

# 2

## Differentiation

In this chapter you will study one of the most important processes of calculus—*differentiation*. In each section, you will learn new methods and rules for finding derivatives of functions. Then you will apply these rules to find such things as velocity, acceleration, and the rates of change of two or more related variables.

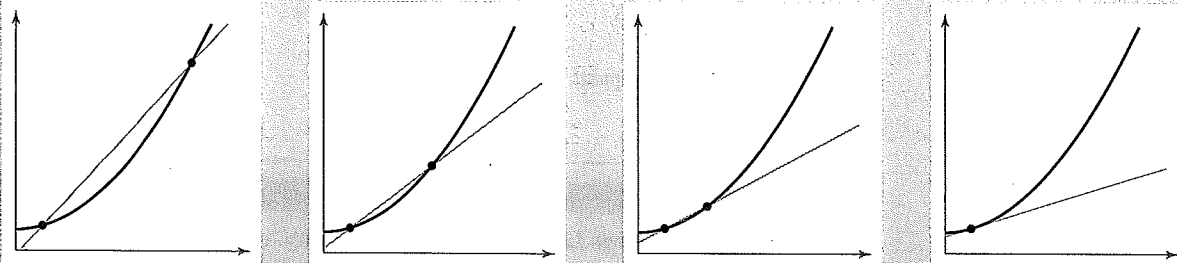
In this chapter, you should learn the following.

- How to find the derivative of a function using the limit definition and understand the relationship between differentiability and continuity. (2.1)
- How to find the derivative of a function using basic differentiation rules. (2.2)
- How to find the derivative of a function using the Product Rule and the Quotient Rule. (2.3)
- How to find the derivative of a function using the Chain Rule and the General Power Rule. (2.4)
- How to find the derivative of a function using implicit differentiation. (2.5)
- How to find a related rate. (2.6)



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When jumping from a platform, a diver's velocity is briefly positive because of the upward movement, but then becomes negative when falling. How can you use calculus to determine the velocity of a diver at impact? (See Section 2.2, Example 10.)



To approximate the slope of a tangent line to a graph at a given point, find the slope of the secant line through the given point and a second point on the graph. As the second point approaches the given point, the approximation tends to become more accurate. (See Section 2.1.)

## 2.1 The Derivative and the Tangent Line Problem

- Find the slope of the tangent line to a curve at a point.
- Use the limit definition to find the derivative of a function.
- Understand the relationship between differentiability and continuity.

### The Tangent Line Problem

Calculus grew out of four major problems that European mathematicians were working on during the seventeenth century.

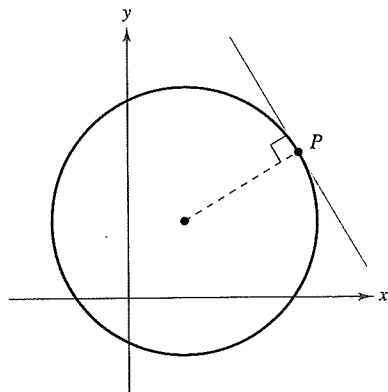
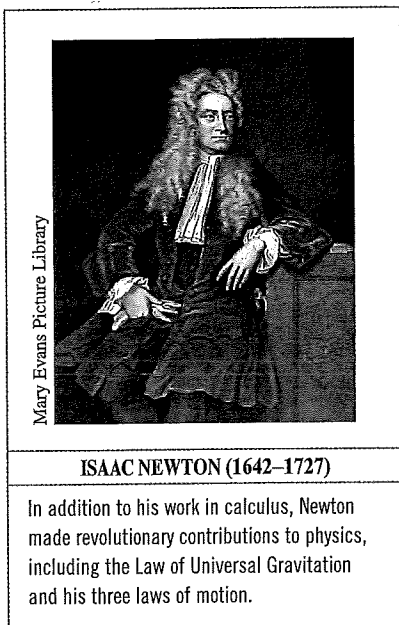
1. The tangent line problem (Section 1.1 and this section)
2. The velocity and acceleration problem (Sections 2.2 and 2.3)
3. The minimum and maximum problem (Section 3.1)
4. The area problem (Sections 1.1 and 4.2)

Each problem involves the notion of a limit, and calculus can be introduced with any of the four problems.

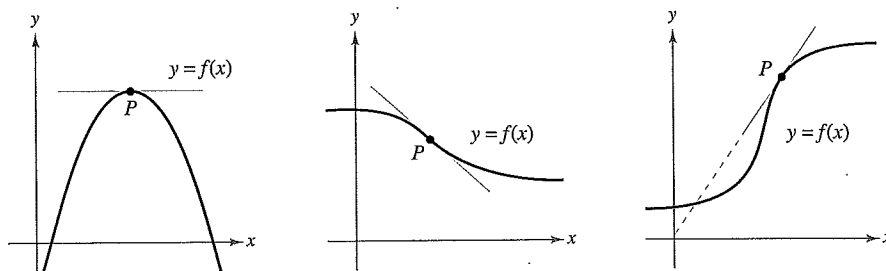
A brief introduction to the tangent line problem is given in Section 1.1. Although partial solutions to this problem were given by Pierre de Fermat (1601–1665), René Descartes (1596–1650), Christian Huygens (1629–1695), and Isaac Barrow (1630–1677), credit for the first general solution is usually given to Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716). Newton's work on this problem stemmed from his interest in optics and light refraction.

What does it mean to say that a line is tangent to a curve at a point? For a circle, the tangent line at a point  $P$  is the line that is perpendicular to the radial line at point  $P$ , as shown in Figure 2.1.

For a general curve, however, the problem is more difficult. For example, how would you define the tangent lines shown in Figure 2.2? You might say that a line is tangent to a curve at a point  $P$  if it touches, but does not cross, the curve at point  $P$ . This definition would work for the first curve shown in Figure 2.2, but not for the second. *Or* you might say that a line is tangent to a curve if the line touches or intersects the curve at exactly one point. This definition would work for a circle but not for more general curves, as the third curve in Figure 2.2 shows.



