + C$$

53.

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	-4	Undef.	0	1	$\frac{4}{3}$	2

54.

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	6	2	4	2	2	0

55.

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	$-2\sqrt{2}$	-2	0	0	$-2\sqrt{2}$	-8

56.

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	$\sqrt{3}$	0	$-\sqrt{3}$	$-\sqrt{3}$	0	$\sqrt{3}$

57. $\frac{dy}{dx} = \sin 2x$

For $x = 0$, $\frac{dy}{dx} = 0$. Matches (b).

58. $\frac{dy}{dx} = \frac{1}{2}\cos x$

For $x = 0$, $\frac{dy}{dx} = \frac{1}{2}$. Matches (c).

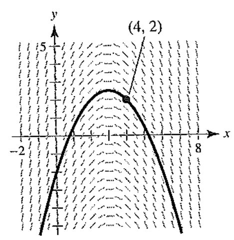
59. $\frac{dy}{dx} = e^{-2x}$

As $x \to \infty$, $\frac{dy}{dx} \to 0$. Matches (d).

60. $\frac{dy}{dx} = \frac{1}{x}$

For $x = 0$, $\frac{dy}{dx}$ is undefined (vertical tangent). Matches (a).

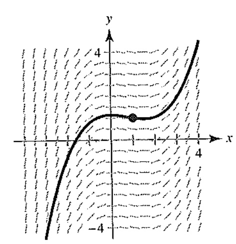
61. (a), (b)



(c) As $x \to \infty$, $y \to -\infty$

As $x \to -\infty$, $y \to -\infty$

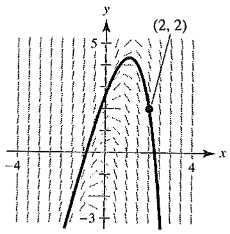
62. (a), (b)



(c) As $x \to \infty$, $y \to \infty$

As $x \to -\infty$, $y \to -\infty$

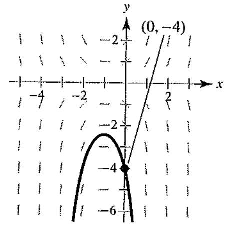
63. (a), (b)



(c) As $x \to \infty$, $y \to -\infty$

As $x \to -\infty$, $y \to -\infty$

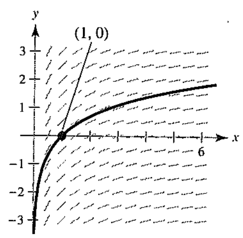
64. (a), (b)



(c) As $x \to \infty$, $y \to -\infty$

As $x \to -\infty$, $y \to -\infty$

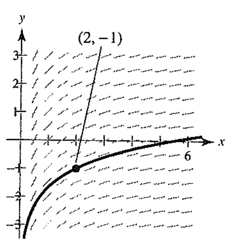
65. (a) $y' = \dfrac{1}{x}, (1, 0)$



As $x \to \infty$, $y \to \infty$

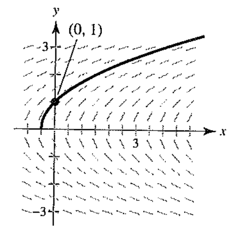
[Note: The solution is $y = \ln x$.]

(b) $y' = \dfrac{1}{x}, (2, -1)$



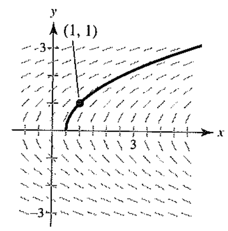
As $x \to \infty$, $y \to \infty$

66. (a) $y' = \dfrac{1}{y}, (0, 1)$



As $x \to \infty$, $y \to \infty$

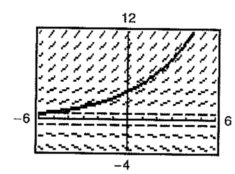
(b) $y' = \dfrac{1}{y}, (1, 1)$



As $x \to \infty$, $y \to \infty$

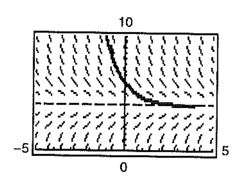
67. $\dfrac{dy}{dx} = 0.25y, \ y(0) = 4$

(a), (b)



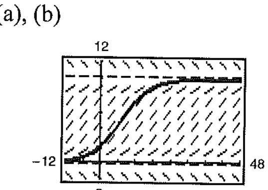
68. $\dfrac{dy}{dx} = 4 - y, \ y(0) = 6$

(a), (b)



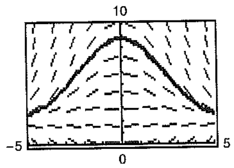
69. $\dfrac{dy}{dx} = 0.02y(10 - y), \ y(0) = 2$

(a), (b)



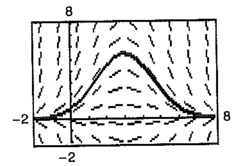
70. $\dfrac{dy}{dx} = 0.2x(2 - y),\ y(0) = 9$

(a), (b)



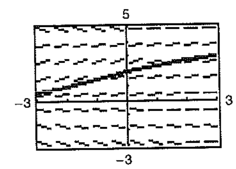
71. $\dfrac{dy}{dx} = 0.4y(3 - x),\ y(0) = 1$

(a), (b)



72. $\dfrac{dy}{dx} = \dfrac{1}{2}e^{-x/8}\sin\dfrac{\pi y}{4},\ y(0) = 2$

(a), (b)



73. $y' = x + y, \quad y(0) = 2, \quad n = 10, \quad h = 0.1$

$y_1 = y_0 + hF(x_0, y_0) = 2 + (0.1)(0 + 2) = 2.2$

$y_2 = y_1 + hF(x_1, y_1) = 2.2 + (0.1)(0.1 + 2.2) = 2.43$, etc.

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	2	2.2	2.43	2.693	2.992	3.332	3.715	4.146	4.631	5.174	5.781

74. $y' = x + y, \quad y(0) = 2, \quad n = 20, \quad h = 0.05$

$y_1 = y_0 + hF(x_0, y_0) = 2 + (0.05)(0 + 2) = 2.1$

$y_2 = y_1 + hF(x_1, y_1) = 2.1 + (0.05)(0.05 + 2.1) = 2.2075$, etc.

The table shows the values for $n = 0, 2, 4, \ldots, 20$.

n	0	2	4	6	8	10	12	14	16	18	20
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	2	2.208	2.447	2.720	3.032	3.387	3.788	4.240	4.749	5.320	5.960

75. $y' = 3x - 2y, \quad y(0) = 3, \quad n = 10, \quad h = 0.05$

$y_1 = y_0 + hF(x_0, y_0) = 3 + (0.05)(3(0) - 2(3)) = 2.7$

$y_2 = y_1 + hF(x_1, y_1) = 2.7 + (0.05)(3(0.05) + 2(2.7)) = 2.4375$, etc.

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
y_n	3	2.7	2.438	2.209	2.010	1.839	1.693	1.569	1.464	1.378	1.308

76. $y' = 0.5x(3 - y), \quad y(0) = 1, \quad n = 5, \quad h = 0.4$

$y_1 = y_0 + hF(x_0, y_0) = 1 + (0.4)(0.5(0)(3 - 1)) = 1$

$y_2 = y_1 + hF(x_1, y_1) = 1 + (0.4)(0.5(0.4)(3 - 1)) = 1.16$, etc.

n	0	1	2	3	4	5
x_n	0	0.4	0.8	1.2	1.6	2.0
y_n	1	1	1.16	1.454	1.825	2.201

77. $y' = e^{xy}, \quad y(0) = 1, \quad n = 10, \quad h = 0.1$

$y_1 = y_0 + hF(x_0, y_0) = 1 + (0.1)e^{0(1)} = 1.1$

$y_2 = y_1 + hF(x_1, y_1) = 1.1 + (0.1)e^{(0.1)(1.1)} \approx 1.2116$, etc.

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	1	1.1	1.212	1.339	1.488	1.670	1.900	2.213	2.684	3.540	5.958

78. $y' = \cos x + \sin y, y(0) = 5, n = 10, h = 0.1$

$y_1 = y_0 + hF(x_0, y_0) = 5 + (0.1)(\cos 0 + \sin 5) \approx 5.0041$

$y_2 = y_1 + hF(x_1, y_1) = 5.0041 + (0.1)(\cos(0.1) + \sin(5.0041)) \approx 5.0078$, etc.

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	5	5.004	5.008	5.010	5.010	5.007	4.999	4.985	4.965	4.938	4.903

79. $\dfrac{dy}{dx} = y, y = 3e^x, (0, 3)$

x	0	0.2	0.4	0.6	0.8	1
$y(x)$ (exact)	3	3.6642	4.4755	5.4664	6.6766	8.1548
$y(x)$ $(h = 0.2)$	3	3.6000	4.3200	5.1840	6.2208	7.4650
$y(x)$ $(h = 0.1)$	3	3.6300	4.3923	5.3147	6.4308	7.7812

80. $\dfrac{dy}{dx} = \dfrac{2x}{y}, y = \sqrt{2x^2 + 4}, \quad (0, 2)$

x	0	0.2	0.4	0.6	0.8	1
$y(x)$ (exact)	2	2.0199	2.0785	2.1726	2.2978	2.4495
$y(x)$ $(h = 0.2)$	2	2.000	2.0400	2.1184	2.2317	2.3751
$y(x)$ $(h = 0.1)$	2	2.0100	2.0595	2.1460	2.2655	2.4131

81. $\dfrac{dy}{dx} = y + \cos x,\ y = \dfrac{1}{2}(\sin x - \cos x + e^x), \quad (0, 0)$

x	0	0.2	0.4	0.6	0.8	1
$y(x)$ (exact)	0	0.2200	0.4801	0.7807	1.1231	1.5097
$y(x)$ $(h = 0.2)$	0	0.2000	0.4360	0.7074	1.0140	1.3561
$y(x)$ $(h = 0.1)$	0	0.2095	0.4568	0.7418	1.0649	1.4273

82. As h increases (from 0.1 to 0.2), the error increases.

83. $\dfrac{dy}{dt} = -\dfrac{1}{2}(y - 72), \quad (0, 140), h = 0.1$

(a)

t	0	1	2	3
Euler	140	112.7	96.4	86.6

(b) $y = 72 + 68e^{-t/2}$ exact

t	0	1	2	3
Exact	140	113.24	97.016	87.173

(c) $\dfrac{dy}{dt} = -\dfrac{1}{2}(y - 72), \quad (0, 140), h = 0.05$

t	0	1	2	3
Euler	140	112.98	96.7	86.9

The approximations are better using $h = 0.05$.

84. When $x = 0$, $y' = 0$, therefore (d) is not possible.

When $x, y > 0$, $y' < 0$ (decreasing function) therefore (c) is the equation.

85. The general solution is a family of curves that satisfies the differential equation. A particular solution is one member of the family that satisfies given conditions.

86. A slope field for the differential equation $y' = F(x, y)$ consists of small line segments at various points (x, y) in the plane. The line segment equals the slope $y' = F(x, y)$ of the solution y at the point (x, y).

87. Consider $y' = F(x, y)$, $y(x_0) = y_0$. Begin with a point (x_0, y_0) that satisfies the initial condition, $y(x_0) = y_0$. Then, using a step size of h, find the point $(x_1, y_1) = (x_0 + h, y_0 + hF(x_0, y_0))$. Continue generating the sequence of points $(x_{n+1}, y_{n+1}) = (x_n + h, y_n + hF(x_n, y_n))$.

88. $y = Ce^{kx}$

$\dfrac{dy}{dx} = Cke^{kx}$

Because $dy/dx = 0.07y$, you have $Cke^{kx} = 0.07Ce^{kx}$.

So, $k = 0.07$.

C cannot be determined.

89. False. Consider Example 2. $y = x^3$ is a solution to $xy' - 3y = 0$, but $y = x^3 + 1$ is not a solution.

90. True

91. True

92. False. The slope field could represent many different differential equations, such as $y' = 2x + 4y$.

93. $\dfrac{dy}{dx} = -2y,\ y(0) = 4,\ y = 4e^{-2x}$

(a)

x	0	0.2	0.4	0.6	0.8	1
y	4	2.6813	1.7973	1.2048	0.8076	0.5413
y_1	4	2.5600	1.6384	1.0486	0.6711	0.4295
y_2	4	2.4000	1.4400	0.8640	0.5184	0.3110
e_1	0	0.1213	0.1589	0.1562	0.1365	0.1118
e_2	0	0.2813	0.3573	0.3408	0.2892	0.2303
r		0.4312	0.4447	0.4583	0.4720	0.4855

(b) If h is halved, then the error is approximately halved $(r \approx 0.5)$.

(c) When $h = 0.05$, the errors will again be approximately halved.

94. $\dfrac{dy}{dx} = x - y,\ y(0) = 1,\ y = x - 1 + 2e^{-x}$

(a)

x	0	0.2	0.4	0.6	0.8	1
y	1	0.8375	0.7406	0.6976	0.6987	0.7358
y_1	1	0.8200	0.7122	0.6629	0.6609	0.6974
y_2	1	0.8000	0.6800	0.6240	0.6192	0.6554
e_1	0	0.0175	0.0284	0.0347	0.0378	0.0384
e_2	0	0.0375	0.0606	0.0736	0.0795	0.0804
r		0.47	0.47	0.47	0.48	0.48

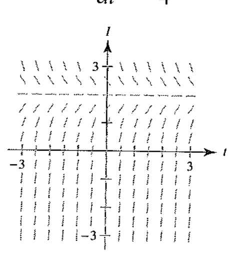
(b) If h is halved, then the error is halved $(r \approx 0.5)$.

(c) When $h = 0.05$, the error will again be approximately halved.

95. (a) $L\dfrac{dI}{dt} + RI = E(t)$

$$4\frac{dI}{dt} + 12I = 24$$

$$\frac{dI}{dt} = \frac{1}{4}(24 - 12I) = 6 - 3I$$



(b) As $t \to \infty$, $I \to 2$. That is, $\lim_{t\to\infty} I(t) = 2$. In fact, $I = 2$ is a solution to the differential equation.

96.

$$y = e^{kt}$$
$$y' = ke^{kt}$$
$$y'' = k^2e^{kt}$$
$$y'' - 16y = 0$$
$$k^2e^{kt} - 16e^{kt} = 0$$
$$k^2 - 16 = 0 \qquad \left(\text{because } e^{kt} \neq 0\right)$$
$$k = \pm 4$$

97. $y = A\sin \omega t$

$$y' = A\omega \cos \omega t$$
$$y'' = -A\omega^2 \sin \omega t$$
$$y'' + 16y = 0$$
$$-A\omega^2 \sin \omega t + 16A \sin \omega t = 0$$
$$A \sin \omega t\left[16 - \omega^2\right] = 0$$

If $A \neq 0$, then $\omega = \pm 4$

98.
$$f(x) + f''(x) = -xg(x)f'(x), \qquad g(x) \geq 0$$
$$2f(x)f'(x) + 2f'(x)f''(x) = -2xg(x)[f'(x)]^2$$
$$\frac{d}{dx}\left[f(x)^2 + f'(x)^2\right] = -2x\,g(x)[f'(x)]^2$$

For $x < 0, -2x\,g(x)[f'(x)]^2 \geq 0$

For $x > 0, -2x\,g(x)[f'(x)]^2 \leq 0$

So, $f(x)^2 + f'(x)^2$ is increasing for $x < 0$ and decreasing for $x > 0$.

$f(x)^2 + f'(x)^2$ has a maximum at $x = 0$. So, it is bounded by its value at $x = 0$, $f(0)^2 + f'(0)^2$. So, f (and f') is bounded.

99. Let the vertical line $x = k$ cut the graph of the solution $y = f(x)$ at (k, t). The tangent line at (k, t) is

$$y - t = f'(k)(x - k)$$

Because $y' + p(x)y = q(x)$, you have

$$y - t = [q(k) - p(k)t](x - k)$$

For any value of t, this line passes through the point $\left(k + \frac{1}{p(k)}, \frac{q(k)}{p(k)}\right)$.

To see this, note that

$$\frac{q(k)}{p(k)} - t \overset{?}{=} [q(k) - p(k)t]\left(k + \frac{1}{p(k)} - k\right)$$
$$\overset{?}{=} q(k)k - p(k)tk + \frac{q(k)}{p(k)} - t - kq(k) + p(k)kt = \frac{q(k)}{p(k)} - t.$$

Section 6.2 Differential Equations: Growth and Decay

1. $\frac{dy}{dx} = x + 3$

$$y = \int (x + 3)\,dx = \frac{x^2}{2} + 3x + C$$

2. $\frac{dy}{dx} = 6 - x$

$$y = \int (6 - x)\,dx = 6x - \frac{x^2}{2} + C$$

3.
$$\frac{dy}{dx} = y + 3$$
$$\frac{dy}{y + 3} = dx$$
$$\int \frac{1}{y + 3}\,dy = \int dx$$
$$\ln|y + 3| = x + C_1$$
$$y + 3 = e^{x + C_1} = Ce^x$$
$$y = Ce^x - 3$$

4.
$$\frac{dy}{dx} = 6 - y$$
$$\frac{dy}{6 - y} = dx$$
$$\int \frac{-1}{6 - y}\,dy = \int -dx$$
$$\ln|6 - y|dy = -x + C_1$$
$$6 - y = e^{-x + C_1} = Ce^{-x}$$
$$y = 6 - Ce^{-x}$$

5.
$$y' = \frac{5x}{y}$$
$$yy' = 5x$$
$$\int yy'\,dx = \int 5x\,dx$$
$$\int y\,dy = \int 5x\,dx$$
$$\frac{1}{2}y^2 = \frac{5}{2}x^2 + C_1$$
$$y^2 - 5x^2 = C$$

6.
$$y' = \frac{\sqrt{x}}{7y}$$
$$7yy' = \sqrt{x}$$
$$\int 7yy'\,dx = \int \sqrt{x}\,dx$$
$$\frac{7y^2}{2} = \frac{2}{3}x^{3/2} + C_1$$
$$21y^2 - 4x^{3/2} = C$$

7.
$$y' = \sqrt{x}y$$
$$\frac{y'}{y} = \sqrt{x}$$
$$\int \frac{y'}{y}\,dx = \int \sqrt{x}\,dx$$
$$\int \frac{dy}{y} = \int \sqrt{x}\,dx$$
$$\ln|y| = \frac{2}{3}x^{3/2} + C_1$$
$$y = e^{(2/3)x^{3/2} + C_1}$$
$$= e^{C_1}e^{(2/3)x^{3/2}}$$
$$= Ce^{\left(2x^{3/2}\right)/3}$$

8.
$$y' = x(1 + y)$$
$$\frac{y'}{1 + y} = x$$
$$\int \frac{y'}{1 + y}\,dx = \int x\,dx$$
$$\int \frac{dy}{1 + y} = \int x\,dx$$
$$\ln(1 + y) = \frac{x^2}{2} + C_1$$
$$1 + y = e^{\left(x^2/2\right) + C_1}$$
$$y = e^{C_1}e^{x^2/2} - 1$$
$$= Ce^{x^2/2} - 1$$

9.
$$\left(1 + x^2\right)y' - 2xy = 0$$
$$y' = \frac{2xy}{1 + x^2}$$
$$\frac{y'}{y} = \frac{2x}{1 + x^2}$$
$$\int \frac{y'}{y}\,dx = \int \frac{2x}{1 + x^2}\,dx$$
$$\int \frac{dy}{y} = \int \frac{2x}{1 + x^2}\,dx$$
$$\ln|y| = \ln\left(1 + x^2\right) + C_1$$
$$\ln|y| = \ln\left(1 + x^2\right) + \ln C$$
$$\ln|y| = \ln\left[C\left(1 + x^2\right)\right]$$
$$y = C\left(1 + x^2\right)$$

10.
$$xy + y' = 100x$$
$$y' = 100x + xy = x(100 - y)$$
$$\frac{y'}{100 - y} = x$$
$$\int \frac{y'}{100 - y}\,dx = \int x\,dx$$
$$\int \frac{1}{100 - y}\,dy = \int x\,dx$$
$$-\ln(100 - y) = \frac{x^2}{2} + C_1$$
$$\ln(100 - y) = -\frac{x^2}{2} - C_1$$
$$100 - y = e^{-\left(x^2/2\right) - C_1}$$
$$-y = e^{-C_1}e^{-x^2/2} - 100$$
$$y = 100 - Ce^{-x^2/2}$$

11.
$$\frac{dQ}{dt} = \frac{k}{t^2}$$
$$\int \frac{dQ}{dt}\,dt = \int \frac{k}{t^2}\,dt$$
$$\int dQ = -\frac{k}{t} + C$$
$$Q = -\frac{k}{t} + C$$

12.
$$\frac{dP}{dt} = k(25 - t)$$
$$\int \frac{dP}{dt}\,dt = \int k(25 - t)\,dt$$
$$\int dP = -\frac{k}{2}(25 - t)^2 + C$$
$$P = -\frac{k}{2}(25 - t)^2 + C$$

13. $$\frac{dN}{ds} = k(500 - s)$$

$$\int \frac{dN}{ds}\,ds = \int k(500 - s)\,ds$$

$$\int dN = -\frac{k}{2}(500 - s)^2 + C$$

$$N = -\frac{k}{2}(500 - s)^2 + C$$

14. $$\frac{dy}{dx} = kx(L - y)$$

$$\frac{1}{L - y}\frac{dy}{dx} = kx$$

$$\int \frac{1}{L - y}\frac{dy}{dx}\,dx = \int kx\,dx$$

$$\int \frac{1}{L - y}\,dy = \frac{kx^2}{2} + C_1$$

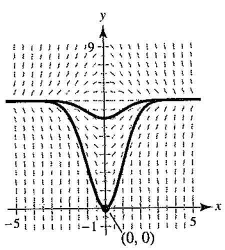
$$-\ln(L - y) = \frac{kx^2}{2} + C_1$$

$$L - y = e^{-(kx^2/2) - C_1}$$

$$-y = -L + e^{-C_1}e^{-kx^2/2}$$

$$y = L - Ce^{-kx^2/2}$$

15. (a)



(b) $$\frac{dy}{dx} = x(6 - y), \quad (0, 0)$$

$$\frac{dy}{y - 6} = -x\,dx$$

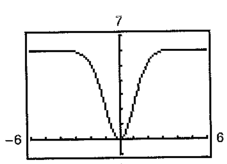
$$\ln|y - 6| = \frac{-x^2}{2} + C$$

$$y - 6 = e^{-x^2/2 + C} = C_1 e^{-x^2/2}$$

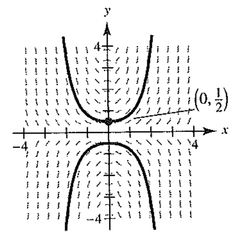
$$y = 6 + C_1 e^{-x^2/2}$$

$$(0, 0)\!: \; 0 = 6 + C_1 \Rightarrow C_1 = -6$$

$$y = 6 - 6e^{-x^2/2}$$



16. (a)



(b) $$\frac{dy}{dx} = xy, \quad \left(0, \frac{1}{2}\right)$$

$$\frac{dy}{y} = x\,dx$$

$$\ln|y| = \frac{x^2}{2} + C$$

$$y = e^{x^2/2 + C} = C_1 e^{x^2/2}$$

$$\left(0, \frac{1}{2}\right)\!: \; \frac{1}{2} = C_1 e^0 \Rightarrow C_1 = \frac{1}{2}$$

$$y = \frac{1}{2}e^{x^2/2}$$

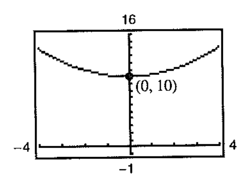
17. $$\frac{dy}{dt} = \frac{1}{2}t, \quad (0, 10)$$

$$\int dy = \int \frac{1}{2}t\,dt$$

$$y = \frac{1}{4}t^2 + C$$

$$10 = \frac{1}{4}(0)^2 + C \Rightarrow C = 10$$

$$y = \frac{1}{4}t^2 + 10$$



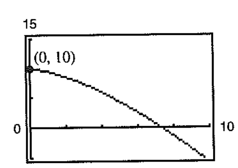
18. $$\frac{dy}{dt} = -\frac{3}{4}\sqrt{t}, \quad (0, 10)$$

$$\int dy = \int -\frac{3}{4}\sqrt{t}\,dt$$

$$y = -\frac{1}{2}t^{3/2} + C$$

$$10 = -\frac{1}{2}(0)^{3/2} + C \Rightarrow C = 10$$

$$y = -\frac{1}{2}t^{3/2} + 10$$



19. $\dfrac{dy}{dt} = -\dfrac{1}{2}y, \quad (0, 10)$

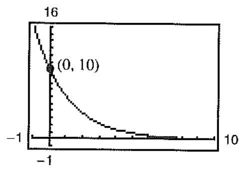
$$\int \frac{dy}{y} = \int -\frac{1}{2}\,dt$$

$$\ln|y| = -\frac{1}{2}t + C_1$$

$$y = e^{-(t/2)+C_1} = e^{C_1}e^{-t/2} = Ce^{-t/2}$$

$$10 = Ce^0 \Rightarrow C = 10$$

$$y = 10e^{-t/2}$$



20. $\dfrac{dy}{dt} = \dfrac{3}{4}y, \quad (0, 10)$

$$\int \frac{dy}{y} = \int \frac{3}{4}\,dt$$

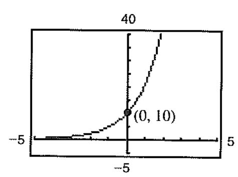
$$\ln y = \frac{3}{4}t + C_1$$

$$y = e^{(3/4)t + C_1}$$

$$= e^{C_1}e^{(3/4)t} = Ce^{3t/4}$$

$$10 = Ce^0 \Rightarrow C = 10$$

$$y = 10e^{3t/4}$$



21. $\dfrac{dy}{dx} = ky$

$$y = Ce^{kx} \quad \text{(Theorem 6.1)}$$

$$(0, 6):\ 6 = Ce^{k(0)} = C$$

$$(4, 15):\ 15 = 6e^{k(4)} \Rightarrow k = \frac{1}{4}\ln\left(\frac{5}{2}\right)$$

$$y = 6e^{[1/4\ln(5/2)]x} \approx 6e^{0.2291x}$$

When $x = 8$,

$$y = 6e^{1/4\ln(5/2)(8)} = 6e^{\ln(5/2)^2} = 6\left(\frac{25}{4}\right) = \frac{75}{2}$$

22. $\dfrac{dN}{dt} = kN$

$$N = Ce^{kt} \quad \text{(Theorem 6.1)}$$

$$(0, 250):\ C = 250$$

$$(1, 400):\ 400 = 250e^k \Rightarrow k = \ln\frac{400}{250} = \ln\frac{8}{5}$$

$$N = 250e^{\ln(8/5)t} \approx 250e^{0.4700t}$$

When $t = 4$, $N = 250e^{4\ln(8/5)} = 250e^{\ln(8/5)^4}$

$$= 250\left(\frac{8}{5}\right)^4 = \frac{8192}{5}.$$

23. $\dfrac{dV}{dt} = kV$

$$V = Ce^{kt} \quad \text{(Theorem 6.1)}$$

$$(0, 20{,}000):\ C = 20{,}000$$

$$(4, 12{,}500):\ 12{,}500 = 20{,}000e^{4k} \Rightarrow k = \frac{1}{4}\ln\left(\frac{5}{8}\right)$$

$$V = 20{,}000e^{[1/4\ln(5/8)]t} \approx 20{,}000e^{-0.1175t}$$

When $t = 6$, $V = 20{,}000e^{(1/4)\ln(5/8)(6)} = 20{,}000e^{\ln(5/8)^{3/2}}$

$$= 20{,}000\left(\frac{5}{8}\right)^{3/2} \approx 9882.118.$$

24. $\dfrac{dP}{dt} = kP$

$$P = Ce^{kt} \quad \text{(Theorem 6.1)}$$

$$(0, 5000):\ C = 5000$$

$$(1, 4750):\ 4750 = 5000e^k \Rightarrow k = \ln\left(\frac{19}{20}\right)$$

$$P = 5000e^{\ln(19/20)t} \approx 5000e^{-0.0513t}$$

When $t = 5$, $P = 5000e^{\ln(19/20)(5)}$

$$= 5000\left(\frac{19}{20}\right)^5 \approx 3868.905.$$

25. $y = Ce^{kt}, \quad \left(0, \dfrac{1}{2}\right), (5, 5)$

$$C = \frac{1}{2}$$

$$y = \frac{1}{2}e^{kt}$$

$$5 = \frac{1}{2}e^{5k}$$

$$k = \frac{\ln 10}{5}$$

$$y = \frac{1}{2}e^{[(\ln 10)/5]t} = \frac{1}{2}\left(10^{t/5}\right) \text{ or } y \approx \frac{1}{2}e^{0.4605t}$$

26. $y = Ce^{kt}, \quad (0, 4), \left(5, \frac{1}{2}\right)$

$C = 4$

$y = 4e^{kt}$

$\frac{1}{2} = 4e^{5k}$

$k = \frac{\ln(1/8)}{5} \approx -0.4159$

$y = 4e^{-0.4159t}$

27. $y = Ce^{kt}, \quad (1, 5), (5, 2)$

$5 = Ce^{k} \Rightarrow 10 = 2Ce^{k}$

$2 = Ce^{5k} \Rightarrow 10 = 5Ce^{k}$

$2Ce^{k} = 5Ce^{5k}$

$2e^{k} = 5e^{5k}$

$\frac{2}{5} = e^{4k}$

$k = \frac{1}{4}\ln\left(\frac{2}{5}\right) = \ln\left(\frac{2}{5}\right)^{1/4}$

$C = 5e^{-k} = 5e^{-1/4\ln(2/5)} = 5\left(\frac{2}{5}\right)^{-1/4} = 5\left(\frac{5}{2}\right)^{1/4}$

$y = 5\left(\frac{5}{2}\right)^{1/4} e^{[1/4\ln(2/5)]t} \approx 6.2872\, e^{-0.2291t}$

28. $y = Ce^{kt}, \quad \left(3, \frac{1}{2}\right), (4, 5)$

$\frac{1}{2} = Ce^{3k} \Rightarrow 1 = 2Ce^{3k}$

$5 = Ce^{4k} \Rightarrow 1 = \frac{1}{5}Ce^{4k}$

$2Ce^{3k} = \frac{1}{5}Ce^{4k}$

$10e^{3k} = e^{4k}$

$10 = e^{k}$

$k = \ln 10 \approx 2.3026$

$y = Ce^{2.3026t}$

$5 = Ce^{2.3026(4)}$

$C \approx 0.0005$

$y = 0.0005e^{2.3026t}$

29. In the model $y = Ce^{kt}$, C represents the initial value of y (when $t = 0$). k is the proportionality constant.

30. $y' = \frac{dy}{dt} = ky$

31. $\frac{dy}{dx} = \frac{1}{2}xy$

$\frac{dy}{dx} > 0$ when $xy > 0$. Quadrants I and III.

32. $\frac{dy}{dx} = \frac{1}{2}x^2y$

$\frac{dy}{dx} > 0$ when $y > 0$. Quadrants I and II.

33. Because the initial quantity is 20 grams,

$y = 20e^{kt}$.

Because the half-life is 1599 years,

$10 = 20e^{k(1599)}$

$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right)$.

So, $y = 20e^{[\ln(1/2)/1599]t}$.

When $t = 1000$, $y = 20e^{[\ln(1/2)/1599](1000)} \approx 12.96$g.

When $t = 10{,}000$, $y \approx 0.26$g.

34. Because the half-life is 1599 years,

$\frac{1}{2} = 1e^{k(1599)}$

$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right)$.

Because there are 1.5 g after 1000 years,

$1.5 = Ce^{[\ln(1/2)/1599](1000)}$

$C \approx 2.314$.

So, the initial quantity is approximately 2.314 g.

When $t = 10{,}000$, $y = 2.314e^{[\ln(1/2)/1599](10{,}000)} \approx 0.03$ g.

35. Because the half-life is 1599 years,

$\frac{1}{2} = 1e^{k(1599)}$

$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right)$.

Because there are 0.1 gram after 10,000 years,

$0.1 = Ce^{[\ln(1/2)/1599](10{,}000)}$

$C \approx 7.63$.

So, the initial quantity is approximately 7.63 g.

When $t = 1000$, $y = 7.63e^{[\ln(1/2)/1599](1000)} \approx 4.95$ g.

36. Because the half-life is 5715 years,

$$\frac{1}{2} = 1e^{k(5715)}$$
$$k = \frac{1}{5715}\ln\left(\frac{1}{2}\right).$$

Because there are 3 grams after 10,000 years,

$$3 = Ce^{[\ln(1/2)/5715](10{,}000)}$$
$$C \approx 10.089.$$

So, the initial quantity is approximately 10.09 g.

When $t = 1000$, $y = 10.09e^{[\ln(1/2)/5715](1000)} \approx 8.94$ g.

37. Because the initial quantity is 5 grams, $C = 5$.

Because the half-life is 5715 years,

$$2.5 = 5e^{k(5715)}$$
$$k = \frac{1}{5715}\ln\left(\frac{1}{2}\right).$$

When $t = 1000$ years, $y = 5e^{[\ln(1/2)/5715](1000)} \approx 4.43$ g.

When $t = 10{,}000$ years, $y = 5e^{[\ln(1/2)/5715](10{,}000)} \approx 1.49$ g.

38. Because the half-life is 5715 years,

$$\frac{1}{2} = 1e^{k(5715)}$$
$$k = \frac{1}{5715}\ln\left(\frac{1}{2}\right).$$

Because there are 1.6 grams when $t = 1000$ years,

$$1.6 = Ce^{[\ln(1/2)/5715](1000)}$$
$$C \approx 1.806.$$

So, the initial quantity is approximately 1.806 g.

When $t = 10{,}000$, $y = 1.806e^{[\ln(1/2)/5715](10{,}000)} \approx 0.54$ g.

39. Because the half-life is 24,100 years,

$$\frac{1}{2} = 1e^{k(24{,}100)}$$
$$k = \frac{1}{24{,}100}\ln\left(\frac{1}{2}\right).$$

Because there are 2.1 grams after 1000 years,

$$2.1 = Ce^{[\ln(1/2)/24{,}100](1000)}$$
$$C \approx 2.161.$$

So, the initial quantity is approximately 2.161 g.

When $t = 10{,}000$, $y = 2.161e^{[\ln(1/2)/24{,}100](10{,}000)} \approx 1.62$ g.

40. Because the half-life is 24,100 years,

$$\frac{1}{2} = 1e^{k(24{,}100)}$$
$$k = \frac{1}{24{,}100}\ln\left(\frac{1}{2}\right).$$

Because there are 0.4 grams after 10,000 years,

$$0.4 = Ce^{[\ln(1/2)/24{,}100](10{,}000)}$$
$$C \approx 0.533.$$

So, the initial quantity is approximately 0.533 g.

When $t = 1000$, $y = 0.533e^{[\ln(1/2)/24{,}100](1000)} \approx 0.52$ g.

41.

$$y = Ce^{kt}$$
$$\frac{1}{2}C = Ce^{k(1599)}$$
$$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right)$$

When $t = 100$, $y = Ce^{[\ln(1/2)/1599](100)} \approx 0.9576\,C$

Therefore, 95.76% remains after 100 years.

42.

$$y = Ce^{kt}$$
$$\frac{1}{2}C = Ce^{k(5715)}$$
$$k = \frac{1}{5715}\ln\left(\frac{1}{2}\right)$$
$$0.15C = Ce^{[\ln(1/2)/5715]t}$$
$$\ln(0.15) = \frac{\ln\left(\frac{1}{2}\right)t}{5715}$$
$$t \approx 15{,}641.8 \text{ years}$$

43. Because $A = 4000e^{0.06t}$, the time to double is given by

$$8000 = 4000e^{0.06t}$$
$$2 = e^{0.06t}$$
$$\ln 2 = 0.06t$$
$$t = \frac{\ln 2}{0.06} \approx 11.55 \text{ years.}$$

Amount after 10 years: $A = 4000e^{(0.06)(10)} \approx \$7288.48$

44. Because $A = 18{,}000e^{0.055t}$, the time to double is given by

$$36{,}000 = 18{,}000e^{0.055t}$$
$$2 = e^{0.055t}$$
$$\ln 2 = 0.055t$$
$$t = \frac{\ln 2}{0.055} \approx 12.6 \text{ years.}$$

Amount after 10 years:
$A = 18{,}000e^{(0.055)(10)} \approx \$31{,}198.55$

45. Because $A = 750e^{rt}$ and $A = 1500$ when $t = 7.75$, you have the following.

$$1500 = 750e^{7.75r}$$
$$r = \frac{\ln 2}{7.75} \approx 0.0894 = 8.94\%$$

Amount after 10 years: $A = 750e^{0.0894(10)} \approx \$1833.67$

46. Because $A = 12{,}500e^{rt}$ and $A = 25{,}000$ when $t = 5$, you have the following.

$$25{,}000 = 12{,}500e^{5r}$$
$$r = \frac{\ln 2}{5} \approx 0.1386 = 13.86\%$$

Amount after 10 years:
$$A = 12{,}500e^{[(\ln 2)/5](10)} = \$50{,}000$$

47. Because $A = 500e^{rt}$ and $A = 1292.85$ when $t = 10$, you have the following.

$$1292.85 = 500e^{10r}$$
$$r = \frac{\ln(1292.85/500)}{10} \approx 0.0950 = 9.50\%$$

The time to double is given by

$$1000 = 500e^{0.0950t}$$
$$t = \frac{\ln 2}{0.095} \approx 7.30 \text{ years.}$$

48. Because $A = 2000e^{rt}$ and $A = 5436.56$ when $t = 10$, you have the following.

$$5436.56 = 2000e^{10r}$$
$$r = \frac{\ln(5436.56/2000)}{10} \approx 0.10 = 10\%$$

The time to double is given by

$$4000 = 2000e^{0.10t}$$
$$t = \frac{\ln 2}{0.10} \approx 6.93 \text{ years.}$$

49. $1{,}000{,}000 = P\left(1 + \frac{0.075}{12}\right)^{(12)(20)}$

$$P = 1{,}000{,}000\left(1 + \frac{0.075}{12}\right)^{-240}$$
$$\approx \$224{,}174.18$$

50. $1{,}000{,}000 = P\left(1 + \frac{0.06}{12}\right)^{(12)(40)}$

$$P = 1{,}000{,}000(1.005)^{-480} \approx \$91{,}262.08$$

51. $1{,}000{,}000 = P\left(1 + \frac{0.08}{12}\right)^{(12)(35)}$

$$P = 1{,}000{,}000\left(1 + \frac{0.08}{12}\right)^{-420}$$
$$= \$61{,}377.75$$

52. $1{,}000{,}000 = P\left(1 + \frac{0.09}{12}\right)^{(12)(25)}$

$$P = 1{,}000{,}000\left(1 + \frac{0.09}{12}\right)^{-300}$$
$$\approx \$106{,}287.83$$

53. (a) $2000 = 1000(1 + 0.07)^t$

$$2 = 1.07^t$$
$$\ln 2 = t \ln 1.07$$
$$t = \frac{\ln 2}{\ln 1.07} \approx 10.24 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.07}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.007}{12}\right)^{12t}$$
$$\ln 2 = 12t \ln\left(1 + \frac{0.07}{12}\right)$$
$$t = \frac{\ln 2}{12 \ln(1 + (0.07/12))} \approx 9.93 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.07}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.07}{365}\right)^{365t}$$
$$\ln 2 = 365t \ln\left(1 + \frac{0.07}{365}\right)$$
$$t = \frac{\ln 2}{365 \ln(1 + (0.07/365))} \approx 9.90 \text{ years}$$

(d) $2000 = 1000e^{(0.07)t}$

$$2 = e^{0.07t}$$
$$\ln 2 = 0.07t$$
$$t = \frac{\ln 2}{0.07} \approx 9.90 \text{ years}$$

54. (a) $2000 = 1000(1 + 0.6)^t$

$2 = 1.06^t$

$\ln 2 = t \ln 1.06$

$t = \dfrac{\ln 2}{\ln 1.06} \approx 11.90$ years

(b) $2000 = 1000\left(1 + \dfrac{0.06}{12}\right)^{12t}$

$2 = \left(1 + \dfrac{0.06}{12}\right)^{12t}$

$\ln 2 = 12t \ln\left(1 + \dfrac{0.06}{12}\right)$

$t = \dfrac{1}{12}\dfrac{\ln 2}{\ln\left(1 + \dfrac{0.06}{12}\right)} \approx 11.58$ years

(c) $2000 = 1000\left(1 + \dfrac{0.06}{365}\right)^{365t}$

$2 = \left(1 + \dfrac{0.06}{365}\right)^{365t}$

$\ln 2 = 365t \ln\left(1 + \dfrac{0.06}{365}\right)$

$t = \dfrac{1}{365}\dfrac{\ln 2}{\ln\left(1 + \dfrac{0.06}{365}\right)} \approx 11.55$ years

(d) $2000 = 1000e^{0.06t}$

$2 = e^{0.06t}$

$\ln 2 = 0.06t$

$t = \dfrac{\ln 2}{0.06} \approx 11.55$ years

55. (a) $2000 = 1000(1 + 0.085)^t$

$2 = 1.085^t$

$\ln 2 = t \ln 1.085$

$t = \dfrac{\ln 2}{\ln 1.085} \approx 8.50$ years

(b) $2000 = 1000\left(1 + \dfrac{0.085}{12}\right)^{12t}$

$2 = \left(1 + \dfrac{0.085}{12}\right)^{12t}$

$\ln 2 = 12t \ln\left(1 + \dfrac{0.085}{12}\right)$

$t = \dfrac{1}{12}\dfrac{\ln 2}{\ln\left(1 + \dfrac{0.085}{12}\right)} \approx 8.18$ years

(c) $2000 = 1000\left(1 + \dfrac{0.085}{365}\right)^{365t}$

$2 = \left(1 + \dfrac{0.085}{365}\right)^{365t}$

$\ln 2 = 365t \ln\left(1 + \dfrac{0.085}{365}\right)$

$t = \dfrac{1}{365}\dfrac{\ln 2}{\ln\left(1 + \dfrac{0.085}{365}\right)} \approx 8.16$ years

(d) $2000 = 1000e^{0.085t}$

$2 = e^{0.085t}$

$\ln 2 = 0.085t$

$t = \dfrac{\ln 2}{0.085} \approx 8.15$ years

56. (a) $2000 = 1000(1 + 0.055)^t$

$2 = 1.055^t$

$\ln 2 = t \ln 1.055$

$t = \dfrac{\ln 2}{\ln 1.055} \approx 12.95$ years

(b) $2000 = 1000\left(1 + \dfrac{0.055}{12}\right)^{12t}$

$2 = \left(1 + \dfrac{0.055}{12}\right)^{12t}$

$\ln 2 = 12t \ln\left(1 + \dfrac{0.055}{12}\right)$

$t = \dfrac{1}{12}\dfrac{\ln 2}{\ln\left(1 + \dfrac{0.055}{12}\right)} \approx 12.63$ years

(c) $2000 = 1000\left(1 + \dfrac{0.055}{365}\right)^{365t}$

$2 = \left(1 + \dfrac{0.055}{365}\right)^{365t}$

$\ln 2 = 365t \ln\left(1 + \dfrac{0.055}{365}\right)$

$t = \dfrac{1}{365}\dfrac{\ln 2}{\ln\left(1 + \dfrac{0.055}{365}\right)} \approx 12.60$ years

(d) $2000 = 1000e^{0.055t}$

$2 = e^{0.055t}$

$\ln 2 = 0.055t$

$t = \dfrac{\ln 2}{0.055} \approx 12.60$ years

57. (a) $P = Ce^{kt} = Ce^{-0.006t}$

$P(7) = 2.3 = Ce^{-0.006(7)} \Rightarrow C \approx 2.40$

$P = 2.40e^{-0.006t}$

(b) For $t = 15$, $P = 2.40e^{-0.006(15)} \approx 2.19$ million

(c) Because $k < 0$, the population is decreasing.

58. (a) $P = Ce^{kt} = Ce^{0.017t}$

$P(7) = 80.3 = Ce^{0.017(7)} \Rightarrow C \approx 71.29$

$P = 71.29e^{0.017t}$

(b) For $t = 15$, $P = 71.29e^{0.017(15)} \approx 92.0$ million

(c) Because $k > 0$, the population is increasing.

59. (a) $P = Ce^{kt} = Ce^{0.024t}$

$P(7) = 6.7 = Ce^{0.024(7)} \Rightarrow C \approx 5.66$

$P = 5.66e^{0.024t}$

(b) For $t = 15$, $P = 5.66e^{0.024(15)} \approx 8.11$ million

(c) Because $k > 0$, the population is increasing.

60. (a) $P = Ce^{kt} = Ce^{-0.003t}$

$P(7) = 10.0 = Ce^{-0.003(7)} \Rightarrow C \approx 10.21$

$P = 10.21e^{-0.003t}$

(b) For $t = 15$, $P = 10.21e^{-0.003(15)} \approx 9.76$ million

(c) Because $k < 0$, the population is decreasing.

61. (a) $P = Ce^{kt} = Ce^{0.036t}$

$P(7) = 30.3 = Ce^{0.036(7)} \Rightarrow C \approx 23.55$

$P = 23.55e^{0.036t}$

(b) For $t = 15$, $P = 23.55e^{0.036(15)} \approx 40.41$ million

(c) Because $k > 0$, the population is increasing.

62. (a) Because the population increases by a constant each month, the rate of change from month to month will always be the same. So, the slope is constant, and the model is linear.

(b) Although the percentage increase is constant each month, the rate of growth is not constant. The rate of change of y is given by

$$\frac{dy}{dt} = ry$$

which is an exponential model.

63. (a) $N = 100.1596(1.2455)^t$

(b) $N = 400$ when $t = 6.3$ hours (graphing utility)

Analytically,

$$400 = 100.1596(1.2455)^t$$

$$1.2455^t = \frac{400}{100.1596} = 3.9936$$

$$t \ln 1.2455 = \ln 3.9936$$

$$t = \frac{\ln 3.9936}{\ln 1.2455} \approx 6.3 \text{ hours.}$$

64. (a) Let $y = Ce^{kt}$.