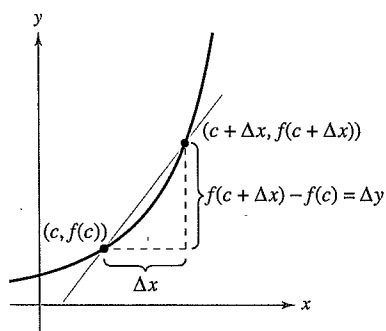
Tangent line to a circle  
Figure 2.1



Tangent line to a curve at a point  
Figure 2.2

### EXPLORATION

**Identifying a Tangent Line** Use a graphing utility to graph the function  $f(x) = 2x^3 - 4x^2 + 3x - 5$ . On the same screen, graph  $y = x - 5$ ,  $y = 2x - 5$ , and  $y = 3x - 5$ . Which of these lines, if any, appears to be tangent to the graph of  $f$  at the point  $(0, -5)$ ? Explain your reasoning.



The secant line through  $(c, f(c))$  and  $(c + \Delta x, f(c + \Delta x))$

Figure 2.3

Essentially, the problem of finding the tangent line at a point  $P$  boils down to the problem of finding the *slope* of the tangent line at point  $P$ . You can approximate this slope using a **secant line**\* through the point of tangency and a second point on the curve, as shown in Figure 2.3. If  $(c, f(c))$  is the point of tangency and  $(c + \Delta x, f(c + \Delta x))$  is a second point on the graph of  $f$ , the slope of the secant line through the two points is given by substitution into the slope formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{(c + \Delta x) - c} \quad \begin{array}{l} \text{Change in } y \\ \text{Change in } x \end{array}$$

$$m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{\Delta x} \quad \text{Slope of secant line}$$

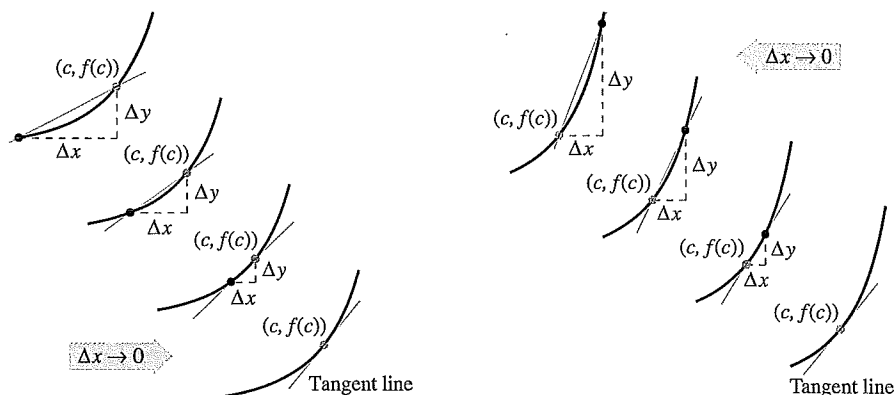
The right-hand side of this equation is a **difference quotient**. The denominator  $\Delta x$  is the **change in  $x$** , and the numerator  $\Delta y = f(c + \Delta x) - f(c)$  is the **change in  $y$** .

The beauty of this procedure is that you can obtain more and more accurate approximations of the slope of the tangent line by choosing points closer and closer to the point of tangency, as shown in Figure 2.4.

#### THE TANGENT LINE PROBLEM

In 1637, mathematician René Descartes stated this about the tangent line problem:

"And I dare say that this is not only the most useful and general problem in geometry that I know, but even that I ever desire to know."



Tangent line approximations

Figure 2.4

#### DEFINITION OF TANGENT LINE WITH SLOPE $m$

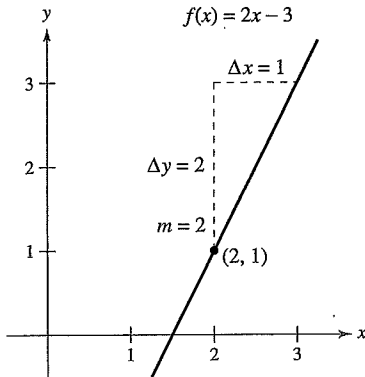
If  $f$  is defined on an open interval containing  $c$ , and if the limit

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through  $(c, f(c))$  with slope  $m$  is the **tangent line** to the graph of  $f$  at the point  $(c, f(c))$ .

The slope of the tangent line to the graph of  $f$  at the point  $(c, f(c))$  is also called the **slope of the graph of  $f$  at  $x = c$** .

\* This use of the word secant comes from the Latin *secare*, meaning to cut, and is not a reference to the trigonometric function of the same name.



The slope of  $f$  at  $(2, 1)$  is  $m = 2$ .

Figure 2.5

### EXAMPLE 1 The Slope of the Graph of a Linear Function

Find the slope of the graph of

$$f(x) = 2x - 3$$

at the point  $(2, 1)$ .

**Solution** To find the slope of the graph of  $f$  when  $c = 2$ , you can apply the definition of the slope of a tangent line, as shown.

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{[2(2 + \Delta x) - 3] - [2(2) - 3]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{4 + 2\Delta x - 3 - 4 + 3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2\Delta x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} 2 \\ &= 2 \end{aligned}$$

The slope of  $f$  at  $(c, f(c)) = (2, 1)$  is  $m = 2$ , as shown in Figure 2.5. ■

**NOTE** In Example 1, the limit definition of the slope of  $f$  agrees with the definition of the slope of a line as discussed in Section P.2. ■

The graph of a linear function has the same slope at any point. This is not true of nonlinear functions, as shown in the following example.

### EXAMPLE 2 Tangent Lines to the Graph of a Nonlinear Function

Find the slopes of the tangent lines to the graph of

$$f(x) = x^2 + 1$$

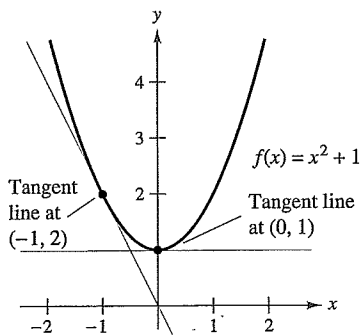
at the points  $(0, 1)$  and  $(-1, 2)$ , as shown in Figure 2.6.

**Solution** Let  $(c, f(c))$  represent an arbitrary point on the graph of  $f$ . Then the slope of the tangent line at  $(c, f(c))$  is given by

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{[(c + \Delta x)^2 + 1] - (c^2 + 1)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{c^2 + 2c(\Delta x) + (\Delta x)^2 + 1 - c^2 - 1}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2c(\Delta x) + (\Delta x)^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (2c + \Delta x) \\ &= 2c. \end{aligned}$$

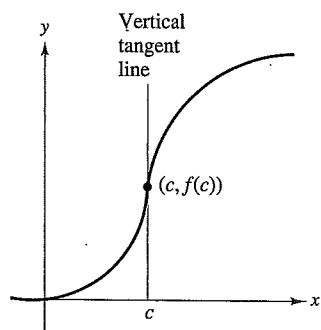
So, the slope at any point  $(c, f(c))$  on the graph of  $f$  is  $m = 2c$ . At the point  $(0, 1)$ , the slope is  $m = 2(0) = 0$ , and at  $(-1, 2)$ , the slope is  $m = 2(-1) = -2$ . ■

**NOTE** In Example 2, note that  $c$  is held constant in the limit process (as  $\Delta x \rightarrow 0$ ). ■



The slope of  $f$  at any point  $(c, f(c))$  is  $m = 2c$ .

Figure 2.6



The graph of  $f$  has a vertical tangent line at  $(c, f(c))$ .

Figure 2.7

The definition of a tangent line to a curve does not cover the possibility of a vertical tangent line. For vertical tangent lines, you can use the following definition. If  $f$  is continuous at  $c$  and

$$\lim_{\Delta x \rightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \infty \quad \text{or} \quad \lim_{\Delta x \rightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = -\infty$$

the vertical line  $x = c$  passing through  $(c, f(c))$  is a **vertical tangent line** to the graph of  $f$ . For example, the function shown in Figure 2.7 has a vertical tangent line at  $(c, f(c))$ . If the domain of  $f$  is the closed interval  $[a, b]$ , you can extend the definition of a vertical tangent line to include the endpoints by considering continuity and limits from the right (for  $x = a$ ) and from the left (for  $x = b$ ).

## The Derivative of a Function

You have now arrived at a crucial point in the study of calculus. The limit used to define the slope of a tangent line is also used to define one of the two fundamental operations of calculus—**differentiation**.

### DEFINITION OF THE DERIVATIVE OF A FUNCTION

The **derivative** of  $f$  at  $x$  is given by

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all  $x$  for which this limit exists,  $f'$  is a function of  $x$ .

Be sure you see that the derivative of a function of  $x$  is also a function of  $x$ . This “new” function gives the slope of the tangent line to the graph of  $f$  at the point  $(x, f(x))$ , provided that the graph has a tangent line at this point.

The process of finding the derivative of a function is called **differentiation**. A function is **differentiable** at  $x$  if its derivative exists at  $x$  and is **differentiable on an open interval  $(a, b)$**  if it is differentiable at every point in the interval.

In addition to  $f'(x)$ , which is read as “ $f$  prime of  $x$ ,” other notations are used to denote the derivative of  $y = f(x)$ . The most common are

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad D_x[y]. \quad \text{Notation for derivatives}$$

The notation  $dy/dx$  is read as “the derivative of  $y$  with respect to  $x$ ” or simply “ $dy, dx$ .” Using limit notation, you can write

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= f'(x). \end{aligned}$$

### ■ FOR FURTHER INFORMATION

For more information on the crediting of mathematical discoveries to the first “discoverers,” see the article “Mathematical Firsts—Who Done It?” by Richard H. Williams and Roy D. Mazzagatti in *Mathematics Teacher*. To view this article, go to the website [www.matharticles.com](http://www.matharticles.com).

### EXAMPLE 3 Finding the Derivative by the Limit Process

Find the derivative of  $f(x) = x^3 + 2x$ .

**Solution**

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} && \text{Definition of derivative} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 + 2(x + \Delta x) - (x^3 + 2x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x + 2\Delta x - x^3 - 2x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2\Delta x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x[3x^2 + 3x\Delta x + (\Delta x)^2 + 2]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} [3x^2 + 3x\Delta x + (\Delta x)^2 + 2] \\
 &= 3x^2 + 2
 \end{aligned}$$

**STUDY TIP** When using the definition to find a derivative of a function, the key is to rewrite the difference quotient so that  $\Delta x$  does not occur as a factor of the denominator.

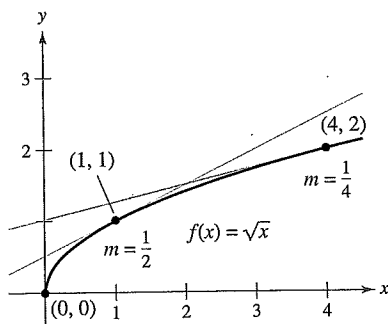
Remember that the derivative of a function  $f$  is itself a function, which can be used to find the slope of the tangent line at the point  $(x, f(x))$  on the graph of  $f$ .

### EXAMPLE 4 Using the Derivative to Find the Slope at a Point

Find  $f'(x)$  for  $f(x) = \sqrt{x}$ . Then find the slopes of the graph of  $f$  at the points  $(1, 1)$  and  $(4, 2)$ . Discuss the behavior of  $f$  at  $(0, 0)$ .