At time 2: $125 = Ce^{k(2)} \Rightarrow C = 125e^{-2k}$

At time 4:

$$350 = Ce^{k(4)} \Rightarrow 350 = \left(125e^{-2k}\right)\left(e^{4k}\right)$$

$$\tfrac{14}{5} = e^{2k}$$

$$2k = \ln\tfrac{14}{5}$$

$$k = \tfrac{1}{2}\ln\tfrac{14}{5} \approx 0.5148$$

$$C = 125e^{-2k}$$

$$= 125e^{-2(1/2)\ln(14/5)}$$

$$= 125\left(\tfrac{5}{14}\right) = \tfrac{625}{14} \approx 44.64$$

Approximately 45 bacteria at time 0.

(b) $y = \frac{625}{14}e^{(1/2)\ln(14/5)t} \approx 44.64e^{0.5148t}$

(c) When $t = 8$,

$$y = \tfrac{625}{14}e^{(1/2)\ln(14/5)8} = \tfrac{625}{14}\left(\tfrac{14}{5}\right)^4 = 2744.$$

(d) $25{,}000 = \frac{625}{14}e^{(1/2)\ln(14/5)t} \Rightarrow t \approx 12.29$ hours

65. (a)

$$19 = 30\left(1 - e^{20k}\right)$$

$$30e^{20k} = 11$$

$$k = \frac{\ln(11/30)}{20} \approx -0.0502$$

$$N \approx 30\left(1 - e^{-0.0502t}\right)$$

(b)

$$25 = 30\left(1 - e^{-0.0502t}\right)$$

$$e^{-0.0502t} = \frac{1}{6}$$

$$t = \frac{-\ln 6}{-0.0502} \approx 36 \text{ days}$$

66. (a)

$$20 = 30(1 - e^{30k})$$
$$30e^{30k} = 10$$
$$k = \frac{\ln(1/3)}{30} = \frac{-\ln 3}{30} \approx -0.0366$$
$$N \approx 30(1 - e^{-0.0366t})$$

(b)

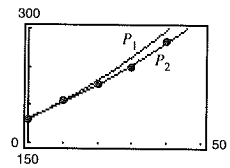
$$25 = 30(1 - e^{-0.0366t})$$
$$e^{-0.0366t} = \frac{1}{6}$$
$$t = \frac{-\ln 6}{-0.0366} \approx 49 \text{ days}$$

67. (a)

$$P_1 = Ce^{kt} = 181e^{kt}$$
$$205 = 181e^{10k} \Rightarrow k = \frac{1}{10}\ln\left(\tfrac{205}{181}\right) \approx 0.01245$$
$$P_1 \approx 181e^{0.01245t} \approx 181(1.01253)^t$$

(b) Using a graphing utility, $P_2 \approx 182.3248(1.01091)^t$

(c)



The model P_2 fits the data better.

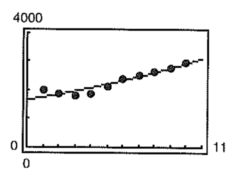
(d) Using the model P_2,

$$320 = 182.3248(1.01091)^t$$
$$\frac{320}{182.3248} = (1.01091)^t$$
$$t = \frac{\ln(320/182.3248)}{\ln(1.01091)}$$
$$\approx 51.8 \text{ years, or } 2011.$$

68. (a) $R = 1654.2353(1.0590)^t = 1654.2353\, e^{0.0573t}$

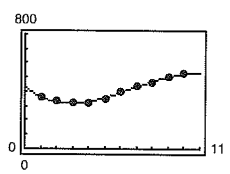
$I = 0.01942t^4 - 1.3690t^3 + 21.970t^2 - 93.66t + 435.6$

(b)

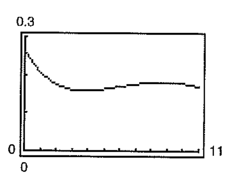


According to the model, $R'(t) \approx 94.79e^{0.0573t}$

(c)



(d) $P(t) = \dfrac{I}{R}$



69. $\beta(I) = 10 \log_{10}\dfrac{I}{I_0}$, $I_0 = 10^{-16}$

(a) $\beta(10^{-14}) = 10 \log_{10}\dfrac{10^{-14}}{10^{-16}} = 20$ decibels

(b) $\beta(10^{-9}) = 10 \log_{10}\dfrac{10^{-9}}{10^{-16}} = 70$ decibels

(c) $\beta(10^{-6.5}) = 10 \log_{10}\dfrac{10^{-6.5}}{10^{-16}} = 95$ decibels

(d) $\beta(10^{-4}) = 10 \log_{10}\dfrac{10^{-4}}{10^{-16}} = 120$ decibels

70. $93 = 10 \log_{10}\dfrac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$-6.7 = \log_{10} I \Rightarrow I = 10^{-6.7}$

$80 = 10 \log_{10}\dfrac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$-8 = \log_{10} I \Rightarrow I = 10^{-8}$

Percentage decrease: $\left(\dfrac{10^{-6.7} - 10^{-8}}{10^{-6.7}}\right)(100) \approx 95\%$

71. $A(t) = V(t)e^{-0.10t}$

$= 100{,}000e^{0.8\sqrt{t}}e^{-0.10t}$

$= 100{,}000e^{0.8\sqrt{t}-0.10t}$

$\dfrac{dA}{dt} = 100{,}000\left(\dfrac{0.4}{\sqrt{t}} - 0.10\right)e^{0.8\sqrt{t}-0.10t} = 0$

when $t = 16$.

The timber should be harvested in the year 2024, $(2008 + 16)$. **Note:** You could also use a graphing utility to graph $A(t)$ and find the maximum of $A(t)$. Use the viewing rectangle $0 \le x \le 30$ and $0 \le y \le 600{,}000$.

72. $R = \dfrac{\ln I - \ln I_0}{\ln 10} = \dfrac{\ln I - 0}{\ln 10}, I = e^{R \ln 10} = 10^R$

(a) $8.3 = \dfrac{\ln I - \ln I_0}{\ln 10}$

$I = 10^{8.3} \approx 199{,}526{,}231.5$

(b) $2R = \dfrac{\ln I - \ln I_0}{\ln 10}$

$I = e^{2R \ln 10} = e^{2R \ln 10} = \left(e^{R \ln 10}\right)^2 = \left(10^R\right)^2$

Increases by a factor of $e^{2R \ln 10}$ or 10^R.

(c) $R = \dfrac{\ln I - \ln I_0}{\ln 10}$

$\dfrac{dR}{dI} = \dfrac{1}{I \ln(10)}$

73. Because $\dfrac{dy}{dt} = k(y - 80)$

$\int \dfrac{1}{y - 80}\,dy = \int k\,dt$

$\ln(y - 80) = kt + C.$

When $t = 0$, $y = 1500$. So, $C = \ln 1420$.

When $t = 1$, $y = 1120$. So,

$k(1) + \ln 1420 = \ln(1120 - 80)$

$k = \ln 1040 - \ln 1420 = \ln \dfrac{104}{142}.$

So, $y = 1420e^{[\ln(104/142)]t} + 80.$

When $t = 5$, $y \approx 379.2$°F.

74. $\dfrac{dy}{dt} = k(y - 20)$

$y = 20 + Ce^{kt}$ (See Example 6)

$160 = 20 + Ce^{k(0)} \Rightarrow C = 140$

$60 = 20 + 140e^{k(5)}$

$\dfrac{2}{7} = e^{5k}$

$k = \dfrac{1}{5}\ln\left(\dfrac{2}{7}\right) \approx -0.25055$

$30 = 20 + 140e^{(1/5)\ln(2/7)t}$

$\dfrac{1}{14} = e^{\ln(2/7)t/5} = \left(\dfrac{2}{7}\right)^{t/5}$

$\ln \dfrac{1}{14} = \dfrac{t}{5} \ln \dfrac{2}{7}$

$t = \dfrac{5 \ln \frac{1}{14}}{\ln \frac{2}{7}} = \dfrac{5 \ln 14}{\ln \frac{7}{2}} \approx 10.53$ minutes

It will take $10.53 - 5 = 5.53$ minutes longer.

75. False. If $y = Ce^{kt}$, $y' = Cke^{kt} \neq$ constant.

76. True

77. False. The prices are rising at a rate of 6.2% per year.

78. True

Section 6.3 Separation of Variables and the Logistic Equation

1. $\frac{dy}{dx} = \frac{x}{y}$

$\int y\,dy = \int x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + C_1$

$y^2 - x^2 = C$

2. $\frac{dy}{dx} = \frac{3x^2}{y^2}$

$\int y^2\,dy = \int 3x^2\,dx$

$\frac{y^3}{3} = x^3 + C_1$

$y^3 - 3x^3 = C$

3. $x^2 + 5y\frac{dy}{dx} = 0$

$5y\frac{dy}{dx} = -x^2$

$\int 5y\,dy = \int -x^2\,dx$

$\frac{5y^2}{2} = \frac{-x^3}{3} + C_1$

$15y^2 + 2x^3 = C$

4. $\frac{dy}{dx} = \frac{x^2 - 3}{6y^2}$

$\int 6y^2\,dy = \int (x^2 - 3)\,dx$

$2y^3 = \frac{x^3}{3} - 3x + C_1$

$6y^3 - x^3 + 9x = C$

5. $\frac{dr}{ds} = 0.75\,r$

$\int \frac{dr}{r} = \int 0.75\,ds$

$\ln|r| = 0.75\,s + C_1$

$r = e^{0.75s + C_1}$

$r = Ce^{0.75s}$

6. $\frac{dr}{ds} = 0.75\,s$

$\int dr = \int 0.75\,s\,ds$

$r = 0.75\frac{s^2}{2} + C$

$r = 0.375\,s^2 + C$

7. $(2 + x)y' = 3y$

$\int \frac{dy}{y} = \int \frac{3}{2 + x}\,dx$

$\ln|y| = 3\ln|2 + x| + \ln C = \ln\left|C(2 + x)^3\right|$

$y = C(x + 2)^3$

8. $xy' = y$

$\int \frac{dy}{y} = \int \frac{dx}{x}$

$\ln y = \ln x + \ln C = \ln Cx$

$y = Cx$

9. $yy' = 4\sin x$

$y\frac{dy}{dx} = 4\sin x$

$\int y\,dy = \int 4\sin x\,dx$

$\frac{y^2}{2} = -4\cos x + C_1$

$y^2 = C - 8\cos x$

10. $yy' = -8\cos(\pi x)$

$y\frac{dy}{dx} = -8\cos(\pi x)$

$\int y\,dy = \int -8\cos(\pi x)\,dx$

$\frac{y^2}{2} = \frac{-8\sin(\pi x)}{\pi} + C$

$y^2 = \frac{-16}{\pi}\sin(\pi x) + C$

11. $\sqrt{1 - 4x^2}\,y' = x$

$dy = \frac{x}{\sqrt{1 - 4x^2}}\,dx$

$\int dy = \int \frac{x}{\sqrt{1 - 4x^2}}\,dx$

$= -\frac{1}{8}\int (1 - 4x^2)^{-1/2}(-8x\,dx)$

$y = -\frac{1}{4}\sqrt{1 - 4x^2} + C$

12. $\sqrt{x^2 - 16}\,y' = 11x$

$\frac{dy}{dx} = \frac{11x}{\sqrt{x^2 - 16}}$

$\int dy = \int \frac{11x}{\sqrt{x^2 - 16}}\,dx$

$y = 11\sqrt{x^2 - 16} + C$

13. $y \ln x - xy' = 0$

$$\int \frac{dy}{y} = \int \frac{\ln x}{x}\,dx \quad \left(u = \ln x,\ du = \frac{dx}{x}\right)$$

$$\ln|y| = \frac{1}{2}(\ln x)^2 + C_1$$

$$y = e^{(1/2)(\ln x)^2 + C_1} = Ce^{(\ln x)^2/2}$$

14. $12yy' - 7e^x = 0$

$$12y\frac{dy}{dx} = 7e^x$$

$$\int 12y\,dy = \int 7e^x\,dx$$

$$6y^2 = 7e^x + C$$

15. $yy' - 2e^x = 0$

$$y\frac{dy}{dx} = 2e^x$$

$$\int y\,dy = \int 2e^x\,dx$$

$$\frac{y^2}{2} = 2e^x + C$$

Initial condition $(0, 3)$: $\frac{9}{2} = 2 + C \Rightarrow C = \frac{5}{2}$

Particular solution: $\frac{y^2}{2} = 2e^x + \frac{5}{2}$

$$y^2 = 4e^x + 5$$

16. $\sqrt{x} + \sqrt{y}\,y' = 0$

$$\int y^{1/2}dy = -\int x^{1/2}\,dx$$

$$\frac{2}{3}y^{3/2} = -\frac{2}{3}x^{3/2} + C_1$$

$$y^{3/2} + x^{3/2} = C$$

Initial condition $(1, 9)$:

$$(9)^{3/2} + (1)^{3/2} = 27 + 1 = 28 = C$$

Particular solution: $y^{3/2} + x^{3/2} = 28$

17. $y(x + 1) + y' = 0$

$$\int \frac{dy}{y} = -\int (x + 1)\,dx$$

$$\ln|y| = -\frac{(x + 1)^2}{2} + C_1$$

$$y = Ce^{-(x+1)^2/2}$$

Initial condition $(-2, 1)$: $1 = Ce^{-1/2}, C = e^{1/2}$

Particular solution: $y = e^{\left[1-(x+1)^2\right]/2} = e^{-(x^2+2x)/2}$

18. $2xy' - \ln x^2 = 0$

$$2x\frac{dy}{dx} = 2\ln x$$

$$\int dy = \int \frac{\ln x}{x}\,dx$$

$$y = \frac{(\ln x)^2}{2} + C$$

Initial condition $(1, 2)$: $2 = C$

Particular solution: $y = \frac{1}{2}(\ln x)^2 + 2$

19. $y(1 + x^2)y' = x(1 + y^2)$

$$\frac{y}{1 + y^2}\,dy = \frac{x}{1 + x^2}\,dx$$

$$\frac{1}{2}\ln(1 + y^2) = \frac{1}{2}\ln(1 + x^2) + C_1$$

$$\ln(1 + y^2) = \ln(1 + x^2) + \ln C = \ln\left[C(1 + x^2)\right]$$

$$1 + y^2 = C(1 + x^2)$$

Initial condition $(0, \sqrt{3})$: $1 + 3 = C \Rightarrow C = 4$

Particular solution: $1 + y^2 = 4(1 + x^2)$

$$y^2 = 3 + 4x^2$$

20. $y\sqrt{1 - x^2}\,\frac{dy}{dx} = x\sqrt{1 - y^2}$

$$\int (1 - y^2)^{-1/2}y\,dy = \int (1 - x^2)^{-1/2}x\,dx$$

$$-(1 - y^2)^{1/2} = -(1 - x^2)^{1/2} + C$$

Initial condition $(0, 1)$: $0 = -1 + C \Rightarrow C = 1$

Particular solution: $\sqrt{1 - y^2} = \sqrt{1 - x^2} - 1$

21. $\frac{du}{dv} = uv\sin v^2$

$$\int \frac{du}{u} = \int v\sin v^2\,dv$$

$$\ln|u| = -\frac{1}{2}\cos v^2 + C_1$$

$$u = Ce^{-(\cos v^2)/2}$$

Initial condition: $u(0) = 1$: $C = \frac{1}{e^{-1/2}} = e^{1/2}$

Particular solution: $u = e^{(1-\cos v^2)/2}$

22. $\dfrac{dr}{ds} = e^{r-2s}$

$\int e^{-r}\,dr = \int e^{-2s}\,ds$

$-e^{-r} = -\dfrac{1}{2}e^{-2s} + C$

Initial condition:

$r(0) = 0: -1 = -\dfrac{1}{2} + C \Rightarrow C = -\dfrac{1}{2}$

Particular solution:

$-e^{-r} = -\dfrac{1}{2}e^{-2s} - \dfrac{1}{2}$

$e^{-r} = \dfrac{1}{2}e^{-2s} + \dfrac{1}{2}$

$-r = \ln\left(\dfrac{1}{2}e^{-2s} + \dfrac{1}{2}\right) = \ln\left(\dfrac{1 + e^{-2s}}{2}\right)$

$r = \ln\left(\dfrac{2}{1 + e^{-2s}}\right)$

23. $dP - kP\,dt = 0$

$\int \dfrac{dP}{P} = k\int dt$

$\ln|P| = kt + C_1$

$P = Ce^{kt}$

Initial condition: $P(0) = P_0, P_0 = Ce^0 = C$

Particular solution: $P = P_0e^{kt}$

24. $dT + k(T - 70)\,dt = 0$

$\int \dfrac{dT}{T - 70} = -k\int dt$

$\ln(T - 70) = -kt + C_1$

$T - 70 = Ce^{-kt}$

Initial condition:

$T(0) = 140: 140 - 70 = 70 = Ce^0 = C$

Particular solution:

$T - 70 = 70e^{-kt}, T = 70(1 + e^{-kt})$

25. $y' = \dfrac{dy}{dx} = \dfrac{x}{4y}$

$\int 4y\,dy = \int x\,dx$

$2y^2 = \dfrac{x^2}{2} + C$

Initial condition $(0, 2)$: $2(2^2) = 0 + C \Rightarrow C = 8$

Particular solution: $2y^2 = \dfrac{x^2}{2} + 8$

$4y^2 - x^2 = 16$

26. $\dfrac{dy}{dx} = \dfrac{-9x}{16y}$

$\int 16y\,dy = -\int 9x\,dx$

$8y^2 = \dfrac{-9}{2}x^2 + C$

Initial condition $(1, 1)$: $8 = -\dfrac{9}{2} + C, C = \dfrac{25}{2}$

Particular solution: $8y^2 = \dfrac{-9}{2}x^2 + \dfrac{25}{2}$

$16y^2 + 9x^2 = 25$

27. $y' = \dfrac{dy}{dx} = \dfrac{y}{2x}$

$\int \dfrac{2}{y}\,dy = \int \dfrac{1}{x}\,dx$

$2\ln|y| = \ln|x| + C_1 = \ln|x| + \ln C$

$y^2 = Cx$

Initial condition $(9, 1)$: $1 = 9C \Rightarrow C = \dfrac{1}{9}$

Particular solution: $y^2 = \dfrac{1}{9}x$

$9y^2 - x = 0$

$y = \dfrac{1}{3}\sqrt{x}$

28. $\dfrac{dy}{dx} = \dfrac{2y}{3x}$

$\int \dfrac{3}{y}\,dy = \int \dfrac{2}{x}\,dx$

$\ln y^3 = \ln x^2 + \ln C$

$y^3 = Cx^2$

Initial condition $(8, 2)$: $2^3 = C(8^2), C = \dfrac{1}{8}$

Particular solution: $8y^3 = x^2, y = \dfrac{1}{2}x^{2/3}$

29. $m = \dfrac{dy}{dx} = \dfrac{0 - y}{(x + 2) - x} = -\dfrac{y}{2}$

$\int \dfrac{dy}{y} = \int -\dfrac{1}{2}\,dx$

$\ln|y| = -\dfrac{1}{2}x + C_1$

$y = Ce^{-x/2}$

30. $m = \dfrac{dy}{dx} = \dfrac{y - 0}{x - 0} = \dfrac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + C_1 = \ln x + \ln C = \ln Cx$$

$$y = Cx$$

31. $f(x, y) = x^3 - 4xy^2 + y^3$

$$f(tx, ty) = t^3x^3 - 4txt^2y^2 + t^3y^3 = t^3(x^3 - 4xy^2 + y^3)$$

Homogeneous of degree 3

32. $f(x, y) = x^3 + 3x^2y^2 - 2y^2$

$$f(tx, ty) = t^3x^3 + 3t^4x^2y^2 - 2t^2y^2$$

Not homogeneous

33. $f(x, y) = \dfrac{x^2y^2}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{t^4x^2y^2}{\sqrt{t^2x^2 + t^2y^2}} = t^3\frac{x^2y^2}{\sqrt{x^2 + y^2}}$$

Homogeneous of degree 3

34. $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{tx\,ty}{\sqrt{t^2x^2 + t^2y^2}} = \frac{t^2xy}{t\sqrt{x^2 + y^2}} = t\frac{xy}{\sqrt{x^2 + y^2}}$$

Homogeneous of degree 1

35. $f(x, y) = 2\ln xy$

$$f(tx, ty) = 2\ln[txty] = 2\ln[t^2xy] = 2(\ln t^2 + \ln xy)$$

Not homogeneous

36. $f(x, y) = \tan(x + y)$

$$f(tx, ty) = \tan(tx + ty) = \tan[t(x + y)]$$

Not homogeneous

37. $f(x, y) = 2\ln\dfrac{x}{y}$

$$f(tx, ty) = 2\ln\frac{tx}{ty} = 2\ln\frac{x}{y}$$

Homogeneous of degree 0

38. $f(x, y) = \tan\dfrac{y}{x}$

$$f(tx, ty) = \tan\frac{ty}{tx} = \tan\frac{y}{x}$$

Homogeneous of degree 0

39. $y' = \dfrac{x + y}{2x},\ y = vx$

$$v + x\frac{dv}{dx} = \frac{x + vx}{2x}$$

$$x\frac{dv}{dx} = \frac{1 + v}{2} - v = \frac{1 - v}{2}$$

$$2\int \frac{dv}{1 - v} = \int \frac{dx}{x}$$

$$-\ln(1 - v)^2 = \ln|x| + \ln C = \ln|Cx|$$

$$\frac{1}{(1 - v)^2} = |Cx|$$

$$\frac{1}{[1 - (y/x)]^2} = |Cx|$$

$$\frac{x^2}{(x - y)^2} = |Cx|$$

$$|x| = C(x - y)^2$$

40. $y' = \dfrac{(x^3 + y^3)}{xy^2}$

$$xy^2dy = (x^3 + y^3)\,dx$$

$$y = vx, \quad dy = x\,dv + v\,dx$$

$$x(vx)^2(x\,dv + v\,dx) = (x^3 + (vx)^3)\,dx$$

$$x^4v^2\,dv + x^3v^3\,dx = x^3\,dx + v^3x^3\,dx$$

$$xv^2\,dv = dx$$

$$\int v^2\,dv = \int \frac{1}{x}\,dx$$

$$\frac{v^3}{3} = \ln|x| + C$$

$$\left(\frac{y}{x}\right)^3 = 3\ln|x| + C$$

$$y^3 = 3x^3\ln|x| + Cx^3$$

41.

$$y' = \frac{x-y}{x+y}, \quad y = vx$$

$$v + x\frac{dv}{dx} = \frac{x - xv}{x + xv}$$

$$v\,dx + x\,dv = \frac{1-v}{1+v}\,dx$$

$$x\,dv = \left(\frac{1-v}{1+v} - v\right)dx = \frac{1 - 2v - v^2}{1+v}\,dx$$

$$\int \frac{v+1}{v^2 + 2v - 1}\,dv = -\int \frac{dx}{x}$$

$$\frac{1}{2}\ln\left|v^2 + 2v - 1\right| = -\ln|x| + \ln C_1 = \ln\left|\frac{C_1}{x}\right|$$

$$\left|v^2 + 2v - 1\right| = \frac{C}{x^2}$$

$$\left|\frac{y^2}{x^2} + 2\frac{y}{x} - 1\right| = \frac{C}{x^2}$$

$$\left|y^2 + 2xy - x^2\right| = C$$

42.