**Solution** Use the procedure for rationalizing numerators, as discussed in Section 1.3.


$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} && \text{Definition of derivative} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \left( \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \right) \left( \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \right) \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} \\
 &= \frac{1}{2\sqrt{x}}, \quad x > 0
 \end{aligned}$$



The slope of  $f$  at  $(x, f(x))$ ,  $x > 0$ , is  $m = 1/(2\sqrt{x})$ .

Figure 2.8

At the point  $(1, 1)$ , the slope is  $f'(1) = \frac{1}{2}$ . At the point  $(4, 2)$ , the slope is  $f'(4) = \frac{1}{4}$ . See Figure 2.8. At the point  $(0, 0)$ , the slope is undefined. Moreover, the graph of  $f$  has a vertical tangent line at  $(0, 0)$ .

The icon  indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple and Mathematica.

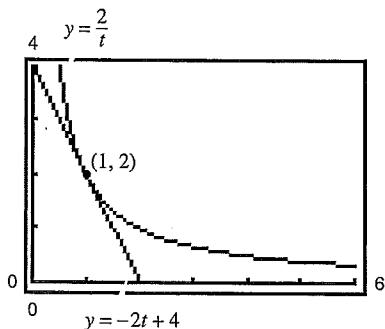
In many applications, it is convenient to use a variable other than  $x$  as the independent variable, as shown in Example 5.

**EXAMPLE 5 Finding the Derivative of a Function**

Find the derivative with respect to  $t$  for the function  $y = 2/t$ .

**Solution** Considering  $y = f(t)$ , you obtain

$$\begin{aligned} \frac{dy}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} && \text{Definition of derivative} \\ &= \lim_{\Delta t \rightarrow 0} \frac{\frac{2}{t + \Delta t} - \frac{2}{t}}{\Delta t} && f(t + \Delta t) = 2/(t + \Delta t) \text{ and } f(t) = 2/t \\ &= \lim_{\Delta t \rightarrow 0} \frac{2t - 2(t + \Delta t)}{t(t + \Delta t)\Delta t} && \text{Combine fractions in numerator.} \\ &= \lim_{\Delta t \rightarrow 0} \frac{-2\Delta t}{\Delta t(t)(t + \Delta t)} && \text{Divide out common factor of } \Delta t. \\ &= \lim_{\Delta t \rightarrow 0} \frac{-2}{t(t + \Delta t)} && \text{Simplify.} \\ &= -\frac{2}{t^2}. && \text{Evaluate limit as } \Delta t \rightarrow 0. \end{aligned}$$



At the point  $(1, 2)$ , the line  $y = -2t + 4$  is tangent to the graph of  $y = 2/t$ .

Figure 2.9

**TECHNOLOGY** A graphing utility can be used to reinforce the result given in Example 5. For instance, using the formula  $dy/dt = -2/t^2$ , you know that the slope of the graph of  $y = 2/t$  at the point  $(1, 2)$  is  $m = -2$ . Using the point-slope form, you can find that the equation of the tangent line to the graph at  $(1, 2)$  is

$$y - 2 = -2(t - 1) \quad \text{or} \quad y = -2t + 4$$

as shown in Figure 2.9.

**Differentiability and Continuity**

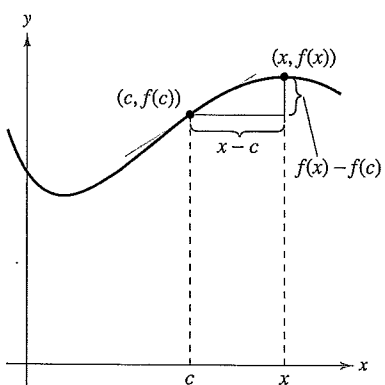
The following alternative limit form of the derivative is useful in investigating the relationship between differentiability and continuity. The derivative of  $f$  at  $c$  is

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \quad \text{Alternative form of derivative}$$

provided this limit exists (see Figure 2.10). (A proof of the equivalence of this form is given in Appendix A.) Note that the existence of the limit in this alternative form requires that the one-sided limits

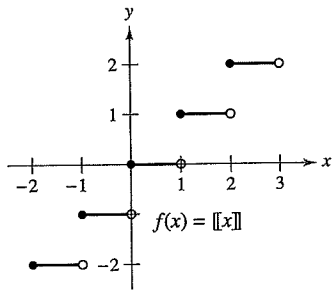
$$\lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} \quad \text{and} \quad \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c}$$

exist and are equal. These one-sided limits are called the **derivatives from the left and from the right**, respectively. It follows that  $f$  is **differentiable on the closed interval  $[a, b]$**  if it is differentiable on  $(a, b)$  and if the derivative from the right at  $a$  and the derivative from the left at  $b$  both exist.



As  $x$  approaches  $c$ , the secant line approaches the tangent line.

Figure 2.10



The greatest integer function is not differentiable at  $x = 0$ , because it is not continuous at  $x = 0$ .

Figure 2.11

If a function is not continuous at  $x = c$ , it is also not differentiable at  $x = c$ . For instance, the greatest integer function

$$f(x) = \llbracket x \rrbracket$$

is not continuous at  $x = 0$ , and so it is not differentiable at  $x = 0$  (see Figure 2.11). You can verify this by observing that

$$\lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{\llbracket x \rrbracket - 0}{x} = \infty \quad \text{Derivative from the left}$$

and

$$\lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{\llbracket x \rrbracket - 0}{x} = 0. \quad \text{Derivative from the right}$$

Although it is true that differentiability implies continuity (as shown in Theorem 2.1 on the next page), the converse is not true. That is, it is possible for a function to be continuous at  $x = c$  and *not* differentiable at  $x = c$ . Examples 6 and 7 illustrate this possibility.