$$y' = \frac{x^2 + y^2}{2xy}, \quad y = vx$$

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2x^2v}$$

$$2v\,dx + 2x\,dv = \frac{1 + v^2}{v}\,dx$$

$$\int \frac{2v}{v^2 - 1}\,dv = -\int \frac{dx}{x}$$

$$\ln\left(v^2 - 1\right) = -\ln x + \ln C = \ln\frac{C}{x}$$

$$v^2 - 1 = \frac{C}{x}$$

$$\frac{y^2}{x^2} - 1 = \frac{C}{x}$$

$$y^2 - x^2 = Cx$$

43.

$$y' = \frac{xy}{x^2 - y^2}, \quad y = vx$$

$$v + x\frac{dv}{dx} = \frac{x^2v}{x^2 - x^2v^2}$$

$$v\,dx + x\,dv = \frac{v}{1 - v^2}\,dx$$

$$x\,dv = \left(\frac{v}{1 - v^2} - v\right)dx = \left(\frac{v^3}{1 - v^2}\right)dx$$

$$\int \frac{1 - v^2}{v^3}\,dv = \int \frac{dx}{x}$$

$$-\frac{1}{2v^2} - \ln|v| = \ln|x| + \ln C_1 = \ln|C_1x|$$

$$\frac{-1}{2v^2} = \ln|C_1xv|$$

$$\frac{-x^2}{2y^2} = \ln|C_1y|$$

$$y = Ce^{-x^2/2y^2}$$

44.

$$y' = \frac{2x + 3y}{x}, \quad y = vx$$

$$v + x\frac{dv}{dx} = \frac{2x + 3vx}{x} = 2 + 3v$$

$$x\frac{dv}{dx} = 2 + 2v \Rightarrow \int \frac{dv}{1+v} = 2\int \frac{dx}{x}$$

$$\ln|1 + v| = \ln x^2 + \ln C = \ln x^2C$$

$$1 + v = x^2C$$

$$1 + \frac{y}{x} = x^2C$$

$$\frac{y}{x} = x^2C - 1$$

$$y = Cx^3 - x$$

45. $x\,dy - \left(2xe^{-y/x} + y\right)dx = 0, \quad y = vx$

$$x\left(v\,dx + x\,dv\right) - \left(2xe^{-v} + vx\right)dx = 0$$

$$\int e^v dv = \int \frac{2}{x}\,dx$$

$$e^v = \ln C_1x^2$$

$$e^{y/x} = \ln C_1 + \ln x^2$$

$$e^{y/x} = C + \ln x^2$$

Initial condition $(1, 0)$: $1 = C$

Particular solution: $e^{y/x} = 1 + \ln x^2$

46. $-y^2\,dx + x(x + y)\,dy = 0, \quad y = vx$

$$-x^2 v^2\,dx + \left(x^2 + x^2v\right)\left(v\,dx + x\,dv\right) = 0$$

$$\int \frac{1 + v}{v}\,dv = -\int \frac{dx}{x}$$

$$v + \ln v = -\ln x + \ln C_1 = \ln \frac{C_1}{x}$$

$$v = \ln \frac{C_1}{xv}$$

$$\frac{C_1}{vx} = e^v$$

$$\frac{C_1}{y} = e^{y/x}$$

$$y = Ce^{-y/x}$$

Initial condition $(1, 1)$: $1 = Ce^{-1} \Rightarrow C = e$

Particular solution: $y = e^{1-y/x}$

47. $\left(x\sec\frac{y}{x} + y\right)dx - x\,dy = 0, \quad y = vx$

$$\left(x\sec v + xv\right)dx - x\left(v\,dx + x\,dv\right) = 0$$

$$\left(\sec v + v\right)dx = v\,dx + x\,dv$$

$$\int \cos v\,dv = \int \frac{dx}{x}$$

$$\sin v = \ln x + \ln C_1$$

$$x = Ce^{\sin v}$$

$$= Ce^{\sin(y/x)}$$

Initial condition $(1, 0)$: $1 = Ce^0 = C$

Particular solution: $x = e^{\sin(y/x)}$

48. $\left(2x^2 + y^2\right)dx + xy\,dy = 0$

Let $y = vx$, $dy = x\,dv + v\,dx$.

$$\left(2x^2 + v^2x^2\right)dx + x(vx)\left(x\,dv + v\,dx\right) = 0$$

$$\left(2x^2 + 2x^2v^2\right)dx + x^3v\,dv = 0$$

$$\left(2 + 2v^2\right)dx = -xv\,dv$$

$$\frac{-2}{x}\,dx = \frac{v}{1 + v^2}dv$$

$$-2\ln x = \frac{1}{2}\ln\left(1 + v^2\right) + C_1$$

$$\ln x^{-2} = \ln\left(1 + v^2\right)^{1/2} + \ln C$$

$$x^{-2} = C\left(1 + v^2\right)^{1/2}$$

$$\frac{1}{x^2} = C\left(1 + \frac{y^2}{x^2}\right)^{1/2} = \frac{C}{x}\left(x^2 + y^2\right)^{1/2}$$

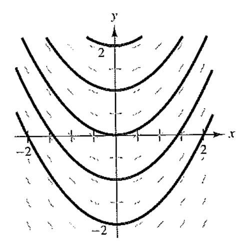
$$\frac{1}{x} = C\left(x^2 + y^2\right)^{1/2}$$

Initial condition $(1, 0)$: $1 = C(1 + 0) \Rightarrow C = 1$

Particular solution: $\frac{1}{x} = \sqrt{x^2 + y^2}$

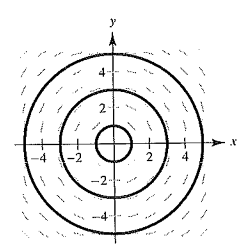
$$1 = x\sqrt{x^2 + y^2}$$

49. $\frac{dy}{dx} = x$



$$y = \int x\,dx = \frac{1}{2}x^2 + C$$

50. $\frac{dy}{dx} = -\frac{x}{y}$

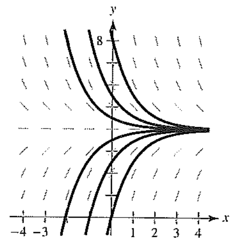


$$y\,dy = -x\,dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C_1$$

$$y^2 + x^2 = C$$

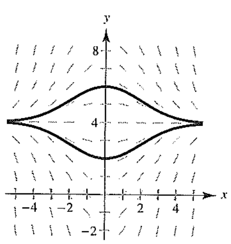
51. $\dfrac{dy}{dx} = 4 - y$



$$\int \frac{dy}{4-y} = \int dx$$
$$\ln|4 - y| = -x + C_1$$
$$4 - y = e^{-x+C_1}$$
$$y = 4 + Ce^{-x}$$

52. $\dfrac{dy}{dx} = 0.25x(4 - y)$

$$\frac{dy}{4-y} = 0.25x\,dx$$
$$\int \frac{dy}{y-4} = \int -0.25x\,dx = -\frac{1}{4}\int x\,dx$$
$$\ln|y - 4| = -\frac{1}{8}x^2 + C_1$$
$$y - 4 = e^{C_1 - (1/8)x^2} = Ce^{-(1/8)x^2}$$
$$y = 4 + Ce^{-(1/8)x^2}$$



53. (a) Euler's Method gives $y \approx 0.1602$ when $x = 1$.

(b) $\dfrac{dy}{dx} = -6xy$

$$\int \frac{dy}{y} = \int -6x$$
$$\ln|y| = -3x^2 + C_1$$
$$y = Ce^{-3x^2}$$
$$y(0) = 5 \Rightarrow C = 5$$
$$y = 5e^{-3x^2}$$

(c) At $x = 1$, $y = 5e^{-3(1)} \approx 0.2489$.

Error: $0.2489 - 0.1602 \approx 0.0887$

54. (a) Euler's Method gives $y \approx 0.2622$ when $x = 1$.

(b) $\dfrac{dy}{dx} = -6xy^2$

$$\int \frac{dy}{y^2} = \int -6x\,dx$$
$$\frac{-1}{y} = -3x^2 + C_1$$
$$y = \frac{1}{3x^2 + C}$$
$$3 = \frac{1}{C} \Rightarrow C = \frac{1}{3}$$
$$y = \frac{1}{3x^2 + \frac{1}{3}} = \frac{3}{9x^2 + 1}$$

(c) At $x = 1$, $y = \dfrac{3}{9(1) + 1} = \dfrac{3}{10} = 0.3$.

Error: $0.3 - 0.2622 = 0.0378$

55. (a) Euler's Method gives $y \approx 3.0318$ when $x = 2$.

(b) $\dfrac{dy}{dx} = \dfrac{2x + 12}{3y^2 - 4}$

$$\int (3y^2 - 4)\,dy = \int (2x + 12)\,dx$$
$$y^3 - 4y = x^2 + 12x + C$$
$$y(1) = 2: 2^3 - 4(2) = 1 + 12 + C \Rightarrow C = -13$$
$$y^3 - 4y = x^2 + 12x - 13$$

(c) At $x = 2$,

$$y^3 - 4y = 2^2 + 12(2) - 13 = 15$$
$$y^3 - 4y - 15 = 0$$
$$(y - 3)(y^2 + 3y + 5) = 0 \Rightarrow y = 3.$$

Error: $3.0318 - 3 = 0.0318$

56. (a) Euler's Method gives $y \approx 1.7270$ when $x = 1.5$.

(b) $\dfrac{dy}{dx} = 2x(1 + y^2)$

$$\int \frac{dy}{1 + y^2} = \int 2x\,dx$$
$$\arctan y = x^2 + C$$
$$\arctan(0) = 1^2 + C \Rightarrow C = -1$$
$$\arctan(y) = x^2 - 1$$
$$y = \tan(x^2 - 1)$$

(c) At $x = 1.5$, $y = \tan(1.5^2 - 1) \approx 3.0096$.

Error: $1.7270 - 3.0096 = -1.2826$

57. $\dfrac{dy}{dt} = ky, \quad y = Ce^{kt}$

Initial amount: $y(0) = y_0 = C$

Half-life: $\dfrac{y_0}{2} = y_0 e^{k(1599)}$

$$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right)$$

$y = Ce^{[\ln(1/2)/1599]t}$

When $t = 50$, $y = 0.9786C$ or 97.86%.

58. $\dfrac{dy}{dt} = ky, \ y = Ce^{kt}$

Initial conditions: $y(0) = 40, y(1) = 35$

$$40 = Ce^0 = C$$
$$35 = 40e^k$$
$$k = \ln\frac{7}{8}$$

Particular solution: $y = 40e^{t\ln(7/8)}$

When 75% has been changed:

$10 = 40e^{t\ln(7/8)}$

$\dfrac{1}{4} = e^{t\ln(7/8)}$

$t = \dfrac{\ln(1/4)}{\ln(7/8)} \approx 10.38$ hours

59. (a) $\dfrac{dy}{dx} = k(y - 4)$

(b) The direction field satisfies $(dy/dx) = 0$ along $y = 4$; but not along $y = 0$. Matches (a).

60. (a) $\dfrac{dy}{dx} = k(x - 4)$

(b) The direction field satisfies $(dy/dx) = 0$ along $x = 4$. Matches (b).

61. (a) $\dfrac{dy}{dx} = ky(y - 4)$

(b) The direction field satisfies $(dy/dx) = 0$ along $y = 0$ and $y = 4$. Matches (c).

62. (a) $\dfrac{dy}{dx} = ky^2$

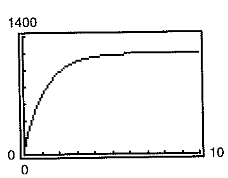
(b) The direction field satisfies $(dy/dx) = 0$ along $y = 0$, and grows more positive as y increases. Matches (d).

63.
$$\frac{dw}{dt} = k(1200 - w)$$
$$\int \frac{dw}{1200 - w} = \int k\,dt$$
$$\ln|1200 - w| = -kt + C_1$$
$$1200 - w = e^{-kt + C_1} = Ce^{-kt}$$
$$w = 1200 - Ce^{-kt}$$

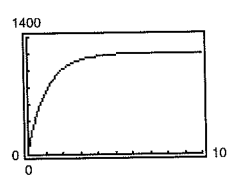
$w(0) = 60 = 1200 - C \Rightarrow C = 1200 - 60 = 1140$

$w = 1200 - 1140e^{-kt}$

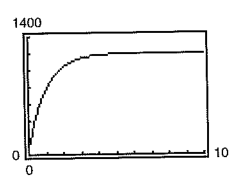
(a)



$k = 0.8$



$k = 0.9$



$k = 1$

(b) $k = 0.8$: $t = 1.31$ years

$k = 0.9$: $t = 1.16$ years

$k = 1.0$: $t = 1.05$ years

(c) Maximum weight: 1200 pounds

$\lim\limits_{x \to \infty} w = 1200$

64. From Exercise 63:

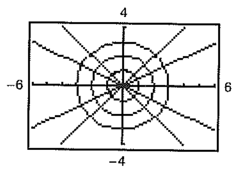
$$w = 1200 - Ce^{-kt}, k = 1$$
$$w = 1200 - Ce^{-t}$$
$$w(0) = w_0 = 1200 - C \Rightarrow C = 1200 - w_0$$
$$w = 1200 - (1200 - w_0)e^{-t}$$

65. Given family (circles): $x^2 + y^2 = C$

$$2x + 2yy' = 0$$
$$y' = -\frac{x}{y}$$

Orthogonal trajectory (lines): $y' = \frac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$
$$\ln|y| = \ln|x| + \ln K$$
$$y = Kx$$

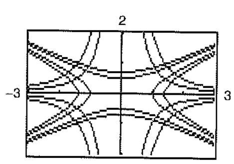


66. Given family (hyperbolas): $x^2 - 2y^2 = C$

$$2x - 4yy' = 0$$
$$y' = \frac{x}{2y}$$

Orthogonal trajectory: $y' = \frac{-2y}{x}$

$$\int \frac{dy}{y} = -\int \frac{2}{x}\,dx$$
$$\ln y = -2 \ln x + \ln k$$
$$y = kx^{-2} = \frac{k}{x^2}$$

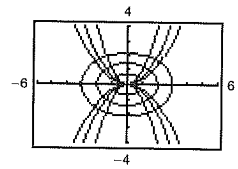


67. Given family (parabolas): $x^2 = Cy$

$$2x = Cy'$$
$$y' = \frac{2x}{C} = \frac{2x}{x^2/y} = \frac{2y}{x}$$

Orthogonal trajectory (ellipses): $y' = -\frac{x}{2y}$

$$2\int y\,dy = -\int x\,dx$$
$$y^2 = -\frac{x^2}{2} + K_1$$
$$x^2 + 2y^2 = K$$

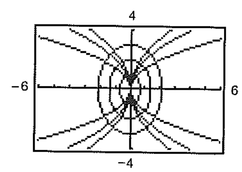


68. Given family (parabolas): $y^2 = 2Cx$

$$2yy' = 2C$$
$$y' = \frac{C}{y} = \frac{y^2}{2x}\left(\frac{1}{y}\right) = \frac{y}{2x}$$

Orthogonal trajectory (ellipse): $y' = -\frac{2x}{y}$

$$\int y\,dy = -\int 2x\,dx$$
$$\frac{y^2}{2} = -x^2 + K_1$$
$$2x^2 + y^2 = K$$

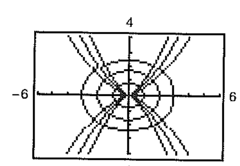


69. Given family: $y^2 = Cx^3$

$$2yy' = 3Cx^2$$
$$y' = \frac{3Cx^2}{2y} = \frac{3x^2}{2y}\left(\frac{y^2}{x^3}\right) = \frac{3y}{2x}$$

Orthogonal trajectory (ellipses): $y' = -\frac{2x}{3y}$

$$3\int y\,dy = -2\int x\,dx$$
$$\frac{3y^2}{2} = -x^2 + K_1$$
$$3y^2 + 2x^2 = K$$



70. Given family (exponential functions): $y = Ce^x$

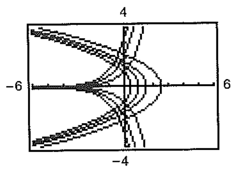
$$y' = Ce^x = y$$

Orthogonal trajectory (parabolas): $y' = -\frac{1}{y}$

$$\int y\,dy = -\int dx$$

$$\frac{y^2}{2} = -x + K_1$$

$$y^2 = -2x + K$$



71. $y = \frac{12}{1 + e^{-x}}$

Because $y(0) = 6$, it matches (c) or (d).

Because (d) approaches its horizontal asymptote slower than (c), it matches (d).

72. $y = \frac{12}{1 + 3e^{-x}}$

Because $y(0) = \frac{12}{4} = 3$, it matches (a).

73. $y = \frac{12}{1 + \frac{1}{2}e^{-x}}$

Because $y(0) = \frac{12}{\left(\frac{3}{2}\right)} = 8$, it matches (b).

74. $y = \frac{12}{1 + e^{-2x}}$

Because $y(0) = 6$, it matches (c) or (d).

Because y approaches $L = 12$ faster for (c), it matches (c).

75. $P(t) = \frac{2100}{1 + 29e^{-0.75t}}$

(a) $k = 0.75$

(b) $L = 2100$

(c) $P(0) = \frac{2100}{1 + 29} = 70$

(d)
$$1050 = \frac{2100}{1 + 29e^{-0.75t}}$$
$$1 + 29e^{-0.75t} = 2$$
$$e^{-0.75t} = \frac{1}{29}$$
$$-0.75t = \ln\left(\frac{1}{29}\right) = -\ln 29$$
$$t = \frac{\ln 29}{0.75} \approx 4.4897 \text{ yr}$$

(e) $\frac{dP}{dt} = 0.75P\left(1 - \frac{P}{2100}\right), \quad P(0) = 70$

76. $P(t) = \frac{5000}{1 + 39e^{-0.2t}}$

(a) $k = 0.2$

(b) $L = 5000$

(c) $P(0) = \frac{5000}{1 + 39} = 125$

(d)
$$2500 = \frac{5000}{1 + 39e^{-0.2t}}$$
$$1 + 39e^{-0.2t} = 2$$
$$e^{-0.2t} = \frac{1}{39}$$
$$-0.2t = \ln\left(\frac{1}{39}\right) = -\ln 39$$
$$t = \frac{\ln 39}{0.2} \approx 18.3178$$

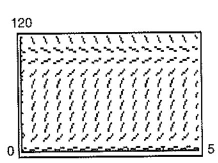
(e) $\frac{dP}{dt} = 0.2P\left(1 - \frac{P}{5000}\right), \quad P(0) = 125$

77. $\frac{dP}{dt} = 3P\left(1 - \frac{P}{100}\right)$

(a) $k = 3$

(b) $L = 100$

(c)



(d) $\frac{d^2P}{dt^2} = 3P'\left(1 - \frac{P}{100}\right) + 3P\left(\frac{-P'}{100}\right)$

$= 3\left[3P\left(1 - \frac{P}{100}\right)\right]\left(1 - \frac{P}{100}\right) - \frac{3P}{100}\left[3P\left(1 - \frac{P}{100}\right)\right] = 9P\left(1 - \frac{P}{100}\right)\left(1 - \frac{P}{100} - \frac{P}{100}\right) = 9P\left(1 - \frac{P}{100}\right)\left(1 - \frac{2P}{100}\right)$

$\frac{d^2P}{dt^2} = 0$ for $P = 50$, and by the first Derivative Test, this is a maximum. $\left(\text{Note: } P = 50 = \frac{L}{2} = \frac{100}{2}\right)$

78. $\frac{dP}{dt} = 0.1P - 0.0004P^2$

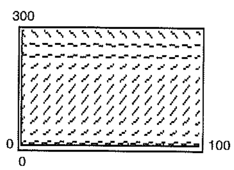
$= 0.1P(1 - 0.004P)$

$= 0.1P\left(1 - \frac{P}{250}\right)$

(a) $k = 0.1 = \frac{1}{10}$

(b) $L = 250$

(c)



(d) $P = \frac{250}{2} = 125$. (Same argument as in Exercise 77)

79. $\frac{dy}{dt} = y\left(1 - \frac{y}{36}\right), \quad y(0) = 4$

$k = 1, L = 36$

$y = \frac{L}{1 + be^{-kt}} = \frac{36}{1 + be^{-t}}$

$(0, 4)$: $4 = \frac{36}{1 + b} \Rightarrow b = 8$

Solution: $y = \frac{36}{1 + 8e^{-t}}$

80. $\frac{dy}{dt} = 2.8y\left(1 - \frac{y}{10}\right), \quad y(0) = 7$

$k = 2.8, L = 10$

$y = \frac{L}{1 + be^{-kt}} = \frac{10}{1 + be^{-2.8t}}$

$(0, 7)$: $7 = \frac{10}{1 + b} \Rightarrow 1 + b = \frac{10}{7} \Rightarrow b = \frac{3}{7}$

Solution: $y = \frac{10}{1 + \left(\frac{3}{7}\right)e^{-2.8t}}$

81. $\frac{dy}{dt} = \frac{4y}{5} - \frac{y^2}{150} = \frac{4}{5}y\left(1 - \frac{y}{120}\right), \quad y(0) = 8$

$k = \frac{4}{5} = 0.8, L = 120$

$y = \frac{L}{1 + be^{-kt}} = \frac{120}{1 + be^{-0.8t}}$

$(0, 8)$: $8 = \frac{120}{1 + b} \Rightarrow b = 14$

Solution: $y = \frac{120}{1 + 14e^{-0.8t}}$

82. $\frac{dy}{dt} = \frac{3y}{20} - \frac{y^2}{1600} = \frac{3}{20}y\left(1 - \frac{y}{240}\right), \quad y(0) = 15$

$k = \frac{3}{20}, L = 240$

$y = \frac{L}{1 + be^{-kt}} = \frac{240}{1 + be^{(-3/20)t}}$

$(0, 15)$: $15 = \frac{240}{1 + b} \Rightarrow b = 15$

Solution: $y = \frac{240}{1 + 15e^{(-3/20)t}}$

83. (a) $P = \frac{L}{1 + be^{-kt}}, L = 200, P(0) = 25$

$25 = \frac{200}{1 + b} \Rightarrow b = 7$

$39 = \frac{200}{1 + 7e^{-k(2)}}$

$1 + 7e^{-2k} = \frac{200}{39}$

$e^{-2k} = \frac{23}{39}$

$k = -\frac{1}{2}\ln\left(\frac{23}{39}\right) = \frac{1}{2}\ln\left(\frac{39}{23}\right) \approx 0.2640$

$P = \frac{200}{1 + 7e^{-0.2640t}}$

(b) For $t = 5$, $P \approx 70$ panthers.

(c)
$$100 = \frac{200}{1 + 7e^{-0.264t}}$$
$$1 + 7e^{-0.264t} = 2$$
$$-0.264t = \ln\left(\frac{1}{7}\right)$$
$$t \approx 7.37 \text{ years}$$

(d)
$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$
$$= 0.264P\left(1 - \frac{P}{200}\right),\ P(0) = 25$$

Using Euler's Method, $P \approx 65.6$ when $t = 5$.

(e) P is increasing most rapidly where $P = 200/2 = 100$, corresponds to $t \approx 7.37$ years.

84. (a)
$$y = \frac{L}{1 + be^{-kt}},\ L = 20,\ y(0) = 1,\ y(2) = 4$$
$$1 = \frac{20}{1 + b} \Rightarrow b = 19$$
$$4 = \frac{20}{1 + 19e^{-2k}}$$
$$1 + 19e^{-2k} = 5$$
$$19e^{-2k} = 4$$
$$k = -\frac{1}{2}\ln\left(\frac{4}{19}\right) = \frac{1}{2}\ln\left(\frac{19}{4}\right) \approx 0.7791$$
$$y = \frac{20}{1 + 19e^{-0.7791t}}$$

(b) For $t = 5$, $y \approx 14.43$ grams

(c)
$$18 = \frac{20}{1 + 19e^{-0.7791t}}$$
$$1 + 19e^{-0.7791t} = \frac{20}{18} = \frac{10}{9}$$
$$19e^{-0.7791t} = \frac{1}{9}$$
$$e^{-0.7791t} = \frac{1}{171}$$
$$t = \frac{-1}{0.7791}\ln\left(\frac{1}{171}\right) \approx 6.60 \text{ hours}$$

(d)
$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) = \frac{1}{2}\ln\left(\frac{19}{4}\right)y\left(1 - \frac{y}{20}\right)$$

t	0	1	2	3	4	5
Exact	1	2.06	4.00	7.05	10.86	14.43
Euler	1	1.74	2.98	4.95	7.86	11.57

(e) The weight is increasing most rapidly when $y = L/2 = 20/2 = 10$, corresponding to $t \approx 3.78$ hours.

85. A differential equation can be solved by separation of variables if it can be written in the form

$$M(x) + N(y)\frac{dy}{dx} = 0.$$

To solve a separable equation, rewrite as,

$$M(x)\,dx = -N(y)\,dy$$

and integrate both sides.

86. $M(x, y)\,dx + N(x, y)\,dy = 0$, where M and N are homogeneous functions of the same degree. See Example 7a.

87. Two families of curves are mutually orthogonal if each curve in the first family intersects each curve in the second family at right angles.

88. Answers will vary. *Sample answer*: There might be limits on available food or space.

89. $y = \dfrac{1}{1 + be^{-kt}}$

$$y' = \frac{-1}{\left(1 + be^{-kt}\right)^2}\left(-bke^{-kt}\right)$$

$$= \frac{k}{\left(1 + be^{-kt}\right)} \cdot \frac{be^{-kt}}{\left(1 + be^{-kt}\right)}$$

$$= \frac{k}{\left(1 + be^{-kt}\right)} \cdot \frac{1 + be^{-kt} - 1}{\left(1 + be^{-kt}\right)}$$

$$= \frac{k}{\left(1 + be^{-kt}\right)} \cdot \left(1 - \frac{1}{1 + be^{-kt}}\right) = ky(1 - y)$$

90. (a)

$$\frac{dv}{dt} = k(W - v)$$

$$\int \frac{dv}{W - v} = \int k\,dt$$

$$-\ln|W - v| = kt + C_1$$

$$v = W - Ce^{-kt}$$

Initial conditions:

$W = 20$, $v = 0$ when $t = 0$ and $v = 10$ when $t = 0.5$ so, $C = 20$, $k = \ln 4$.

Particular solution:

$$v = 20\left(1 - e^{-(\ln 4)t}\right) = 20\left(1 - \left(\frac{1}{4}\right)^t\right)$$

or

$$v = 20\left(1 - e^{-1.386t}\right)$$

(b) $s = \int 20\left(1 - e^{-1.386t}\right)dt \approx 20\left(t + 0.7215e^{-1.386t}\right) + C$

Because $s(0) = 0$, $C \approx -14.43$ and you have

$$s \approx 20t + 14.43\left(e^{-1.386t} - 1\right).$$

91. False. $\dfrac{dy}{dx} = \dfrac{x}{y}$ is separable, but $y = 0$ is not a solution.

92. True

$$\frac{dy}{dx} = (x - 2)(y + 1)$$

93. False

$$f(tx, ty) = t^2x^2 - 4xyt^2 + 6t^2y^2 + 1$$
$$\neq t^2 f(x, y)$$

94. True

$x^2 + y^2 = 2Cy$ $\qquad$ $x^2 + y^2 = 2Kx$

$\dfrac{dy}{dx} = \dfrac{x}{C - y}$ $\qquad$ $\dfrac{dy}{dx} = \dfrac{K - x}{y}$

$$\frac{x}{C - y} \cdot \frac{K - x}{y} = \frac{Kx - x^2}{Cy - y^2}$$
$$= \frac{2Kx - 2x^2}{2Cy - 2y^2}$$
$$= \frac{x^2 + y^2 - 2x^2}{x^2 + y^2 - 2y^2}$$
$$= \frac{y^2 - x^2}{x^2 - y^2}$$
$$= -1$$

95.

$$fg' + gf' = f'g' \qquad \text{Product Rule}$$
$$(f - f')g' + gf' = 0$$
$$g' + \frac{f'}{f - f'}g = 0$$

Need $f - f' = e^{x^2} - 2xe^{x^2} = (1 - 2x)e^{x^2} \neq 0$, so avoid $x = \dfrac{1}{2}$.

$$\frac{g'}{g} = \frac{f'}{f' - f} = \frac{2xe^{x^2}}{(2x - 1)e^{x^2}} = 1 + \frac{1}{2x - 1}$$

$$\ln|g(x)| = x + \frac{1}{2}\ln|2x - 1| + C_1$$

$$g(x) = Ce^x|2x - 1|^{1/2}$$

So there exists g and interval (a, b), as long as $\dfrac{1}{2} \notin (a, b)$.

Section 6.4 First-Order Linear Differential Equations

1. $x^3y' + xy = e^x + 1$

$y' + \frac{1}{x^2}y = \frac{1}{x^3}(e^x + 1)$

Linear

2. $2xy - y'\ln x = y$

$(\ln x)y' + (1 - 2x)y = 0$

$y' + \frac{(1 - 2x)}{\ln x}y = 0$

Linear

3. $y' - y\sin x = xy^2$

Not linear, because of the xy^2-term.

4. $\frac{2 - y'}{y} = 5x$

$2 - y' = 5xy$

$y' + 5xy = 2$

Linear

5. $\frac{dy}{dx} + \left(\frac{1}{x}\right)y = 6x + 2$

Integrating factor: $e^{\int(1/x)\,dx} = e^{\ln x} = x$

$xy = \int x(6x + 2)\,dx = 2x^3 + x^2 + C$

$y = 2x^2 + x + \frac{C}{x}$

6. $\frac{dy}{dx} + \frac{2}{x}y = 3x - 5$

Integrating factor: $e^{\int 2/x\,dx} = e^{\ln x^2} = x^2$

$x^2y = \int x^2(3x - 5)\,dx = \frac{3}{4}x^4 + \frac{5x^3}{3} + C$

$y = \frac{3}{4}x^2 + \frac{5}{3}x + \frac{C}{x^2}$

7. $y' - y = 16$

Integrating factor: $e^{\int -1\,dx} = e^{-x}$

$e^{-x}y' - e^{-x}y = 16e^{-x}$

$ye^{-x} = \int 16e^{-x}\,dx = -16e^{-x} + C$

$y = -16 + Ce^x$

8. $y' + 2xy = 10x$

Integrating factor: $e^{\int 2x\,dx} = e^{x^2}$

$ye^{x^2} = \int 10xe^{x^2}\,dx = 5e^{x^2} + C$

$y = 5 + Ce^{-x^2}$

9. $(y + 1)\cos x\,dx = dy$

$y' = (y + 1)\cos x = y\cos x + \cos x$

$y' - (\cos x)y = \cos x$

Integrating factor: $e^{\int -\cos x\,dx} = e^{-\sin x}$

$y'e^{-\sin x} - (\cos x)e^{-\sin x}y = (\cos x)e^{-\sin x}$

$ye^{-\sin x} = \int(\cos x)e^{-\sin x}\,dx$

$= -e^{-\sin x} + C$

$y = -1 + Ce^{\sin x}$

10. $[(y - 1)\sin x]\,dx - dy = 0$

$y' - (\sin x)y = -\sin x$

Integrating factor: $e^{\int -\sin x\,dx} = e^{\cos x}$

$ye^{\cos x} = \int -\sin xe^{\cos x}\,dx = e^{\cos x} + C$

$y = 1 + Ce^{-\cos x}$

11. $(x - 1)y' + y = x^2 - 1$

$y' + \left(\frac{1}{x - 1}\right)y = x + 1$

Integrating factor: $e^{\int[1/(x-1)]\,dx} = e^{\ln|x-1|} = x - 1$

$y(x - 1) = \int(x^2 - 1)\,dx = \frac{1}{3}x^3 - x + C_1$

$y = \frac{x^3 - 3x + C}{3(x - 1)}$

12. $y' + 3y = e^{3x}$

Integrating factor: $e^{\int 3\,dx} = e^{3x}$

$ye^{3x} = \int e^{3x}e^{3x}dx = \int e^{6x}dx = \frac{1}{6}e^{6x} + C$

$y = \frac{1}{6}e^{3x} + Ce^{-3x}$

13. $y' - 3x^2y = e^{x^3}$

Integrating factor: $e^{-\int 3x^2\,dx} = e^{-x^3}$

$$ye^{-x^3} = \int e^{x^3}e^{-x^3}dx = \int dx = x + C$$

$$y = (x + C)e^{x^3}$$

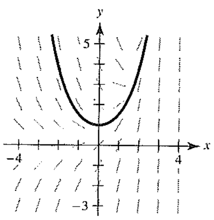
14. $y' + y\tan x = \sec x$

Integrating factor: $e^{\int \tan x\,dx} = e^{-\ln|\cos x|} = \sec x$

$$y\sec x = \int \sec^2 x\,dx = \tan x + C$$

$$y = \sin x + C\cdot\cos x$$

15. (a) Answers will vary.



(b) $\dfrac{dy}{dx} = e^x - y$

$\dfrac{dy}{dx} + y = e^x$ Integrating factor: $e^{\int dx} = e^x$

$$e^xy' + e^xy = e^{2x}$$

$$(ye^x) = \int e^{2x}dx$$

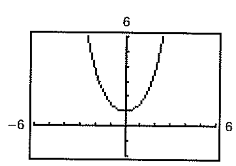
$$ye^x = \frac{1}{2}e^{2x} + C$$

$$y(0) = 1 \Rightarrow 1 = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}$$

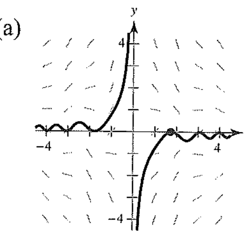
$$ye^x = \frac{1}{2}e^{2x} + \frac{1}{2}$$

$$y = \frac{1}{2}e^x + \frac{1}{2}e^{-x} = \frac{1}{2}(e^x + e^{-x})$$

(c)



16. (a)



(b) $y' + \dfrac{1}{x}y = \sin x^2,\ P(x) = \dfrac{1}{x},\ Q(x) = \sin x^2$

$$u(x) = e^{\int (1/x)\,dx} = e^{\ln x} = x$$

$$y'x + y = x\sin x^2$$

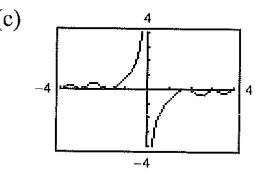
$$yx = \int x\sin x^2\,dx = -\frac{1}{2}\cos x^2 + C$$

$$y = \frac{1}{x}\left[-\frac{1}{2}\cos x^2 + C\right]$$

$$0 = \frac{1}{\sqrt{\pi}}\left[-\frac{1}{2}\cos\pi + C\right] \Rightarrow C = -\frac{1}{2}$$

$$y = \frac{1}{x}\left[-\frac{1}{2}\cos x^2 - \frac{1}{2}\right]$$

(c)



17. $y'\cos^2 x + y - 1 = 0$

$$y' + (\sec^2 x)y = \sec^2 x$$

Integrating factor: $e^{\int \sec^2 x\,dx} = e^{\tan x}$

$$ye^{\tan x} = \int \sec^2 x e^{\tan x}\,dx = e^{\tan x} + C$$

$$y = 1 + Ce^{-\tan x}$$

Initial condition: $y(0) = 5, C = 4$

Particular solution: $y = 1 + 4e^{-\tan x}$

18. $x^3y' + 2y = e^{1/x^2}$

$$y' + \left(\frac{2}{x^3}\right)y = \frac{1}{x^3}e^{1/x^2}$$

Integrating factor: $e^{\int (2/x^3)\,dx} = e^{-(1/x^2)}$

$$ye^{-1/x^2} = \int \frac{1}{x^3}\,dx = -\frac{1}{2x^2} + C_1$$

$$y = e^{1/x^2}\left(\frac{Cx^2 - 1}{2x^2}\right)$$

Initial condition: $y(1) = e, C = 3$

Particular solution: $y = e^{1/x^2}\left(\dfrac{3x^2 - 1}{2x^2}\right)$

19. $y' + y\tan x = \sec x + \cos x$

Integrating factor: $e^{\int \tan x\,dx} = e^{\ln|\sec x|} = \sec x$

$$y\sec x = \int \sec x(\sec x + \cos x)\,dx = \tan x + x + C$$
$$y = \sin x + x\cos x + C\cos x$$

Initial condition: $y(0) = 1, 1 = C$

Particular solution: $y = \sin x + (x + 1)\cos x$

20. $y' + y\sec x = \sec x$

Integrating factor:

$$e^{\int \sec x\,dx} = e^{\ln|\sec x + \tan x|} = \sec x + \tan x$$

$$y(\sec x + \tan x) = \int(\sec x + \tan x)\sec x\,dx$$
$$= \sec x + \tan x + C$$
$$y = 1 + \frac{C}{\sec x + \tan x}$$

Initial condition: $y(0) = 4, 4 = 1 + \frac{C}{1 + 0}, C = 3$

Particular solution:

$$y = 1 + \frac{3}{\sec x + \tan x} = 1 + \frac{3\cos x}{1 + \sin x}$$

21. $y' + \left(\frac{1}{x}\right)y = 0$

Integrating factor: $e^{\int(1/x)\,dx} = e^{\ln|x|} = x$

Separation of variables:

$$\frac{dy}{dx} = -\frac{y}{x}$$
$$\int \frac{1}{y}\,dy = \int -\frac{1}{x}\,dx$$
$$\ln y = -\ln x + \ln C$$
$$\ln xy = \ln C$$
$$xy = C$$

Initial condition: $y(2) = 2, C = 4$

Particular solution: $xy = 4$

22. $y' + (2x - 1)y = 0$

Integrating factor: $e^{\int(2x-1)\,dx} = e^{x^2 - x}$

$$ye^{x^2 - x} = C$$
$$y = Ce^{x - x^2}$$

Separation of variables:

$$\int \frac{1}{y}\,dy = \int(1 - 2x)\,dx$$
$$\ln y + \ln C_1 = x - x^2$$
$$yC_1 = e^{x - x^2}$$
$$y = Ce^{x - x^2}$$

Initial condition: $y(1) = 2, 2 = C$

Particular solution: $y = 2e^{x - x^2}$

23. $x\,dy = (x + y + 2)\,dx$

$$\frac{dy}{dx} = \frac{x + y + 2}{x} = \frac{y}{x} + 1 + \frac{2}{x}$$

$$\frac{dy}{dx} - \frac{1}{x}y = 1 + \frac{2}{x} \qquad \text{Linear}$$

$$u(x) = e^{\int -(1/x)\,dx} = \frac{1}{x}$$

$$y = x\int\left(1 + \frac{2}{x}\right)\frac{1}{x}\,dx = x\int\left(\frac{1}{x} + \frac{2}{x^2}\right)dx$$
$$= x\left[\ln|x| + \frac{-2}{x} + C\right]$$
$$= -2 + x\ln|x| + Cx$$

$$y(1) = 10 = -2 + C \Rightarrow C = 12$$
$$y = -2 + x\ln|x| + 12x$$

24. $2xy' - y = x^3 - x$

$$\frac{dy}{dx} - \frac{1}{2x}y = \frac{x^2}{2} - \frac{1}{2} \qquad \text{Linear}$$

$$u(x) = e^{\int -(1/2x)\,dx} = \frac{1}{x^{1/2}}$$

$$y = x^{1/2}\int\left(\frac{x^2}{2} - \frac{1}{2}\right)\frac{1}{x^{1/2}}\,dx = x^{1/2}\int\left(\frac{x^{3/2}}{2} - \frac{x^{-1/2}}{2}\right)dx$$
$$= x^{1/2}\left[\frac{x^{5/2}}{5} - x^{1/2} + C\right]$$
$$= \frac{x^3}{5} - x + C\sqrt{x}$$

$$y(4) = 2 = \frac{64}{5} - 4 + 2C \Rightarrow C = -\frac{17}{5}$$
$$y = \frac{x^3}{5} - x - \frac{17}{5}\sqrt{x}$$

25.
$$\frac{dP}{dt} = kP + N,\ N \text{ constant}$$
$$\frac{dP}{kP + N} = dt$$
$$\int \frac{1}{kP + N}\,dP = \int dt$$
$$\frac{1}{k}\ln(kP + N) = t + C_1$$
$$\ln(kP + N) = kt + C_2$$
$$kP + N = e^{kt + C_2}$$
$$P = \frac{C_3e^{kt} - N}{k}$$
$$P = Ce^{kt} - \frac{N}{k}$$

When $t = 0$: $P = P_0$

$$P_0 = C - \frac{N}{k} \Rightarrow C = P_0 + \frac{N}{k}$$
$$P = \left(P_0 + \frac{N}{k}\right)e^{kt} - \frac{N}{k}$$

26.
$$\frac{dA}{dt} = rA + P$$
$$\frac{dA}{rA + P} = dt$$
$$\int \frac{dA}{rA + P} = \int dt$$
$$\frac{1}{r}\ln(rA + P) = t + C_1$$
$$\ln(rA + P) = rt + C_2$$
$$rA + P = e^{rt + C_2}$$
$$A = \frac{C_3e^{rt} - P}{r}$$
$$A = Ce^{rt} - \frac{P}{r}$$

When $t = 0$: $A = 0$

$$0 = C - \frac{P}{r} \Rightarrow C = \frac{P}{r}$$
$$A = \frac{P}{r}\left(e^{rt} - 1\right)$$

27. (a) $A = \frac{P}{r}\left(e^{rt} - 1\right)$

$$A = \frac{275{,}000}{0.06}\left(e^{0.08(10)} - 1\right) \approx \$4{,}212{,}796.94$$

(b) $A = \frac{550{,}000}{0.05}\left(e^{0.059(25)} - 1\right) \approx \$31{,}424{,}909.75$

28. $1{,}000{,}000 = \frac{125{,}000}{0.08}\left(e^{0.08t} - 1\right)$

$$1.64 = e^{0.08t}$$
$$t = \frac{\ln(1.64)}{0.08} \approx 6.18 \text{ years}$$

29. (a) $\frac{dQ}{dt} = q - kQ,\ q$ constant

(b) $Q' + kQ = q$

Let $P(t) = k$, $Q(t) = q$, then the integrating factor is $u(t) = e^{kt}$.

$$Q = e^{-kt}\int qe^{kt}\,dt = e^{-kt}\left(\frac{q}{k}e^{kt} + C\right) = \frac{q}{k} + Ce^{-kt}$$

When $t = 0$: $Q = Q_0$

$$Q_0 = \frac{q}{k} + C \Rightarrow C = Q_0 - \frac{q}{k}$$
$$Q = \frac{q}{k} + \left(Q_0 - \frac{q}{k}\right)e^{-kt}$$

(c) $\lim_{t\to\infty} Q = \frac{q}{k}$

30. (a) $\frac{dN}{dt} = k(75 - N)$

(b) $N' + kN = 75k$

Integrating factor: $e^{\int k\,dt} = e^{kt}$

$$N'e^{kt} + kNe^{kt} = 75\,ke^{kt}$$
$$\left(Ne^{kt}\right)' = 75\,ke^{kt}$$
$$Ne^{kt} = \int 75\,ke^{kt} = 75\,e^{kt} + C$$
$$N = 75 + Ce^{-kt}$$

(c) For $t = 1$, $N = 20$:

$$20 = 75 + Ce^{-k} \Rightarrow -55 = Ce^{-k}$$

For $t = 20$, $N = 35$:

$$35 = 75 + Ce^{-20k} \Rightarrow -40 = Ce^{-20k}$$

$$\frac{55}{40} = \frac{Ce^{-k}}{Ce^{-20k}} \Rightarrow e^{19k} = \frac{11}{8} \Rightarrow k = \frac{1}{19}\ln\left(\frac{11}{8}\right) \approx 0.0168$$

$$Ce^{-k} = -55$$
$$C = -55e^{k} \approx -55.9296$$
$$N = 75 - 55.9296\,e^{-0.0168t}$$

31. Let Q be the number of pounds of concentrate in the solution at any time t. Because the number of gallons of solution in the tank at any time t is $v_0 + (r_1 - r_2)t$ and because the tank loses r_2 gallons of solution per minute, it must lose concentrate at the rate

$$\left[\frac{Q}{v_0 + (r_1 - r_2)t}\right]r_2.$$

The solution gains concentrate at the rate $r_1 q_1$. Therefore, the net rate of change is

$$\frac{dQ}{dt} = q_1 r_1 - \left[\frac{Q}{v_0 + (r_1 - r_2)t}\right]r_2$$

or

$$\frac{dQ}{dt} + \frac{r_2 Q}{v_0 + (r_1 - r_2)t} = q_1 r_1.$$

32. From Exercise 31, and using $r_1 = r_2 = r$,

$$\frac{dQ}{dt} + \frac{rQ}{v_0} = q_1 r.$$

33. (a) $Q' + \dfrac{r^2 Q}{v_0 + (r_1 - r_2)t} = q_1 r_1$

$Q(0) = q_0, q_0 = 25, q_1 = 0, v_0 = 200,$

$r_1 = 10, r_2 = 10, Q' + \dfrac{1}{20}Q = 0$

$$\int \frac{1}{Q}\,dQ = \int -\frac{1}{20}\,dt$$

$$\ln Q = -\frac{1}{20}t + \ln C_1$$

$$Q = Ce^{-(1/20)t}$$

Initial condition: $Q(0) = 25, C = 25$

Particular solution: $Q = 25e^{-(1/20)t}$

(b) $15 = 25e^{-(1/20)t}$

$$\ln\left(\frac{3}{5}\right) = -\frac{1}{20}t$$

$$t = -20\ln\left(\frac{3}{5}\right) \approx 10.2 \text{ min}$$

(c) $\lim_{t\to\infty} 25e^{-(1/20)t} = 0$

34. (a) $Q' + \dfrac{r^2 Q}{v_0 + (r_1 - r_2)t} = q_1 r_1$

$Q(0) = q_0 = 25, q_1 = 0.04, v_0 = 200,$

$r_1 = 10, r_2 = 10, Q' + \dfrac{1}{20}Q = 0.4$

Integrating factor: $e^{(1/20)t}$

$$Qe^{(-1/20)t} = \int 0.4e^{(1/20)t}\,dt = 8e^{(1/20)t} + C$$

$$Q = 8 + Ce^{(-1/20)t}$$

$$Q = (0) = 25 = 8 + C \Rightarrow C = 17$$

$$Q = 8 + 17e^{(-1/20)t}$$

(b) $15 = 8 + 17e^{(-1/20)t}$

$7 = 17e^{(-1/20)t}$

$$\ln\left(\frac{7}{17}\right) = -\frac{1}{20}t \Rightarrow t = -20\ln\left(\frac{7}{17}\right) \approx 17.75 \text{ min}$$

(c) $\lim_{t\to\infty} Q(t) = 8$ lbs

35. (a) The volume of the solution in the tank is given by $v_0 + (r_1 - r_2)t$. Therefore, $100 + (5 - 3)t = 200$ or $t = 50$ minutes.