### EXAMPLE 6 A Graph with a Sharp Turn

The function

$$f(x) = |x - 2|$$

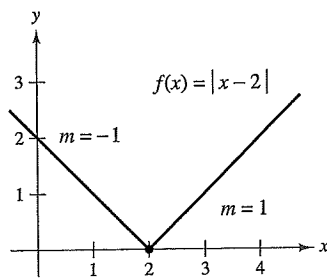
shown in Figure 2.12 is continuous at  $x = 2$ . However, the one-sided limits

$$\lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{|x - 2| - 0}{x - 2} = -1 \quad \text{Derivative from the left}$$

and

$$\lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^+} \frac{|x - 2| - 0}{x - 2} = 1 \quad \text{Derivative from the right}$$

are not equal. So,  $f$  is not differentiable at  $x = 2$  and the graph of  $f$  does not have a tangent line at the point  $(2, 0)$ .



$f$  is not differentiable at  $x = 2$ , because the derivatives from the left and from the right are not equal.

Figure 2.12

### EXAMPLE 7 A Graph with a Vertical Tangent Line

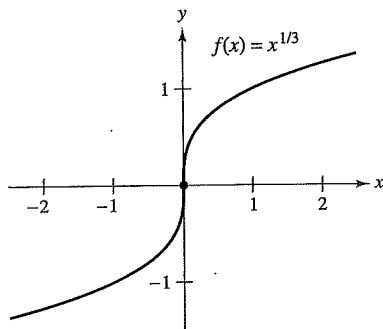
The function

$$f(x) = x^{1/3}$$

is continuous at  $x = 0$ , as shown in Figure 2.13. However, because the limit

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \rightarrow 0} \frac{x^{1/3} - 0}{x} \\ &= \lim_{x \rightarrow 0} \frac{1}{x^{2/3}} \\ &= \infty \end{aligned}$$

is infinite, you can conclude that the tangent line is vertical at  $x = 0$ . So,  $f$  is not differentiable at  $x = 0$ . ■



$f$  is not differentiable at  $x = 0$ , because  $f$  has a vertical tangent line at  $x = 0$ .

Figure 2.13

From Examples 6 and 7, you can see that a function is not differentiable at a point at which its graph has a sharp turn *or* a vertical tangent line.

**TECHNOLOGY** Some graphing utilities, such as *Maple*, *Mathematica*, and the *TI-89*, perform symbolic differentiation. Others perform *numerical differentiation* by finding values of derivatives using the formula

$$f'(x) \approx \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

where  $\Delta x$  is a small number such as 0.001. Can you see any problems with this definition? For instance, using this definition, what is the value of the derivative of  $f(x) = |x|$  when  $x = 0$ ?

**THEOREM 2.1 DIFFERENTIABILITY IMPLIES CONTINUITY**

If  $f$  is differentiable at  $x = c$ , then  $f$  is continuous at  $x = c$ .

**PROOF** You can prove that  $f$  is continuous at  $x = c$  by showing that  $f(x)$  approaches  $f(c)$  as  $x \rightarrow c$ . To do this, use the differentiability of  $f$  at  $x = c$  and consider the following limit.

$$\begin{aligned} \lim_{x \rightarrow c} [f(x) - f(c)] &= \lim_{x \rightarrow c} \left[ (x - c) \left( \frac{f(x) - f(c)}{x - c} \right) \right] \\ &= \left[ \lim_{x \rightarrow c} (x - c) \right] \left[ \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \right] \\ &= (0)[f'(c)] \\ &= 0 \end{aligned}$$

Because the difference  $f(x) - f(c)$  approaches zero as  $x \rightarrow c$ , you can conclude that  $\lim_{x \rightarrow c} f(x) = f(c)$ . So,  $f$  is continuous at  $x = c$ . ■

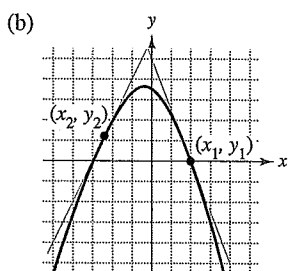
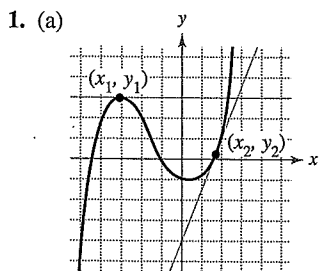
The following statements summarize the relationship between continuity and differentiability.

1. If a function is differentiable at  $x = c$ , then it is continuous at  $x = c$ . So, differentiability implies continuity.
2. It is possible for a function to be continuous at  $x = c$  and not be differentiable at  $x = c$ . So, continuity does not imply differentiability (see Example 6).

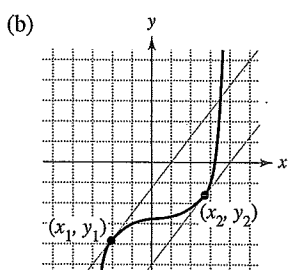
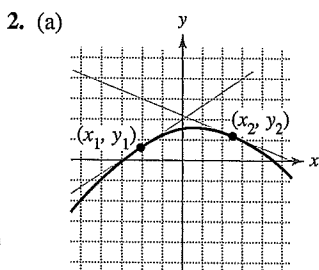
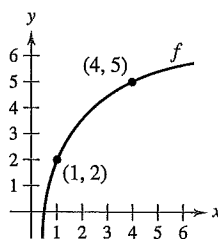
**2.1 Exercises**

See [www.CalcChat.com](http://www.CalcChat.com) for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, estimate the slope of the graph at the points  $(x_1, y_1)$  and  $(x_2, y_2)$ .



In Exercises 3 and 4, use the graph shown in the figure. To print an enlarged copy of the graph, go to the website [www.mathgraphs.com](http://www.mathgraphs.com).



3. Identify or sketch each of the quantities on the figure.

- (a)  $f(1)$  and  $f(4)$                       (b)  $f(4) - f(1)$

(c)  $y = \frac{f(4) - f(1)}{4 - 1}(x - 1) + f(1)$

4. Insert the proper inequality symbol ( $<$  or  $>$ ) between the given quantities.

(a)  $\frac{f(4) - f(1)}{4 - 1}$                        $\frac{f(4) - f(3)}{4 - 3}$


(b)  $\frac{f(4) - f(1)}{4 - 1}$                        $f'(1)$

In Exercises 5–10, find the slope of the tangent line to the graph of the function at the given point.