(b) $Q' + \dfrac{r_2 Q}{v_0 + (r_1 - r_2)t} = q_1 r_1$

$Q(0) = q_0, q_0 = 0, q_1 = 0.5, v_0 = 100, r_1 = 5,$

$r_2 = 3, Q' + \dfrac{3}{100 + 2t}Q = 2.5$

Integrating factor: $e^{\int [3/(100+2t)]\,dt} = (50 + t)^{3/2}$

$$Q(50 + t)^{3/2} = \int 2.5(50 + t)^{3/2}\,dt = (50 + t)^{5/2} + C$$

$$Q = (50 + t) + C(50 + t)^{-3/2}$$

Initial condition:

$Q(0) = 0, 0 = 50 + C(50^{-3/2}), C = -50^{5/2}$

Particular solution:

$$Q = (50 + t) - 50^{-5/2}(50 + t)^{-3/2}$$

$$Q(50) = 100 - 50^{5/2}(100)^{-3/2}$$

$$= 100 - \frac{25}{\sqrt{2}} \approx 82.32 \text{ lbs}$$

(c) The volume of the solution is given by
$v_0 + (r_1 - r_2)t = 100 + (5 - 3)t = 200 \Rightarrow t = 50$ minutes.

$$Q' + \frac{r_2 Q}{v_0 + (r_1 - r_2)t} = q_1 r_1$$

$$Q(0) = q_0 = 0, q_1 = 1, v_0 = 100, r_1 = 5, r_2 = 3$$

$$Q' + \frac{3Q}{100 + 2t} = 5$$

Integrating factor is $(50 + t)^{3/2}$, as in #43.

$$Q(50 + t)^{3/2} = \int 5(50 + t)^{3/2}\, dt = 2(50 + t)^{5/2} + C$$

$$Q = 2(50 + t) + C(50 + t)^{-3/2}$$

$Q(0) = 0$:

$$0 = 100 + C(50)^{-3/2} \Rightarrow C = -100(50)^{3/2} = -2(50)^{5/2}$$

$$Q = 2(50 + t) - 2(50)^{5/2}(50 + t)^{-3/2}$$

When $t = 50$,

$$Q = 200 - 2(50)^{5/2}(100)^{-3/2} = 200 - \frac{50}{\sqrt{2}}$$

≈ 164.64 lbs (double the answer to part (b))

36. $y' + P(x)y = Q(x)$

Integrating factor: $u = e^{\int P(x)\,dx}$

$$y'u + P(x)yu = Q(x)u$$

$$(uy)' = Q(x)u$$

so $u'(x) = P(x)u$

Answer (a)

37. From Example 6,

$$\frac{dv}{dt} + \frac{kv}{m} = g$$

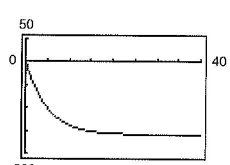
$$v = \frac{mg}{k}\left(1 - e^{-kt/m}\right), \quad \text{Solution}$$

$$g = -32, mg = -8, v(5) = -101, m = \frac{-8}{g} = \frac{1}{4}$$

implies that $-101 = \frac{-8}{k}\left(1 - e^{-5k/(1/4)}\right)$.

Using a graphing utility, $k \approx 0.050165$, and $v = -159.47\left(1 - e^{-0.2007t}\right)$.

As $t \to \infty$, $v \to -159.47$ ft/sec. The graph of v is shown below.



38.

$$s(t) = \int v(t)\, dt$$

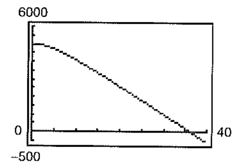
$$= \int -159.47\left(1 - e^{-0.2007t}\right) dt$$

$$= -159.47t - 794.57e^{-0.2007t} + C$$

$$s(0) = 5000 = -794.57 + C \Rightarrow C = 5794.57$$

$$s(t) = -159.47t - 794.57e^{-0.2007t} + 5794.57$$

The graph of $s(t)$ is shown below.



$s(t) = 0$ when $t \approx 36.33$ sec.

39. $L\frac{dI}{dt} + RI = E_0, I' + \frac{R}{L}I = \frac{E_0}{L}$

Integrating factor: $e^{\int (R/L)\,dt} = e^{Rt/L}$

$$I\, e^{Rt/L} = \int \frac{E_0}{L} e^{Rt/L}\, dt = \frac{E_0}{R} e^{Rt/L} + C$$

$$I = \frac{E_0}{R} + Ce^{-Rt/L}$$

40. $I(0) = 0, E_0 = 120$ volts, $R = 600$ ohms, $L = 4$ henrys

$$I = \frac{E_0}{R} + Ce^{-Rt/L}$$

$$(0) = \frac{120}{600} + C \Rightarrow C = -\frac{1}{5}$$

$$I = \frac{1}{5} - \frac{1}{5}e^{-150t}$$

$$\lim_{t \to \infty} I = \frac{1}{5} \text{ amp}$$

$$(0.90)\frac{1}{5} = 0.18 = \frac{1}{5}\left(1 - e^{-150t}\right)$$

$$0.9 = 1 - e^{-150t}$$

$$e^{-150t} = 0.1$$

$$-150t = \ln(0.1)$$

$$t = \frac{\ln(0.1)}{-150} \approx 0.0154 \text{ sec}$$

41. $\frac{dy}{dx} + P(x)y = Q(x)$ Standard form

$u(x) = e^{\int P(x)\,dx}$ Integrating factor

42. $y' + P(x)y = Q(x)y^n$ Standard form

Let $z = y^{1-n}(n \neq 0, 1)$. Multiplying by $(1-n)y^{-n}$ produces

$$(1-n)y^{-n}y' + (1-n)P(x)y^{1-n} = (1-n)Q(x)$$
$$z' + (1-n)P(x)z = (1-n)Q(x). \text{ Linear}$$

43. $y' - 2x = 0$

$$\int dy = \int 2x\,dx$$
$$y = x^2 + C$$

Matches c.

44. $y' - 2y = 0$

$$\int \frac{dy}{y} = \int 2\,dx$$
$$\ln y = 2x + C_1$$
$$y = Ce^{2x}$$

Matches d.

45. $y' - 2xy = 0$

$$\int \frac{dy}{y} = \int 2x\,dx$$
$$\ln y = x^2 + C_1$$
$$y = Ce^{x^2}$$

Matches a.

46. $y' - 2xy = x$

$$\int \frac{dy}{2y+1} = \int x\,dx$$
$$\frac{1}{2}\ln(2y+1) = \frac{1}{2}x^2 + C_1$$
$$2y + 1 = C_2e^{x^2}$$
$$y = -\frac{1}{2} + Ce^{x^2}$$

Matches b.

47. $y' + 3x^2y = x^2y^3$

$n = 3, Q = x^2, P = 3x^2$

$$y^{-2}e^{\int(-2)3x^2\,dx} = \int(-2)x^2e^{\int(-2)3x^2\,dx}\,dx$$
$$y^{-2}e^{-2x^3} = -\int 2x^2e^{-2x^3}\,dx$$
$$y^{-2}e^{-2x^3} = \frac{1}{3}e^{-2x^3} + C$$
$$y^{-2} = \frac{1}{3} + Ce^{2x^3}$$
$$\frac{1}{y^2} + Ce^{2x^3} + \frac{1}{3}$$

48. $y' + xy = xy^{-1}$

$n = -1, Q = x, P = x, e^{\int 2x\,dx} = e^{x^2}$

$$y^2e^{x^2} = \int 2xe^{x^2}\,dx = e^{x^2} + C$$
$$y^2 = 1 + Ce^{-x^2}$$

49. $y' + \left(\frac{1}{x}\right)y = xy^2$

$n = 2, Q = x, P = x^{-1}$

$$e^{\int-(1/x)\,dx} = e^{-\ln|x|} = x^{-1}$$
$$y^{-1}x^{-1} = \int -x\left(x^{-1}\right)dx = -x + C$$
$$\frac{1}{y} = -x^2 + Cx$$
$$y = \frac{1}{Cx - x^2}$$

50. $y' + \left(\frac{1}{x}\right)y = x\sqrt{y}$

$n = \frac{1}{2}, Q = x, P = x^{-1}$

$$e^{(1/2)(1/x)\,dx} = e^{(1/2)\ln x} = \sqrt{x}$$
$$y^{1/2}x^{1/2} = \int \frac{1}{2}x^{1/2}(x)\,dx$$
$$= \frac{1}{5}x^{5/2} + C_1 = \frac{x^{5/2} + C}{5}$$
$$y = \frac{\left(x^{5/2} + C\right)^2}{25x}$$

51. $xy' + y = xy^3$

$$y' + \frac{1}{x}y = y^3$$

$n = 3, Q = 1, P = \frac{1}{x}, \quad e^{\int\frac{-2}{x}dx} = e^{-2\ln x} = x^{-2}$

$$y^{-2}x^{-2} = \int -2x^{-2}\,dx + C = 2x^{-1} + C$$
$$y^{-2} = 2x + Cx^2$$
$$y^2 = \frac{1}{2x + Cx^2} \quad \text{or} \quad \frac{1}{y^2} = 2x + Cx^2$$

52. $y' - y = y^3$

$n = 3, P = -1, Q = 1, \quad e^{\int-2(-1)\,dx} = e^{2x}$

$$y^{-2}e^{2x} = \int(-2)e^{2x}\,dx = -e^{2x} + C$$
$$y^{-2} = -1 + Ce^{-2x}$$
$$y^2 = \frac{1}{-1 + Ce^{-2x}}$$

53. $y' - y = e^x \sqrt[3]{y}, n = \frac{1}{3}, Q = e^x, P = -1$

$$e^{\int -(2/3)\,dx} = e^{-(2/3)x}$$

$$y^{2/3}e^{-(2/3)x} = \int \tfrac{2}{3}e^x e^{-(2/3)x}\,dx = \int \tfrac{2}{3}e^{(1/3)x}\,dx$$

$$y^{2/3}e^{-(2/3)x} = 2e^{(1/3)x} + C$$

$$y^{2/3} = 2e^x + Ce^{2x/3}$$

54. $yy' - 2y^2 = e^x$

$$y' - 2y = e^x y^{-1}$$

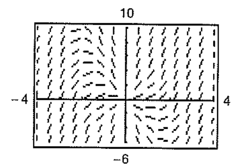
$$n = -1, Q = e^x, P = -2$$

$$e^{\int 2(-2)\,dx} = e^{-4x}$$

$$y^2 e^{-4x} = \int 2e^{-4x}e^x\,dx = -\tfrac{2}{3}e^{-3x} + C$$

$$y^2 = -\tfrac{2}{3}e^x + Ce^{4x}$$

55. (a)



(b) $\dfrac{dy}{dx} - \dfrac{1}{x}y = x^2$

Integrating factor: $e^{-1/x\,dx} = e^{-\ln x} = \dfrac{1}{x}$

$$\frac{1}{x}y' - \frac{1}{x^2}y = x$$

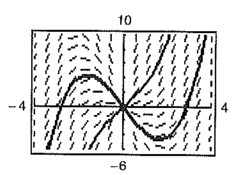
$$\left(\frac{1}{x}y\right) = \int x\,dx = \frac{x^2}{2} + C$$

$$y = \frac{x^3}{2} + Cx$$

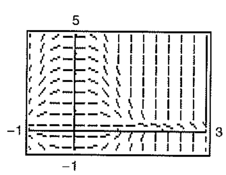
$(-2, 4)$: $4 = \dfrac{-8}{2} - 2C \Rightarrow C = -4 \Rightarrow y = \dfrac{x^3}{2} - 4x = \dfrac{1}{2}x(x^2 - 8)$

$(2, 8)$: $8 = \dfrac{8}{2} + 2C \Rightarrow C = 2 \Rightarrow y = \dfrac{x^3}{2} + 2x = \dfrac{1}{2}x(x^2 + 4)$

(c)



56. (a)



(b) $y' + 4x^3y = x^3$

Integrating factor: $e^{\int 4x^3\,dx} = e^{x^4}$

$$y'e^{x^4} + 4x^3ye^{x^4} = x^3e^{x^4}$$

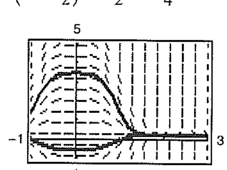
$$ye^{x^4} = \int x^3e^{x^4}\,dx = \tfrac{1}{4}e^{x^4} + C$$

$$y = \tfrac{1}{4} + Ce^{-x^4}$$

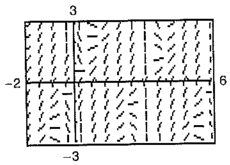
$\left(0, \frac{7}{2}\right)$: $\frac{7}{2} = \frac{1}{4} + C \Rightarrow C = \frac{13}{4} \Rightarrow y = \frac{1}{4} + \frac{13}{4}e^{-x^4}$

$\left(0, -\frac{1}{2}\right)$: $-\frac{1}{2} = \frac{1}{4} + C \Rightarrow C = -\frac{3}{4} \Rightarrow y = \frac{1}{4} - \frac{3}{4}e^{-x^4}$

(c)



57. (a)



(b) $y' + (\cot x)y = 2$

Integrating factor: $e^{\int \cot x\, dx} = e^{\ln|\sin x|} = \sin x$

$$y' \sin x + (\cos x)y = 2 \sin x$$

$$y \sin x = \int 2 \sin x\, dx = -2 \cos x + C$$

$$y = -2 \cot x + C \csc x$$

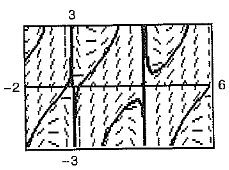
$(1, 1)$: $1 = -2 \cot 1 + C \csc 1 \Rightarrow C = \dfrac{1 + 2 \cot 1}{\csc 1} = \sin 1 + 2 \cos 1$

$$y = -2 \cot x + (\sin 1 + 2 \cos 1) \csc x$$

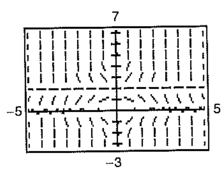
$(3, -1)$: $-1 = -2 \cot 3 + C \csc 3 \Rightarrow C = \dfrac{2 \cot 3 - 1}{\csc 3} = 2 \cos 3 - \sin 3$

$$y = -2 \cot x + (2 \cos 3 - \sin 3) \csc x$$

(c)



58. (a)



(b) $y' + 2xy = xy^2$

Bernoulli equation, $n = 2$ letting $z = y^{1-2} = y^{-1}$, you obtain $e^{-2x\, dx} = e^{-x^2}$ and $\int (-1)xe^{-x^2}\, dx = \frac{1}{2}e^{-x^2}$. The solution is:

$$y^{-1}e^{-x^2} = \frac{1}{2}e^{-x^2} + C$$

$$\frac{1}{y} = \frac{1}{2} + Ce^{x^2} = \frac{1 + 2Ce^{x^2}}{2}$$

$$y = \frac{2}{1 + 2Ce^{x^2}}$$

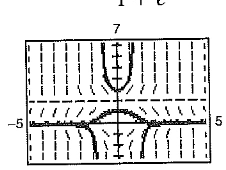
$(0, 3)$: $3 = \dfrac{2}{1 + 2C} \Rightarrow 1 + 2C = \dfrac{2}{3} \Rightarrow C = -\dfrac{1}{6}$

$$y = \frac{2}{1 - (e^{x^2}/3)} = \frac{6}{3 - e^{x^2}}$$

$(0, 1)$: $1 = \dfrac{2}{1 + 2C} \Rightarrow 1 + 2C = 2 \Rightarrow C = \dfrac{1}{2}$

$$y = \frac{2}{1 + e^{x^2}}$$

(c)



59. $e^{2x+y}\,dx - e^{x-y}\,dy = 0$

Separation of variables:

$$e^{2x}e^{y}\,dx = e^{x}e^{-y}\,dy$$
$$\int e^{x}\,dx = \int e^{-2y}\,dy$$
$$e^{x} = -\tfrac{1}{2}e^{-2y} + C_1$$
$$2e^{x} + e^{-2y} = C$$

60. $\dfrac{dy}{dx} = \dfrac{x-3}{y(y+4)}$

Separation of variables:

$$\int (y^2 + 4y)\,dy = \int (x-3)\,dx$$
$$\frac{y^3}{3} + 2y^2 = \frac{x^2}{2} - 3x + C_1$$
$$2y^3 + 12y^2 = 3x^2 - 18x + C$$

61. $(y\cos x - \cos x)\,dx + dy = 0$

Separation of variables:

$$\int \cos x\,dx = \int \frac{-1}{y-1}\,dy$$
$$\sin x = -\ln(y-1) + \ln C$$
$$\ln(y-1) = -\sin x + \ln C$$
$$y = Ce^{-\sin x} + 1$$

62. $y' = 2x\sqrt{1-y^2}$

Separation of variables:

$$\int \frac{1}{\sqrt{1-y^2}}\,dy = \int 2x\,dx$$
$$\arcsin y = x^2 + C$$
$$y = \sin(x^2 + C)$$

63. $(3y^2 + 4xy)\,dx + (2xy + x^2)\,dy = 0$

Homogeneous: $y = vx,\ dy = v\,dx + x\,dv$

$$(3v^2x^2 + 4vx^2)\,dx + (2vx^2 + x^2)(v\,dx + x\,dv) = 0$$
$$\int \frac{5}{x}\,dx + \int\left(\frac{2v+1}{v^2+v}\right)dv = 0$$
$$\ln x^5 + \ln\left|v^2 + v\right| = \ln C$$
$$x^5(v^2 + v) = C$$
$$x^3y^2 + x^4y = C$$

64. $(x+y)\,dx - x\,dy = 0$

Linear: $y' - \dfrac{1}{x}y = 1$

Integrating factor: $e^{\int -(1/x)\,dx} = e^{\ln\left|x^{-1}\right|} = \dfrac{1}{x}$

$$y\frac{1}{x} = \int \frac{1}{x}\,dx = \ln|x| + C$$
$$y = x(\ln|x| + C)$$

65. $(2y - e^{x})\,dx + x\,dy = 0$

Linear: $y' + \left(\dfrac{2}{x}\right)y = \dfrac{1}{x}e^{x}$

Integrating factor: $e^{\int (2/x)\,dx} = e^{\ln x^2} = x^2$

$$yx^2 = \int x^2\frac{1}{x}e^{x}\,dx = e^{x}(x-1) + C$$
$$y = \frac{e^{x}}{x^2}(x-1) + \frac{C}{x^2}$$

66. $(y^2 + xy)\,dx - x^2\,dy = 0$

Homogeneous: $y = vx,\ dy = v\,dx + x\,dv$

$$(v^2x^2 + vx^2)\,dx - x^2(v\,dx + x\,dv) = 0$$
$$v^2\,dx - x\,dv = 0$$
$$\int \frac{1}{x}\,dx = \int \frac{1}{v^2}\,dv$$
$$\ln x = -\frac{1}{v} + C$$
$$y = \frac{x}{C - \ln|x|}$$

67. $(x^2y^4 - 1)\,dx + x^3y^3\,dy = 0$

$$y' + \left(\frac{1}{x}\right)y = x^{-3}y^{-3}$$

Bernoulli: $n = -3,\ Q = x^{-3},\ P = x^{-1}$,

$$e^{\int (4/x)\,dx} = e^{\ln x^4} = x^4$$
$$y^4x^4 = \int 4(x^{-3})(x^4)\,dx = 2x^2 + C$$
$$x^4y^4 - 2x^2 = C$$

68. $y\,dx + (3x + 4y)\,dy = 0$

Homogeneous: $x = vy,\ dx = v\,dy + y\,dv$

$$y(v\,dy + y\,dv) + (3vy + 4y)\,dy = 0$$

$$\int \frac{1}{v+1}\,dv = \int -\frac{4}{y}\,dy$$

$$\ln|v+1| = -\ln y^4 + \ln C$$

$$y^4(v+1) = C$$

$$y^3(x+y) = C$$

69. $3(y - 4x^2)\,dx = -x\,dy$

$$x\frac{dy}{dx} = -3y + 12x^2$$

$$y' + \frac{3}{x}y = 12x$$

Integrating factor: $e^{\int(3/x)\,dx} = e^{3\ln x} = x^3$

$$y'x^3 + \frac{3}{x}x^3y = 12x(x^3) = 12x^4$$

$$yx^3 = \int 12x^4\,dx = \frac{12}{5}x^5 + C$$

$$y = \frac{12}{5}x^2 + \frac{C}{x^3}$$

70. $x\,dx + (y + e^y)(x^2 + 1)\,dy = 0$

Separation of variables:

$$\int \frac{x}{x^2+1}\,dx = \int -(y + e^y)\,dy$$

$$\frac{1}{2}\ln(x^2+1) = -\frac{1}{2}y^2 - e^y + C_1$$

$$\ln(x^2+1) + y^2 + 2e^y = C$$

71. False. The equation contains $\sqrt{y}$.

72. True. $y' + (x - e^x)y = 0$ is linear.

Review Exercises for Chapter 6

1. $y = x^3,\ y' = 3x^2$

$2xy' + 4y = 2x(3x^2) + 4(x^3) = 10x^3.$

Yes, it is a solution.