5.  $f(x) = 3 - 5x$ ,  $(-1, 8)$       6.  $g(x) = \frac{3}{2}x + 1$ ,  $(-2, -2)$   
 7.  $g(x) = x^2 - 9$ ,  $(2, -5)$       8.  $g(x) = 6 - x^2$ ,  $(1, 5)$   
 9.  $f(t) = 3t - t^2$ ,  $(0, 0)$       10.  $h(t) = t^2 + 3$ ,  $(-2, 7)$

In Exercises 11–24, find the derivative by the limit process.

11.  $f(x) = 7$       12.  $g(x) = -3$   
 13.  $f(x) = -10x$       14.  $f(x) = 3x + 2$   
 15.  $h(s) = 3 + \frac{2}{3}s$       16.  $f(x) = 8 - \frac{1}{5}x$   
 17.  $f(x) = x^2 + x - 3$       18.  $f(x) = 2 - x^2$   
 19.  $f(x) = x^3 - 12x$       20.  $f(x) = x^3 + x^2$   
 21.  $f(x) = \frac{1}{x-1}$       22.  $f(x) = \frac{1}{x^2}$   
 23.  $f(x) = \sqrt{x+4}$       24.  $f(x) = \frac{4}{\sqrt{x}}$

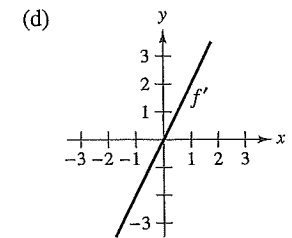
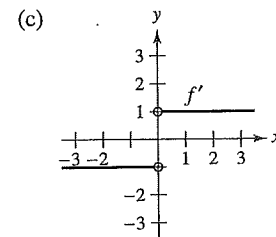
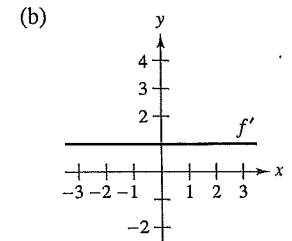
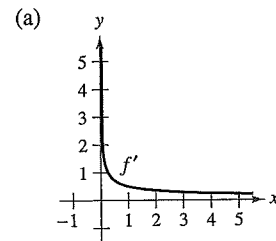
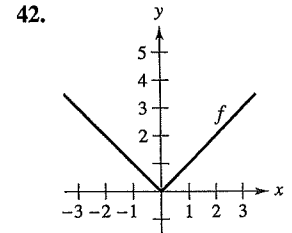
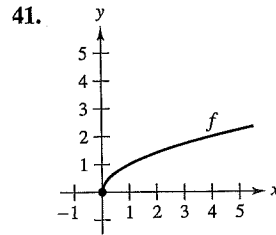
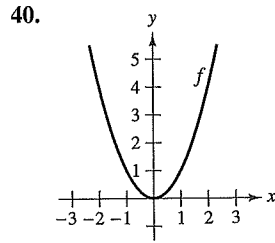
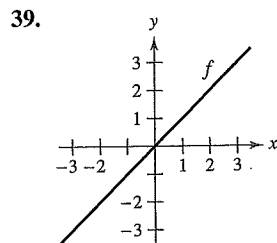
 In Exercises 25–32, (a) find an equation of the tangent line to the graph of  $f$  at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the derivative feature of a graphing utility to confirm your results.

25.  $f(x) = x^2 + 3$ ,  $(1, 4)$   
 26.  $f(x) = x^2 + 3x + 4$ ,  $(-2, 2)$   
 27.  $f(x) = x^3$ ,  $(2, 8)$       28.  $f(x) = x^3 + 1$ ,  $(1, 2)$   
 29.  $f(x) = \sqrt{x}$ ,  $(1, 1)$       30.  $f(x) = \sqrt{x-1}$ ,  $(5, 2)$   
 31.  $f(x) = x + \frac{4}{x}$ ,  $(4, 5)$       32.  $f(x) = \frac{1}{x+1}$ ,  $(0, 1)$

In Exercises 33–38, find an equation of the line that is tangent to the graph of  $f$  and parallel to the given line.

Function	Line
33. $f(x) = x^2$	$2x - y + 1 = 0$
34. $f(x) = 2x^2$	$4x + y + 3 = 0$
35. $f(x) = x^3$	$3x - y + 1 = 0$
36. $f(x) = x^3 + 2$	$3x - y - 4 = 0$
37. $f(x) = \frac{1}{\sqrt{x}}$	$x + 2y - 6 = 0$
38. $f(x) = \frac{1}{\sqrt{x-1}}$	$x + 2y + 7 = 0$

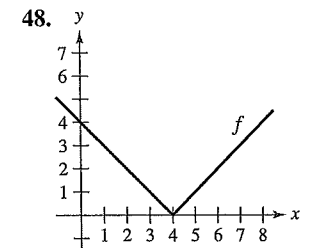
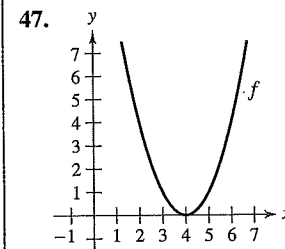
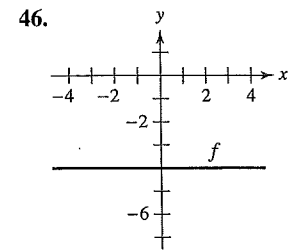
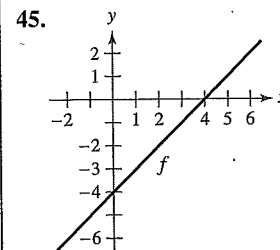
In Exercises 39–42, the graph of  $f$  is given. Select the graph of  $f'$ .