2.

$$y = 2\sin 2x$$

$$y' = 4\cos 2x$$

$$y'' = -8\sin 2x$$

$$y''' = -16\cos 2x$$

$$y''' - 8y = -16\cos 2x - 8(2\sin 2x) \neq 0$$

Not a solution

3. $\dfrac{dy}{dx} = 4x^2 + 7$

$$y = \int(4x^2 + 7)\,dx = \frac{4x^3}{3} + 7x + C$$

4. $\dfrac{dy}{dx} = 3x^3 - 8x$

$$y = \int(3x^3 - 8x)\,dx = \frac{3}{4}x^4 - 4x^2 + C$$

5. $\dfrac{dy}{dx} = \cos 2x$

$$y = \int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$$

6. $\dfrac{dy}{dx} = 2\sin x$

$$y = \int 2\sin x\,dx = -2\cos x + C$$

7. $\dfrac{dy}{dx} = x\sqrt{x-5}$

$$y = \int x\sqrt{x-5}\,dx$$

Let $u = x - 5,\ du = dx,\ x = u + 5$

$$y = \int(u+5)\sqrt{u}\,du$$

$$= \int\left(u^{3/2} + 5u^{1/2}\right)du$$

$$= \frac{2}{5}u^{5/2} + \frac{10}{3}u^{3/2} + C$$

$$= \frac{2}{5}(x-5)^{5/2} + \frac{10}{3}(x-5)^{3/2} + C$$

$$= \frac{1}{15}(x-5)^{3/2}\left[6(x-5) + 50\right] + C$$

$$= \frac{1}{15}(x-5)^{3/2}(6x + 20) + C$$

8. $\dfrac{dy}{dx} = 2x\sqrt{x - 7}$

$y = \int 2x\sqrt{x - 7}\,dx$

Let $u = x - 7$, $du = dx$, $x = u + 7$:

$$y = \int 2(u + 7)u^{1/2}\,du$$
$$= \frac{4}{5}u^{5/2} + \frac{28}{3}u^{3/2} + C$$
$$= \frac{4}{5}(x - 7)^{5/2} + \frac{28}{3}(x - 7)^{3/2} + C$$
$$= \frac{4}{15}(x - 7)^{3/2}(3x + 14) + C$$

9. $\dfrac{dy}{dx} = e^{2-x}$

$y = \int e^{2-x}\,dx = -e^{2-x} + C$

10. $\dfrac{dy}{dx} = 3e^{-x/3}$

$y = \int 3e^{-x/3}\,dx = -9e^{-x/3} + C$

11. $\dfrac{dy}{dx} = 2x - y$

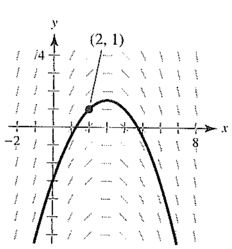
x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	-10	-4	-4	0	2	8

12. $\dfrac{dy}{dx} = x \sin\left(\dfrac{\pi y}{4}\right)$

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	-4	0	0	0	-4	0

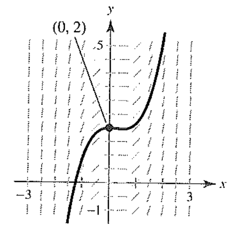
13. $y' = 3 - x, \quad (2, 1)$

(a) and (b)



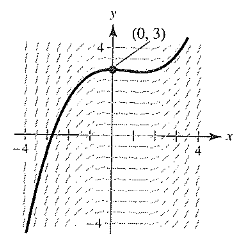
14. $y' = 2x^2 - x, \quad (0, 2)$

(a) and (b)



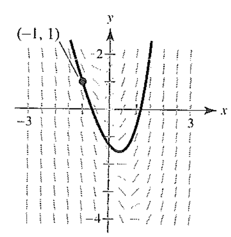
15. $y' = \frac{1}{4}x^2 - \frac{1}{3}x, \quad (0, 3)$

(a) and (b)



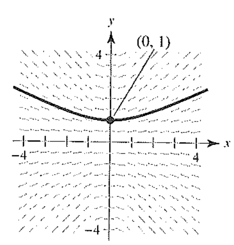
16. $y' = y + 4x, \quad (-1, 1)$

(a) and (b)



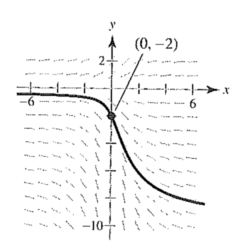
17. $y' = \dfrac{xy}{x^2 + 4}, \quad (0, 1)$

(a) and (b)



18. $y' = \dfrac{y}{x^2 + 1}, \quad (0, -2)$

(a) and (b)



19. $\dfrac{dy}{dx} = 8 - x$

$$y = \int (8 - x)\,dx = 8x - \frac{x^2}{2} + C$$

20. $\dfrac{dy}{dx} = y + 8$

$$\int \frac{dy}{y + 8} = \int dx$$

$$\ln|y + 8| = x + C_1$$

$$|y + 8| = e^{x + C_1} = Ce^x$$

$$y = -8 + Ce^x$$

21. $\dfrac{dy}{dx} = (3 + y)^2$

$$\int (3 + y)^{-2}\,dy = \int dx$$

$$-(3 + y)^{-1} = x + C$$

$$3 + y = \frac{-1}{x + C}$$

$$y = -3 - \frac{1}{x + C}$$

22. $\dfrac{dy}{dx} = 10\sqrt{y}$

$$\int y^{-1/2}\,dy = \int 10\,dx$$

$$2y^{1/2} = 10x + C_1$$

$$y^{1/2} = 5x + C \qquad \left(C = \frac{C_1}{2}\right)$$

$$y = (5x + C)^2$$

23. $(2 + x)y' - xy = 0$

$$(2 + x)\frac{dy}{dx} = xy$$

$$\frac{1}{y}\,dy = \frac{x}{2 + x}dx$$

$$\frac{1}{y}\,dy = \left(1 - \frac{2}{2 + x}\right)dx$$

$$\ln|y| = x - 2\ln|2 + x| + C_1$$

$$y = Ce^x(2 + x)^{-2} = \frac{Ce^x}{(2 + x)^2}$$

24. $xy' - (x + 1)y = 0$

$$x\frac{dy}{dx} = (x + 1)y$$

$$\int \frac{dy}{y} = \int \frac{x + 1}{x}\,dx$$

$$\ln|y| = x + \ln|x| + C_1$$

$$y = Cxe^x$$

25. $y = Ce^{kt}$

$\left(0, \frac{3}{4}\right)$: $\frac{3}{4} = C$

$(5, 5)$: $5 = \frac{3}{4}e^{k(5)}$

$$\frac{20}{3} = e^{5k}$$

$$k = \frac{1}{5}\ln\left(\frac{20}{3}\right)$$

$$y = \frac{3}{4}e^{[\ln(20/3)/5]t} \approx \frac{3}{4}e^{0.379t}$$

26. $y = Ce^{kt}$

$\left(2, \frac{3}{2}\right)$: $\frac{3}{2} = Ce^{2k} \Rightarrow C = \frac{3}{2}e^{-2k}$

$(4, 5)$: $5 = Ce^{4k} = \left(\frac{3}{2}e^{-2k}\right)e^{4k} = \frac{3}{2}e^{2k}$

$$\frac{10}{3} = e^{2k} \Rightarrow k = \frac{1}{2}\ln\left(\frac{10}{3}\right)$$

So, $C = \frac{3}{2}e^{-2(1/2)\ln(10/3)} = \frac{3}{2}\left(\frac{3}{10}\right) = \frac{9}{20}$.

Finally, $y = \frac{9}{20}e^{1/2\ln(10/3)t}$.

27. $y = Ce^{kt}$

$(0, 5)$: $C = 5$

$\left(5, \frac{1}{6}\right)$: $\frac{1}{6} = 5e^{5k}$

$$k = \frac{1}{5}\ln\left(\frac{1}{30}\right) = \frac{-\ln 30}{5}$$

$$y = 5e^{[-t\ln 30]/5} \approx 5e^{-0.680t}$$

28. $y = Ce^{kt}$

$(1, 9)$: $9 = Ce^k \Rightarrow C = 9e^{-k}$

$(6, 2)$: $2 = Ce^{6k} \Rightarrow 2 = \left(9e^{-k}\right)e^{6k} = 9e^{5k}$

$$k = \frac{1}{5}\ln\left(\frac{2}{9}\right) \approx -0.3008$$

So, $C = 9e^{-1/5\ln(2/9)} = 9\left(\frac{2}{9}\right)^{-1/5} \approx 12.15864$.

Finally, $y \approx 12.1586e^{-0.3008t}$.

29. $\dfrac{dP}{dh} = kp, \quad P(0) = 30$

$$P(h) = 30e^{kh}$$

$$P(18{,}000) = 30e^{18{,}000k} = 15$$

$$k = \frac{\ln(1/2)}{18{,}000} = \frac{-\ln 2}{18{,}000}$$

$$P(h) = 30e^{-(h\ln 2)/18{,}000}$$

$$P(35{,}000) = 30e^{-(35{,}000\ln 2)/18{,}000} \approx 7.79 \text{ inches}$$

30. $y = Ce^{kt} = 15e^{kt}$

$$7.5 = 15e^{k(1599)}$$

$$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right) \approx -0.000433$$

When $t = 750$, $y = 15e^{-0.000433(750)} \approx 10.84$ g.

31. $S = Ce^{k/t}$

(a) $S = 5$ when $t = 1$

$$5 = Ce^{k}$$

$$\lim_{t\to\infty} Ce^{k/t} = C = 30$$

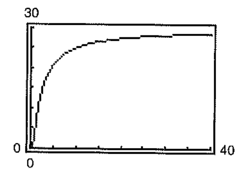
$$5 = 30e^{k}$$

$$k = \ln\frac{1}{6} \approx -1.7918$$

$$S = 30e^{-1.7918/t}$$

(b) When $t = 5$, $S \approx 20.9646$ which is 20,965 units.

(c)



32. $S = 25\left(1 - e^{kt}\right)$

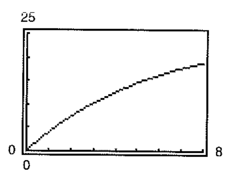
(a) $4 = 25\left(1 - e^{k(1)}\right) \Rightarrow 1 - e^{k} = \frac{4}{25} \Rightarrow e^{k} = \frac{21}{25} \Rightarrow k = \ln\left(\frac{21}{25}\right) \approx -0.1744$

$S = 25\left(1 - e^{-0.1744t}\right)$

(b) 25,000 units $\left(\lim_{t\to\infty} S = 25\right)$

(c) When $t = 5$, $S \approx 14.545$ which is 14,545 units.

(d)



33.

$$P = Ce^{0.0185t}$$

$$2C = Ce^{0.0185t}$$

$$2 = e^{0.0185t}$$

$$\ln 2 = 0.0185t$$

$$t = \frac{\ln 2}{0.0185} \approx 37.5 \text{ years}$$

34. (a)

$$\frac{dy}{ds} = -0.012y,\ s > 50$$

$$\frac{-1}{0.012}\int\frac{dy}{y} = \int ds$$

$$\frac{-1}{0.012}\ln y = s + C_1$$

$$y = Ce^{-0.012s}$$

When $s = 50$, $y = 28 = Ce^{-0.012(50)} \Rightarrow C = 28e^{0.6}$

$y = 28e^{0.6-0.012s}$, $s > 50$.

(b)

Speed (s)	50	55	60	65	70
Miles per Gallon (y)	28	26.4	24.8	23.4	22.0

35. $\frac{dy}{dx} = \frac{x^5 + 7}{x} = x^4 + \frac{7}{x}$

$$y = \int\left(x^4 + \frac{7}{x}\right) dx = \frac{x^5}{5} + 7\ln|x| + C$$

36. $\frac{dy}{dx} = \frac{e^{-2x}}{1 + e^{-2x}}$

$$\int dy = \int \frac{e^{-2x}}{1 + e^{-2x}}\, dx = -\frac{1}{2}\int \frac{-2e^{-2x}}{1 + e^{-2x}}\, dx$$

$$y = -\frac{1}{2}\ln\left(1 + e^{-2x}\right) + C$$

37. $y' - 16xy = 0$

$$\frac{dy}{dx} = 16xy$$

$$\int \frac{1}{y}\, dy = \int 16x\, dx$$

$$\ln|y| = 8x^2 + C_1$$

$$e^{8x^2 + C_1} = y$$

$$y = Ce^{8x^2}$$

38. $y' - e^y \sin x = 0$

$$\frac{dy}{dx} = e^y \sin x$$

$$\int e^{-y}\, dy = \int \sin x\, dx$$

$$-e^{-y} = -\cos x + C_1$$

$$e^y = \frac{1}{\cos x + C} \qquad (C = -C_1)$$

$$y = \ln\left|\frac{1}{\cos x + C}\right| = -\ln|\cos x + C|$$

39. $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$ (homogeneous differential equation)

$$\left(x^2 + y^2\right) dx - 2xy\, dy = 0$$

Let $y = vx,\ dy = x\, dv + v\, dx$.

$$\left(x^2 + v^2x^2\right) dx - 2x(vx)(x\, dv + v\, dx) = 0$$

$$\left(x^2 + v^2x^2 - 2x^2v^2\right) dx - 2x^3v\, dv = 0$$

$$\left(x^2 - x^2v^2\right) dx = 2x^3v\, dv$$

$$\left(1 - v^2\right) dx = 2xv\, dv$$

$$\int \frac{dx}{x} = \int \frac{2v}{1 - v^2}\, dv$$

$$\ln|x| = -\ln\left|1 - v^2\right| + C_1 = -\ln\left|1 - v^2\right| + \ln C$$

$$x = \frac{C}{1 - v^2} = \frac{C}{1 - (y/x)^2} = \frac{Cx^2}{x^2 - y^2}$$

$$1 = \frac{Cx}{x^2 - y^2} \quad \text{or} \quad C_1 = \frac{x}{x^2 - y^2}$$

40. $\dfrac{dy}{dx} = \dfrac{3(x+y)}{x}$ (homogeneous differential equation)

$$3(x+y)\,dx - x\,dy = 0$$

Let $y = vx$, $dy = x\,dv + v\,dx$.

$$3(x+vx)\,dx - x(x\,dv + v\,dx) = 0$$
$$(3x + 2vx)\,dx - x^2\,dv = 0$$
$$(3+2v)\,dx = x\,dv$$
$$\int \frac{1}{x}\,dx = \int \frac{1}{3+2v}\,dv$$
$$\ln|x| = \frac{1}{2}\ln|3+2v| + C_1 = \ln(3+2v)^{1/2} + \ln C_2$$
$$x = C_2(3+2v)^{1/2}$$
$$x^2 = C(3+2v) = C\left(3 + 2\left(\frac{y}{x}\right)\right)$$
$$x^3 = C(3x+2y) = 3Cx + 2Cy$$
$$y = \frac{x^3 - 3Cx}{2C}$$

41. $y = C_1x + C_2x^3$

$y' = C_1 + 3C_2x^2$

$y'' = 6C_2x$

$$x^2y'' - 3xy' + 3y = x^2(6C_2x) - 3x(C_1 + 3C_2x^2) + 3(C_1x + C_2x^3)$$
$$= 6C_2x^3 - 3C_1x - 9C_2x^3 + 3C_1x + 3C_2x^3 = 0$$

$x = 2, y = 0$: $0 = 2C_1 + 8C_2 \Rightarrow C_1 = -4C_2$

$x = 2, y' = 4$: $4 = C_1 + 12C_2$

$$4 = (-4C_2) + 12C_2 = 8C_2 \Rightarrow C_2 = \tfrac{1}{2}, C_1 = -2$$

$y = -2x + \frac{1}{2}x^3$

42. $\dfrac{dv}{dt} = kv - 9.8$

(a) $$\int \frac{dv}{kv - 9.8} = \int dt$$
$$\frac{1}{k}\ln|kv - 9.8| = t + C_1$$
$$\ln|kv - 9.8| = kt + C_2$$
$$kv - 9.8 = e^{kt + C_2} = C_3e^{kt}$$
$$v = \frac{1}{k}\left[9.8 + C_3e^{kt}\right]$$

At $t = 0$, $v_0 = \dfrac{1}{k}(9.8 + C_3) \Rightarrow C_3 = kv_0 - 9.8$

$$v = \frac{1}{k}\left[9.8 + (kv_0 - 9.8)e^{kt}\right].$$

Note that $k < 0$ because the object is moving downward.

(b) $\lim_{t\to\infty} v(t) = \dfrac{9.8}{k}$

(c) $s(t) = \int \frac{1}{k}\left[9.8 + (kv_0 - 9.8)e^{kt}\right] dt = \frac{1}{k}\left[9.8t + \frac{1}{k}(kv_0 - 9.8)e^{kt}\right] + C = \frac{9.8t}{k} + \frac{1}{k^2}(kv_0 - 9.8)e^{kt} + C$

$s(0) = \frac{1}{k^2}(kv_0 - 9.8) + C \Rightarrow C = s_0 - \frac{1}{k^2}(kv_0 - 9.8)$

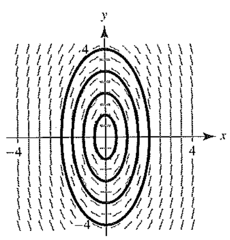
$s(t) = \frac{9.8t}{k} + \frac{1}{k^2}(kv_0 - 9.8)e^{kt} + s_0 - \frac{1}{k^2}(kv_0 - 9.8) = \frac{9.8t}{k} + \frac{1}{k^2}(kv_0 - 9.8)(e^{kt} - 1) + s_0$

43. $\frac{dy}{dx} = \frac{-4x}{y}$

$\int y\, dy = \int -4x\, dx$

$\frac{y^2}{2} = -2x^2 + C_1$

$4x^2 + y^2 = C$ ellipses



44. $\frac{dy}{dx} = 3 - 2y$

$\int \frac{dy}{2y - 3} = \int -dx$

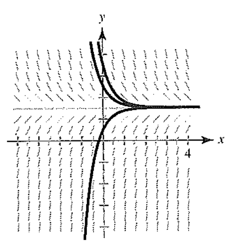
$\frac{1}{2}\ln|2y - 3| = -x + C_1$

$\ln|2y - 3| = -2x + 2C_1$

$|2y - 3| = C_2e^{-2x}$

$2y = 3 + C_2e^{-2x}$

$y = \frac{3}{2} + Ce^{-2x}$



45. $P(t) = \frac{5250}{1 + 34e^{-0.55t}}$

(a) $k = 0.55$

(b) $L = 5250$

(c) $P(0) = \frac{5250}{1 + 34} = 150$

(d) $2625 = \frac{5250}{1 + 34e^{-0.55t}}$

$1 + 34e^{-0.55t} = 2$

$e^{-0.55t} = \frac{1}{34}$

$t = \frac{-1}{0.55}\ln\left(\frac{1}{34}\right) \approx 6.41$ yr

(e) $\frac{dP}{dt} = 0.55P\left(1 - \frac{P}{5250}\right)$

46. $P(t) = \frac{4800}{1 + 14e^{-0.15t}}$

(a) $k = 0.15$

(b) $L = 4800$

(c) $P(0) = \frac{4800}{1 + 14} = 320$

(d) $2400 = \frac{4800}{1 + 14e^{-0.15t}}$

$14e^{-0.15t} = 1$

$t = -\frac{1}{0.15}\ln\left(\frac{1}{14}\right) \approx 17.59$ yr

(e) $\frac{dP}{dt} = 0.15P\left(1 - \frac{P}{4800}\right)$

47. $\frac{dy}{dt} = y\left(1 - \frac{y}{80}\right)$, $(0, 8)$

$k = 1, L = 80$

$y = \frac{L}{1 + be^{-kt}} = \frac{80}{1 + be^{-t}}$

$y(0) = 8$: $8 = \frac{80}{1 + b} \Rightarrow b = 9$

Solution: $y = \frac{80}{1 + 9e^{-t}}$

48. $\frac{dy}{dt} = 1.76y\left(1 - \frac{y}{8}\right), \quad (0, 3)$

$k = 1.76, L = 8$

$y = \frac{L}{1 + be^{-kt}} = \frac{8}{1 + be^{-1.76t}}$

$y(0) = 3: \quad 3 = \frac{8}{1 + b} \Rightarrow b = \frac{5}{3}$

Solution: $y = \frac{8}{1 + \left(\frac{5}{3}\right)e^{-1.76t}}$

49. (a) $L = 20{,}400, y(0) = 1200, y(1) = 2000$

$y = \frac{20{,}400}{1 + be^{-kt}}$

$y(0) = 1200 = \frac{20{,}400}{1 + b} \Rightarrow b = 16$

$y(1) = 2000 = \frac{20{,}400}{1 + 16e^{-k}}$

$16e^{-k} = \frac{46}{5}$

$k = -\ln\frac{23}{40} = \ln\frac{40}{23} \approx 0.553$

$y = \frac{20{,}400}{1 + 16e^{-0.553t}}$

(b) $y(8) \approx 17{,}118$ trout

(c) $10{,}000 = \frac{20{,}400}{1 + 16e^{-0.553t}} \Rightarrow t \approx 4.94$ yr

50. $\frac{dy}{dt} = 0.553y\left(1 - \frac{y}{20{,}400}\right), \quad y(0) = 1200$

Use Euler's method with $h = 1$.

t	0	2	4	6	8
Exact	1200	3241	7414	12,915	17,117
Euler	1200	2743	5853	10,869	16,170

Euler's method gives $y(8) \approx 16{,}170$ trout.

51. $y' - y = 10$

$P(x) = -1, Q(x) = 10$

$u(x) = e^{\int -dx} = e^{-x}$

$y = \frac{1}{e^{-x}}\int 10e^{-x}\,dx$

$= e^{x}\left(-10e^{-x} + C\right)$

$= -10 + Ce^{x}$

52. $e^{x}y' + 4e^{x}y = 1$

$y' + 4y = e^{-x}$

$P(x) = 4, Q(x) = e^{-x}$

$u(x) = e^{\int 4\,dx} = e^{4x}$

$y = \frac{1}{e^{4x}}\int e^{-x}e^{4x}\,dx = e^{-4x}\left(\frac{1}{3}e^{3x} + C\right) = \frac{1}{3}e^{-x} + Ce^{-4x}$

53. $4y' = e^{x/y} + y$

$y' - \frac{1}{4}y = \frac{1}{4}e^{x/4}$

$P(x) = -\frac{1}{4}, Q(x) = \frac{1}{4}e^{x/4}$

$u(x) = e^{\int -(1/4)\,dx} = e^{-(1/4)x}$

$y = \frac{1}{e^{-(1/4)x}}\int \frac{1}{4}e^{x/4}e^{-(1/4)x}\,dx$

$= e^{(1/4)x}\left(\frac{1}{4}x + C\right)$

$= \frac{1}{4}xe^{x/4} + Ce^{x/4}$

54. $\frac{dy}{dx} - \frac{5y}{x^2} = \frac{1}{x^2}$

$P(x) = -\frac{5}{x^2}, Q(x) = \frac{1}{x^2}$

$u(x) = e^{\int -\left(5/x^2\right)dx} = e^{5/x}$

$y = \frac{1}{e^{5/x}}\int \frac{1}{x^2}e^{5/x}\,dx = \frac{1}{e^{5/x}}\left(-\frac{1}{5}e^{5/x} + C\right) = -\frac{1}{5} + Ce^{-5/x}$

55. $(x - 2)y' + y = 1$

$\frac{dy}{dx} + \frac{1}{x - 2}y = \frac{1}{x - 2}$

$P(x) = \frac{1}{x - 2}, Q(x) = \frac{1}{x - 2}$

$u(x) = e^{\int (1/x-2)\,dx} = e^{\ln|x-2|} = x - 2$

$y = \frac{1}{x - 2}\int\left(\frac{1}{x - 2}\right)(x - 2)\,dx = \frac{1}{x - 2}(x + C)$

56. $(x+3)y' + 2y = 2(x+3)^2$

$$\frac{dy}{dx} + \frac{2}{x+3}y = 2(x+3)$$

$$P(x) = \frac{2}{x+3},\ Q(x) = 2(x+3)$$

$$u(x) = e^{\int(2/x+3)\,dx} = e^{2\ln(x+3)} = (x+3)^2$$

$$y = \frac{1}{(x+3)^2}\int 2(x+3)(x+3)^2\,dx$$

$$= \frac{1}{(x+3)^2}\left[\frac{(x+3)^4}{2} + C\right]$$

$$= \frac{(x+3)^2}{2} + \frac{C}{(x+3)^2}$$

57. $(3y + \sin 2x)\,dx - dy = 0$

$$y' - 3y = \sin 2x$$

Integrating factor: $e^{\int -3\,dx} = e^{-3x}$

$$ye^{-3x} = \int e^{-3x}\sin 2x\,dx$$

$$= \tfrac{1}{13}e^{-3x}(-3\sin 2x - 2\cos 2x) + C$$

$$y = -\tfrac{1}{13}(3\sin 2x + 2\cos 2x) + Ce^{3x}$$

58. $dy = (y\tan x + 2e^x)\,dx$

$$\frac{dy}{dx} - (\tan x)y = 2e^x$$

Integrating factor: $e^{-\int \tan x\,dx} = e^{\ln|\cos x|} = \cos x$

$$y\cos x = \int 2e^x\cos x\,dx = e^x(\cos x + \sin x) + C$$

$$y = e^x(1 + \tan x) + C\sec x$$

59. $y' + 5y = e^{5x}$

Integrating factor: $e^{\int 5\,dx} = e^{5x}$

$$ye^{5x} = \int e^{10x}\,dx = \tfrac{1}{10}e^{10x} + C$$

$$y = \tfrac{1}{10}e^{5x} + Ce^{-5x}$$

60. $y' - \left(\dfrac{a}{x}\right)y = bx^3$

Integrating factor: $e^{-\int(a/x)\,dx} = e^{-a\ln x} = x^{-a}$

$$yx^{-a} = \int bx^3(x^{-a})\,dx = \frac{b}{4-a}x^{4-a} + C$$

$$y = \frac{bx^4}{4-a} + Cx^a$$

61. $y' + y = xy^2$ Bernoulli equation

$n = 2$, let $z = y^{1-2} = y^{-1}$, $z' = -y^{-2}y'$.

$$(-y^{-2})y' + (-y^{-2})y = -x$$

$$z' - z = -x \qquad \text{Linear equation}$$

$$u(x) = e^{\int -dx} = e^{-x}$$

$$z = \frac{1}{e^{-x}}\int -xe^{-x}\,dx = e^x\left[xe^{-x} + e^{-x} + C\right]$$

$$y^{-1} = x + 1 + Ce^x$$

$$y = \frac{1}{x + 1 + Ce^x}$$

62. $y' + 2xy = xy^2$ Bernoulli equation

$n = 2$, let $z = y^{1-2} = y^{-1}$, $z' = -y^{-2}y'$.

$$(-y^{-2})y' + 2xy(-y^{-2}) = -x$$

$$z' - 2xz = -x \qquad \text{Linear equation}$$

$$u(x) = e^{\int -2x\,dx} = e^{-x^2}$$

$$z = \frac{1}{e^{-x^2}}\int(-x)e^{-x^2}\,dx = e^{x^2}\left(\frac{1}{2}e^{-x^2} + C\right)$$

$$\frac{1}{y} = \frac{1}{2} + Ce^{x^2}$$

$$y = \frac{1}{\frac{1}{2} + Ce^{x^2}} = \frac{2}{1 + C_1e^{x^2}}$$

63. $y' + \dfrac{1}{x}y = \dfrac{y^3}{x^2}$ Bernoulli equation

$n = 3$, let $z = y^{1-3} = y^{-2}$, $z' = -2y^{-3}y'$.

$$(-2y^{-3})y' + \frac{1}{x}y(-2y^{-3}) = \frac{-2}{x^2}$$

$$z' - \frac{2}{x}z = \frac{-2}{x^2} \qquad \text{Linear equation}$$

$$u(x) = e^{\int -(2/x)\,dx} = e^{-2\ln x} = x^{-2}$$

$$z = \frac{1}{x^{-2}}\int\frac{-2}{x^2}(x^{-2})\,dx = x^2\left(\frac{2x^{-3}}{3} + C\right)$$

$$\frac{1}{y^2} = \frac{2}{3x} + Cx^2$$

64. $xy' + y = xy^2$

$y' + \frac{1}{x}y = y^2$ Bernoulli Equation

$n = 2$, let $z = y^{1-2} = y^{-1}$, $z' = -y^{-2}y'$.

$$-y^{-2}y' + \frac{1}{x}y(-y^{-2}) = y^2(-y^{-2})$$

$$z' - \frac{1}{x}z = -1 \quad \text{Linear equation}$$

$$u(x) = e^{\int -(1/x)\,dx} = \frac{1}{x}$$

$$z = x\int -\frac{1}{x}\,dx = -x[\ln|x| + C]$$

$$\frac{1}{y} = -x\ln x + Cx$$

$$y = \frac{1}{Cx - x\ln x}$$

65. Answers will vary. *Sample answer*: $(x^2 + 3y^2)\,dx - 2xy\,dy = 0$

Solution: Let $y = vx$, $dy = x\,dv + v\,dx$.

$$(x^2 + 3v^2x^2)\,dx - 2x(vx)(x\,dv + v\,dx) = 0$$

$$(x^2 + v^2x^2)\,dx - 2x^3v\,dv = 0$$

$$(1 + v^2)\,dx = 2\,xv\,dv$$

$$\int \frac{dx}{x} = \int \frac{2v}{1 + v^2}\,dv$$

$$\ln|x| = \ln|1 + v^2| + C_1$$

$$x = C(1 + v^2) = C\left(1 + \frac{y^2}{x^2}\right)$$

$$x^3 = C(x^2 + y^2)$$

66. Answers will vary. *Sample answer*: $y' = y\left(1 - \frac{y}{40}\right)$

Solution: $k = 1$, $L = 40$

$$y = \frac{L}{1 + be^{-kt}} = \frac{40}{1 + be^{-t}}$$

67. Answers will vary.

Sample answer: $x^3y' + 2x^2y = 1$

$$y' + \frac{2}{x}y = \frac{1}{x^3}$$

$$u(x) = e^{\int (2/x)\,dx} = x^2$$

$$y = \frac{1}{x^2}\int \frac{1}{x^3}(x^2)\,dx = \frac{1}{x^2}[\ln|x| + C]$$

68. Answers will vary. *Sample answer*: $y' + xy = xy^{-1}$

Solution: $n = -1$, $Q = x$, $P = x$, $e^{\int 2x\,dx} = e^{x^2}$

$$y^2e^{x^2} = \int 2xe^{x^2}\,dx = e^{x^2} + C$$

$$y^2 = 1 + Ce^{-x^2}$$

Problem Solving for Chapter 6

1. (a)
$$\frac{dy}{dt} = y^{1.01}$$
$$\int y^{-1.01}\,dy = \int dt$$
$$\frac{y^{-0.01}}{-0.01} = t + C_1$$
$$\frac{1}{y^{0.01}} = -0.01t + C$$
$$y^{0.01} = \frac{1}{C - 0.01t}$$
$$y = \frac{1}{(C - 0.01t)^{100}}$$
$$y(0) = 1\colon 1 = \frac{1}{C^{100}} \Rightarrow C = 1$$
So, $y = \dfrac{1}{(1 - 0.01t)^{100}}$.

For $T = 100$, $\lim_{t \to T^-} y = \infty$.

(b)
$$\int y^{-(1+\varepsilon)}\,dy = \int k\,dt$$
$$\frac{y^{-\varepsilon}}{-\varepsilon} = kt + C_1$$
$$y^{-\varepsilon} = -\varepsilon kt + C$$
$$y = \frac{1}{(C - \varepsilon kt)^{1/\varepsilon}}$$
$$y(0) = y_0 = \frac{1}{C^{1/\varepsilon}} \Rightarrow C^{1/\varepsilon} = \frac{1}{y_0} \Rightarrow C = \left(\frac{1}{y_0}\right)^{\varepsilon}$$
So, $y = \dfrac{1}{\left(\dfrac{1}{y_0^{\varepsilon}} - \varepsilon kt\right)^{1/\varepsilon}}$.

For $t \to \dfrac{1}{y_0^{\varepsilon}\varepsilon k}$, $y \to \infty$.

2. Because $\dfrac{dy}{dt} = k(y - 20)$,
$$\int \frac{1}{y - 20}\,dy = \int k\,dt$$
$$\ln|y - 20| = kt + C$$
$$y = Ce^{kt} + 20.$$
When $t = 0$, $y = 72$. Therefore, $C = 52$.

When $t = 1$, $y = 48$. Therefore,

$48 = 52e^k + 20$, $e^k = \dfrac{28}{52} = \dfrac{7}{13}$, and $k = \ln\dfrac{7}{13}$. So,

$y = 52e^{[\ln(7/13)]t} + 20$.

When $t = 5$, $y = 52e^{5\ln(7/13)} + 20 \approx 22.35°\text{F}$.

3. (a) $\dfrac{dS}{dt} = k_1 S(L - S)$

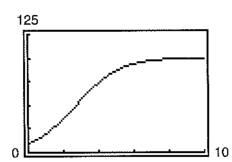
$S = \dfrac{L}{1 + Ce^{-kt}}$ is a solution because
$$\frac{dS}{dt} = -L\left(1 + Ce^{-kt}\right)^{-2}\left(-Cke^{-kt}\right)$$
$$= \frac{LC\,ke^{-kt}}{\left(1 + Ce^{-kt}\right)^2}$$
$$= \left(\frac{k}{L}\right)\frac{L}{1 + Ce^{-kt}} \cdot \frac{C\,Le^{-kt}}{1 + Ce^{-kt}}$$
$$= \left(\frac{k}{L}\right)\frac{L}{1 + Ce^{-kt}} \cdot \left(L - \frac{L}{1 + Ce^{-kt}}\right)$$
$$= k_1 S(L - S), \text{ where } k_1 = \frac{k}{L}.$$
$L = 100$. Also, $S = 10$ when $t = 0 \Rightarrow C = 9$.

And, $S = 20$ when $t = 1 \Rightarrow k = -\ln\dfrac{4}{9}$.

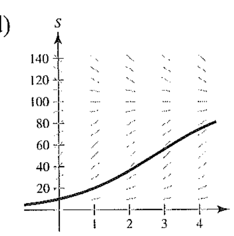
Particular Solution: $S = \dfrac{100}{1 + 9e^{\ln(4/9)t}} = \dfrac{100}{1 + 9e^{-0.8109t}}$

(b)
$$\frac{dS}{dt} = k_1 S(100 - S)$$
$$\frac{d^2S}{dt^2} = k_1\left[S\left(-\frac{dS}{dt}\right) + (100 - S)\frac{dS}{dt}\right]$$
$$= k_1(100 - 2S)\frac{dS}{dt}$$
$$= 0 \text{ when } S = 50 \text{ or } \frac{dS}{dt} = 0.$$
Choosing $S = 50$, you have:
$$50 = \frac{100}{1 + 9e^{\ln(4/9)t}}$$
$$2 = 1 + 9e^{\ln(4/9)t}$$
$$\frac{\ln(1/9)}{\ln(4/9)} = t$$
$$t \approx 2.7 \text{ months}$$
(This is the point of inflection.)

(c)



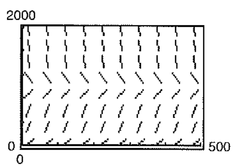
(d)



(e) Sales will decrease toward the line $S = L$.

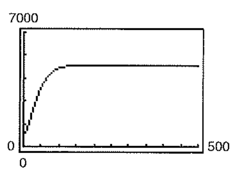
4. (a)
$$\frac{dy}{dt} = k \ln\left(\frac{L}{y}\right)y$$
$$\frac{dy}{y[\ln L - \ln y]} = k\,dt$$
$$\ln[\ln L - \ln y] = -kt + C_1$$
$$\ln\frac{L}{y} = Ce^{-kt}$$
$$\frac{L}{y} = e^{Ce^{-kt}}$$
$$y = Le^{-Ce^{-kt}}$$

(b)



(c) As $t \to \infty$, $y \to L$, the carrying capacity.

(d) $y_0 = 500 = 5000e^{-C} \Rightarrow e^C = 10 \Rightarrow C = \ln 10$



$$\frac{dy}{dt} = k \ln\left(\frac{L}{y}\right)y$$
$$\frac{d^2y}{dt^2} = k \ln\left(\frac{L}{y}\right)\frac{dy}{dt} + ky\frac{1}{(L/y)}\left(\frac{-L}{y^2}\right)\frac{dy}{dt}$$
$$= k\frac{dy}{dt}\left[\ln\left(\frac{L}{y}\right) - 1\right] = k^2 \ln\left(\frac{L}{y}\right)y\left[\ln\left(\frac{L}{y}\right) - 1\right]$$

So, $\dfrac{d^2y}{dt^2} = 0$ when

$$\ln\left(\frac{L}{y}\right) = 1 \Rightarrow \frac{L}{y} = e \Rightarrow y = \frac{L}{e}.$$

$$y = \frac{L}{e} = \frac{5000}{e} \approx 1839.4 \text{ and } t \approx 41.7.$$

The graph is concave upward on $(0, 41.7)$ and downward on $(41.7, \infty)$.

5. Let $u = \dfrac{1}{2}k\left(t - \dfrac{\ln b}{k}\right)$.

$$1 + \tanh u = 1 + \frac{e^4 - e^{-u}}{e^u + e^{-u}} = \frac{2}{1 + e^{-2u}}$$
$$e^{-2u} = e^{-k(t-(\ln b/k))} = e^{\ln b}e^{-kt} = be^{-kt}$$

Finally,

$$\frac{1}{2}L\left[1 + \tanh\left(\frac{1}{2}k\left(t - \frac{\ln b}{k}\right)\right)\right] = \frac{L}{2}[1 + \tanh u]$$
$$= \frac{L}{2}\frac{2}{1 + be^{-kt}}$$
$$= \frac{L}{1 + be^{-kt}}.$$

Notice the graph of the logistics function is just a shift of the graph of the hyperbolic tangent. (See section 5.8.)

6. $[f(x)g(x)]' \stackrel{?}{=} f'(x)g'(x)$

(a) Let $g(x) = x$, $g'(x) = 1$, then

$$[f(x)x]' = f'(x)$$
$$f'(x)x + f(x) = f'(x)$$
$$\frac{df}{dx}(x - 1) = -f(x)$$
$$\int\frac{df}{f} = \int\frac{dx}{1 - x}$$
$$\ln|f(x)| = -\ln|1 - x|$$
$$f(x) = \frac{1}{1 - x}$$

(b)
$$(fg)' = f'g'$$
$$f'g + fg' = f'g'$$
$$f'(g - g') = -fg'$$
$$\frac{f'}{f} = \frac{g'}{g' - g}$$
$$\ln|f| = \int\frac{g'}{g' - g}\,dx$$
$$f = e^{\int\frac{g'}{g'-g}dx}$$

(c) If $g(x) = e^x$, then $g'(x) - g(x) = e^x - e^x = 0$

Therefore, no f can exist.

7. $k = \left(\frac{1}{12}\right)^2 \pi$

$g = 32$

$x^2 + (y-6)^2 = 36$ Equation of tank

$x^2 = 36 - (y-6)^2 = 12y - y^2$

Area of cross section: $A(h) = (12h - h^2)\pi$

$$A(h)\frac{dh}{dt} = -k\sqrt{2gh}$$

$$(12h - h^2)\pi\frac{dh}{dt} = -\frac{1}{144}\pi\sqrt{64h}$$

$$(12h - h^2)\frac{dh}{dt} = -\frac{1}{18}h^{1/2}$$

$$\int\left(18h^{3/2} - 216h^{1/2}\right)dh = \int dt$$

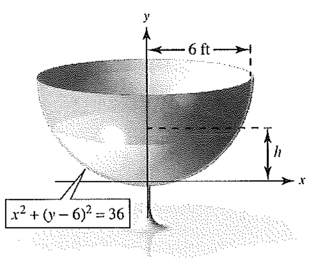
$$\frac{36}{5}h^{5/2} - 144h^{3/2} = t + C$$

$$\frac{h^{3/2}}{5}(36h - 720) = t + C$$

When $h = 6, t = 0$ and $C = \frac{6^{3/2}}{5}(-504) \approx -1481.45$.

The tank is completely drained when

$h = 0 \Rightarrow t = 1481.45 \text{ sec} \approx 24 \text{ min, } 41 \text{ sec}$



8. (a) $A(h)\frac{dh}{dt} = -k\sqrt{2gh}$

$$\pi r^2\frac{dh}{dt} = -k\sqrt{64h}$$

$$h^{-1/2}\,dh = \frac{-8k}{\pi r^2}dt = -C\,dt, \quad C = \frac{8k}{\pi r^2}$$

$$2\sqrt{h} = -Ct + C_1$$

$$2\sqrt{18} = C_1 \quad (\text{at } t = 0, h = 18)$$

So, $2\sqrt{h} = -Ct + 6\sqrt{2}$.

At $t = 30(60) = 1800, h = 12$:

$$2\sqrt{12} = -1800C + 6\sqrt{2}$$

$$\frac{6\sqrt{2} - 4\sqrt{3}}{1800} = C \approx 0.000865$$

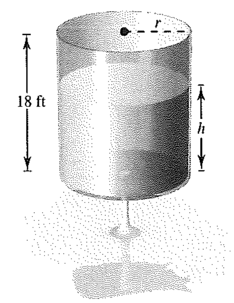
So, $2\sqrt{h} = -0.000865t + 6\sqrt{2}$.

$$h = 0 \Rightarrow t = \frac{6\sqrt{2}}{0.000865}$$

≈ 9809.1 seconds (2 h, 43 min, 29 sec)

(b) $t = 3600 \text{ sec} \Rightarrow 2\sqrt{h} = -0.000865(3600) + 6\sqrt{2}$

$\Rightarrow h \approx 7.21 \text{ ft}$



9. $A(h)\frac{dh}{dt} = -k\sqrt{2gh}$

$$\pi 64\frac{dh}{dt} = \frac{-\pi}{36}8\sqrt{h}$$

$$\int h^{-1/2}\,dh = \int\frac{-1}{288}dt$$

$$2\sqrt{h} = \frac{-t}{288} + C$$

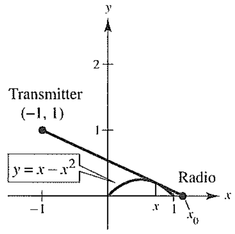
$$h = 20: 2\sqrt{20} = C = 4\sqrt{5}$$

$$2\sqrt{h} = \frac{-t}{288} + 4\sqrt{5}$$

$h = 0 \Rightarrow t = 4\sqrt{5}(288)$

$\approx 2575.95 \text{ sec} \approx 42 \text{ min, } 56 \text{ sec}$

10. Let the radio receiver be located at $(x_0, 0)$. The tangent line to $y = x - x^2$ joins $(-1, 1)$ and $(x_0, 0)$.



(a) If (x, y) is the point of tangency on $y = x - x^2$, then

$$1 - 2x = \frac{y-1}{x+1} = \frac{x - x^2 - 1}{x+1}$$

$$x - 2x^2 + 1 - 2x = x - x^2 - 1$$

$$x^2 + 2x - 2 = 0$$

$$x = \left(\frac{-2 \pm \sqrt{4+8}}{2}\right) = -1 + \sqrt{3}$$

$$y = x - x^2 = 3\sqrt{3} - 5.$$

Then $\dfrac{1-0}{-1-x_0} = \dfrac{1 - 3\sqrt{3} + 5}{-1 + 1 - \sqrt{3}} = \dfrac{6 - 3\sqrt{3}}{-\sqrt{3}}$

$$\sqrt{3} = (1 + x_0)(6 - 3\sqrt{3}) = 6 - 3\sqrt{3} + x_0(6 - 3\sqrt{3})$$

$$x_0 = \frac{4\sqrt{3} - 6}{6 - 3\sqrt{3}} \approx 1.155.$$

(b) Now let the transmitter be located at $(-1, h)$.

$$1 - 2x = \frac{y-h}{x+1} = \frac{x - x^2 - h}{x+1}$$

$$x - 2x^2 + 1 - 2x = x - x^2 - h$$

$$x^2 + 2x - h - 1 = 0$$

$$x = \frac{\left(-2 \pm \sqrt{4 + 4(h+1)}\right)}{2} = -1 + \sqrt{2+h}$$

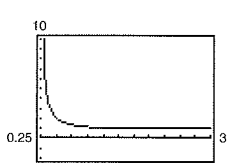
$$y = x - x^2 = 3\sqrt{2+h} - h - 4$$

Then, $\dfrac{h-0}{-1-x_0} = \dfrac{h - \left(3\sqrt{2+h} - h - 4\right)}{-1 - \left(-1 + \sqrt{2+h}\right)} = \dfrac{2h + 4 - 3\sqrt{2+h}}{-\sqrt{2+h}}$

$$\frac{x_0 + 1}{h} = \frac{\sqrt{2+h}}{2h + 4 - 3\sqrt{2+h}}$$

$$x_0 = \frac{h\sqrt{2+h}}{2h + 4 - 3\sqrt{2+h}} - 1.$$

(c)

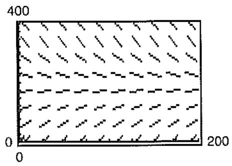


There is a vertical asymptote at $h = \dfrac{1}{4}$, which is the height of the mountain.

11. $\dfrac{ds}{dt} = 3.5 - 0.019s$

(a) $\displaystyle\int \frac{-ds}{3.5 - 0.019s} = -\int dt$

$$\frac{1}{0.019}\ln|3.5 - 0.019s| = -t + C_1$$
$$\ln|3.5 - 0.019s| = -0.019t + C_2$$
$$3.5 - 0.019s = C_3e^{-0.019t}$$
$$0.019s = 3.5 - C_3e^{-0.019t}$$
$$s = 184.21 - Ce^{-0.019t}$$

(b) 

(c) As $t \to \infty$, $Ce^{-0.019t} \to 0$, and $s \to 184.21$.

12. (a) $\displaystyle\int \frac{dC}{C} = \int -\frac{R}{V}\,dt$

$$\ln|C| = -\frac{R}{V}t + K_1$$
$$C = Ke^{-Rt/V}$$

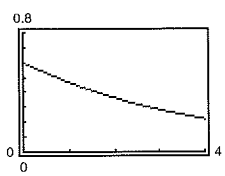
Since $C = C_0$ when $t = 0$, it follows that $K = C_0$ and the function is $C = C_0e^{-Rt/V}$.

(b) Finally, as $t \to \infty$, we have

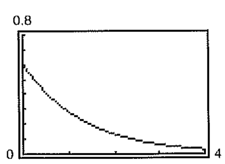
$$\lim_{t\to\infty} C = \lim_{t\to\infty} C_0e^{-Rt/V} = 0.$$

13. From Exercises 12, you have $C = C_0e^{-Rt/V}$.

(a) For $V = 2$, $R = 0.5$, and $C_0 = 0.6$, you have $C = 0.6e^{-0.25t}$



(b) For $V = 2$, $R = 1.5$, and $C_0 = 0.6$, you have $C = 0.6e^{-0.75t}$.



14. (a) $\displaystyle\int \frac{1}{Q - RC}\,dC = \int \frac{1}{V}\,dt$

$$-\frac{1}{R}\ln|Q - RC| = \frac{t}{V} + K_1$$
$$Q - RC = e^{-R[(t/V)+K_1]}$$
$$C = \frac{1}{R}\left(Q - e^{-R[(t/V)+K_1]}\right) = \frac{1}{R}\left(Q - Ke^{-Rt/V}\right)$$

Because $C = 0$ when $t = 0$, it follows that $K = Q$ and you have $C = \dfrac{Q}{R}\left(1 - e^{-Rt/V}\right)$.

(b) As $t \to \infty$, the limit of C is Q/R.