43. The tangent line to the graph of  $y = g(x)$  at the point  $(4, 5)$  passes through the point  $(7, 0)$ . Find  $g(4)$  and  $g'(4)$ .  
 44. The tangent line to the graph of  $y = h(x)$  at the point  $(-1, 4)$  passes through the point  $(3, 6)$ . Find  $h(-1)$  and  $h'(-1)$ .

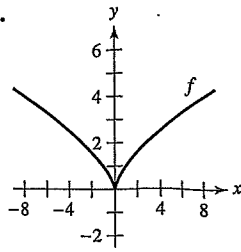
### WRITING ABOUT CONCEPTS

In Exercises 45–50, sketch the graph of  $f'$ . Explain how you found your answer.

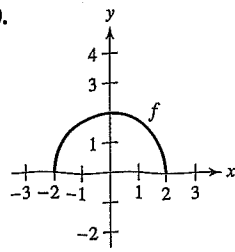


**WRITING ABOUT CONCEPTS** (continued)

49.



50.



51. Sketch a graph of a function whose derivative is always negative. Explain how you found your answer.

52. Sketch a graph of a function whose derivative is always positive. Explain how you found your answer.

In Exercises 53–56, the limit represents  $f'(c)$  for a function  $f$  and a number  $c$ . Find  $f$  and  $c$ .

$$53. \lim_{\Delta x \rightarrow 0} \frac{[5 - 3(1 + \Delta x)] - 2}{\Delta x}$$

$$54. \lim_{\Delta x \rightarrow 0} \frac{(-2 + \Delta x)^3 + 8}{\Delta x}$$

$$55. \lim_{x \rightarrow 6} \frac{-x^2 + 36}{x - 6}$$

$$56. \lim_{x \rightarrow 9} \frac{2\sqrt{x} - 6}{x - 9}$$

In Exercises 57–59, identify a function  $f$  that has the given characteristics. Then sketch the function.

57.  $f(0) = 2$ ;

58.  $f(0) = 4$ ;  $f'(0) = 0$ ;

$f'(x) = -3$ ,  $-\infty < x < \infty$

$f'(x) < 0$  for  $x < 0$ ;

$f'(x) > 0$  for  $x > 0$

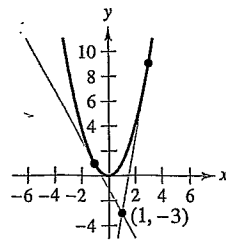
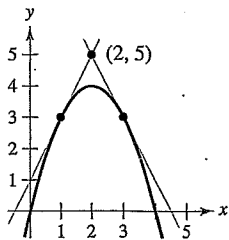
59.  $f(0) = 0$ ;  $f'(0) = 0$ ;  $f'(x) > 0$  for  $x \neq 0$

60. Assume that  $f'(c) = 3$ . Find  $f'(-c)$  if (a)  $f$  is an odd function and if (b)  $f$  is an even function.

In Exercises 61 and 62, find equations of the two tangent lines to the graph of  $f$  that pass through the indicated point.

61.  $f(x) = 4x - x^2$

62.  $f(x) = x^2$



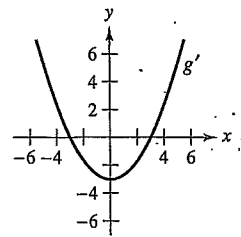
63. **Graphical Reasoning** Use a graphing utility to graph each function and its tangent lines at  $x = -1$ ,  $x = 0$ , and  $x = 1$ . Based on the results, determine whether the slopes of tangent lines to the graph of a function at different values of  $x$  are always distinct.

(a)  $f(x) = x^2$

(b)  $g(x) = x^3$

**CAPSTONE**

64. The figure shows the graph of  $g'$ .



(a)  $g'(0) =$  \_\_\_\_\_ (b)  $g'(3) =$  \_\_\_\_\_

(c) What can you conclude about the graph of  $g$  knowing that  $g'(1) = -\frac{8}{3}$ ?

(d) What can you conclude about the graph of  $g$  knowing that  $g'(-4) = \frac{7}{3}$ ?

(e) Is  $g(6) - g(4)$  positive or negative? Explain.

(f) Is it possible to find  $g(2)$  from the graph? Explain.

65. **Graphical Analysis** Consider the function  $f(x) = \frac{1}{2}x^2$ .

(a) Use a graphing utility to graph the function and estimate the values of  $f'(0)$ ,  $f'(\frac{1}{2})$ ,  $f'(1)$ , and  $f'(2)$ .

(b) Use your results from part (a) to determine the values of  $f'(-\frac{1}{2})$ ,  $f'(-1)$ , and  $f'(-2)$ .

(c) Sketch a possible graph of  $f'$ .

(d) Use the definition of derivative to find  $f'(x)$ .

66. **Graphical Analysis** Consider the function  $f(x) = \frac{1}{3}x^3$ .

(a) Use a graphing utility to graph the function and estimate the values of  $f'(0)$ ,  $f'(\frac{1}{2})$ ,  $f'(1)$ ,  $f'(2)$ , and  $f'(3)$ .

(b) Use your results from part (a) to determine the values of  $f'(-\frac{1}{2})$ ,  $f'(-1)$ ,  $f'(-2)$ , and  $f'(-3)$ .

(c) Sketch a possible graph of  $f'$ .

(d) Use the definition of derivative to find  $f'(x)$ .

67. **Graphical Reasoning** In Exercises 67 and 68, use a graphing utility to graph the functions  $f$  and  $g$  in the same viewing window where

$$g(x) = \frac{f(x + 0.01) - f(x)}{0.01}$$

Label the graphs and describe the relationship between them.

67.  $f(x) = 2x - x^2$

68.  $f(x) = 3\sqrt{x}$

In Exercises 69 and 70, evaluate  $f(2)$  and  $f(2.1)$  and use the results to approximate  $f'(2)$ .

69.  $f(x) = x(4 - x)$

70.  $f(x) = \frac{1}{4}x^3$

71. **Graphical Reasoning** In Exercises 71 and 72, use a graphing utility to graph the function and its derivative in the same viewing window. Label the graphs and describe the relationship between them.

71.  $f(x) = \frac{1}{\sqrt{x}}$

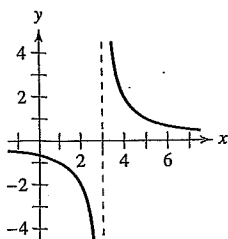
72.  $f(x) = \frac{x^3}{4} - 3x$

In Exercises 73–82, use the alternative form of the derivative to find the derivative at  $x = c$  (if it exists).

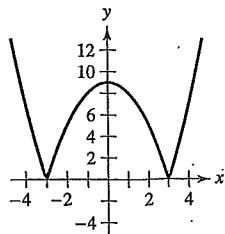
73.  $f(x) = x^2 - 5$ ,  $c = 3$       74.  $g(x) = x(x - 1)$ ,  $c = 1$   
 75.  $f(x) = x^3 + 2x^2 + 1$ ,  $c = -2$   
 76.  $f(x) = x^3 + 6x$ ,  $c = 2$   
 77.  $g(x) = \sqrt{|x|}$ ,  $c = 0$       78.  $f(x) = 2/x$ ,  $c = 5$   
 79.  $f(x) = (x - 6)^{2/3}$ ,  $c = 6$   
 80.  $g(x) = (x + 3)^{1/3}$ ,  $c = -3$   
 81.  $h(x) = |x + 7|$ ,  $c = -7$       82.  $f(x) = |x - 6|$ ,  $c = 6$

In Exercises 83–88, describe the  $x$ -values at which  $f$  is differentiable.

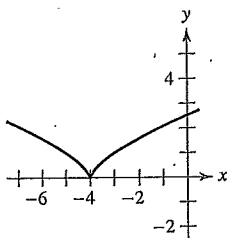
83.  $f(x) = \frac{2}{x - 3}$



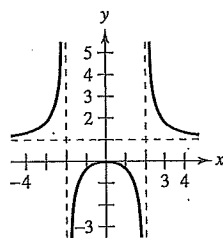
84.  $f(x) = |x^2 - 9|$



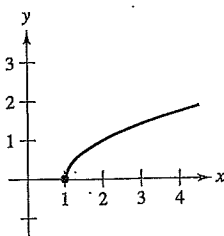
85.  $f(x) = (x + 4)^{2/3}$



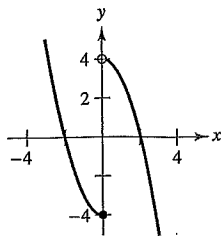
86.  $f(x) = \frac{x^2}{x^2 - 4}$



87.  $f(x) = \sqrt{x - 1}$



88.  $f(x) = \begin{cases} x^2 - 4, & x \leq 0 \\ 4 - x^2, & x > 0 \end{cases}$



**Graphical Analysis** In Exercises 89–92, use a graphing utility to graph the function and find the  $x$ -values at which  $f$  is differentiable.

89.  $f(x) = |x - 5|$

90.  $f(x) = \frac{4x}{x - 3}$

91.  $f(x) = x^{2/5}$

92.  $f(x) = \begin{cases} x^3 - 3x^2 + 3x, & x \leq 1 \\ x^2 - 2x, & x > 1 \end{cases}$

In Exercises 93–96, find the derivatives from the left and from the right at  $x = 1$  (if they exist). Is the function differentiable at  $x = 1$ ?

93.  $f(x) = |x - 1|$

94.  $f(x) = \sqrt{1 - x^2}$

95.  $f(x) = \begin{cases} (x - 1)^3, & x \leq 1 \\ (x - 1)^2, & x > 1 \end{cases}$

96.  $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

In Exercises 97 and 98, determine whether the function is differentiable at  $x = 2$ .

97.  $f(x) = \begin{cases} x^2 + 1, & x \leq 2 \\ 4x - 3, & x > 2 \end{cases}$

98.  $f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 2 \\ \sqrt{2x}, & x \geq 2 \end{cases}$

**99. Graphical Reasoning** A line with slope  $m$  passes through the point  $(0, 4)$  and has the equation  $y = mx + 4$ .

(a) Write the distance  $d$  between the line and the point  $(3, 1)$  as a function of  $m$ .

(b) Use a graphing utility to graph the function  $d$  in part (a). Based on the graph, is the function differentiable at every value of  $m$ ? If not, where is it not differentiable?

**100. Conjecture** Consider the functions  $f(x) = x^2$  and  $g(x) = x^3$ .

- (a) Graph  $f$  and  $f'$  on the same set of axes.  
 (b) Graph  $g$  and  $g'$  on the same set of axes.  
 (c) Identify a pattern between  $f$  and  $g$  and their respective derivatives. Use the pattern to make a conjecture about  $h'(x)$  if  $h(x) = x^n$ , where  $n$  is an integer and  $n \geq 2$ .  
 (d) Find  $f'(x)$  if  $f(x) = x^4$ . Compare the result with the conjecture in part (c). Is this a proof of your conjecture? Explain.

**True or False?** In Exercises 101–104, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

101. The slope of the tangent line to the differentiable function  $f$  at the point  $(2, f(2))$  is  $\frac{f(2 + \Delta x) - f(2)}{\Delta x}$ .  
 102. If a function is continuous at a point, then it is differentiable at that point.  
 103. If a function has derivatives from both the right and the left at a point, then it is differentiable at that point.  
 104. If a function is differentiable at a point, then it is continuous at that point.  
 105. Let  $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$  and  $g(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Show that  $f$  is continuous, but not differentiable, at  $x = 0$ . Show that  $g$  is differentiable at 0, and find  $g'(0)$ .

**106. Writing** Use a graphing utility to graph the two functions  $f(x) = x^2 + 1$  and  $g(x) = |x| + 1$  in the same viewing window. Use the *zoom* and *trace* features to analyze the graphs near the point  $(0, 1)$ . What do you observe? Which function is differentiable at this point? Write a short paragraph describing the geometric significance of differentiability at a point.