

6.3 Separation of Variables and the Logistic Equation

- Recognize and solve differential equations that can be solved by separation of variables.
- Recognize and solve homogeneous differential equations.
- Use differential equations to model and solve applied problems.
- Solve and analyze logistic differential equations.

Separation of Variables

Consider a differential equation that can be written in the form

$$M(x) + N(y) \frac{dy}{dx} = 0$$

where M is a continuous function of x alone and N is a continuous function of y alone. As you saw in the preceding section, for this type of equation, all x terms can be collected with dx and all y terms with dy , and a solution can be obtained by integration. Such equations are said to be **separable**, and the solution procedure is called **separation of variables**. Below are some examples of differential equations that are separable.

<u>Original Differential Equation</u>	<u>Rewritten with Variables Separated</u>
$x^2 + 3y \frac{dy}{dx} = 0$	$3y \, dy = -x^2 \, dx$
$(\sin x)y' = \cos x$	$dy = \cot x \, dx$
$\frac{xy'}{e^y + 1} = 2$	$\frac{1}{e^y + 1} \, dy = \frac{2}{x} \, dx$

EXAMPLE 1 Separation of Variables

Find the general solution of $(x^2 + 4) \frac{dy}{dx} = xy$.

Solution To begin, note that $y = 0$ is a solution. To find other solutions, assume that $y \neq 0$ and separate variables as shown.

$$(x^2 + 4) \, dy = xy \, dx \quad \text{Differential form}$$

$$\frac{dy}{y} = \frac{x}{x^2 + 4} \, dx \quad \text{Separate variables.}$$

Now, integrate to obtain

$$\int \frac{dy}{y} = \int \frac{x}{x^2 + 4} \, dx \quad \text{Integrate.}$$

$$\ln|y| = \frac{1}{2} \ln(x^2 + 4) + C_1$$

$$\ln|y| = \ln\sqrt{x^2 + 4} + C_1$$

$$|y| = e^{C_1} \sqrt{x^2 + 4}$$

$$y = \pm e^{C_1} \sqrt{x^2 + 4}.$$

Because $y = 0$ is also a solution, you can write the general solution as

$$y = C\sqrt{x^2 + 4}. \quad \text{General solution } (C = \pm e^{C_1})$$

NOTE Be sure to check your solutions throughout this chapter. In Example 1, you can check the solution $y = C\sqrt{x^2 + 4}$ by differentiating and substituting into the original equation.

$$(x^2 + 4) \frac{dy}{dx} = xy$$

$$(x^2 + 4) \frac{Cx}{\sqrt{x^2 + 4}} \stackrel{?}{=} x(C\sqrt{x^2 + 4})$$

$$Cx\sqrt{x^2 + 4} = Cx\sqrt{x^2 + 4}$$

So, the solution checks.

FOR FURTHER INFORMATION

For an example (from engineering) of a differential equation that is separable, see the article “Designing a Rose Cutter” by J. S. Hartzler in *The College Mathematics Journal*. To view this article, go to the website www.matharticles.com.

In some cases it is not feasible to write the general solution in the explicit form $y = f(x)$. The next example illustrates such a solution. Implicit differentiation can be used to verify this solution.

EXAMPLE 2 Finding a Particular Solution

Given the initial condition $y(0) = 1$, find the particular solution of the equation

$$xy \, dx + e^{-x^2}(y^2 - 1) \, dy = 0.$$

Solution Note that $y = 0$ is a solution of the differential equation—but this solution does not satisfy the initial condition. So, you can assume that $y \neq 0$. To separate variables, you must rid the first term of y and the second term of e^{-x^2} . So, you should multiply by e^{x^2}/y and obtain the following.

$$\begin{aligned} xy \, dx + e^{-x^2}(y^2 - 1) \, dy &= 0 \\ e^{-x^2}(y^2 - 1) \, dy &= -xy \, dx \\ \int \left(y - \frac{1}{y} \right) dy &= \int -xe^{x^2} \, dx \\ \frac{y^2}{2} - \ln |y| &= -\frac{1}{2}e^{x^2} + C \end{aligned}$$

From the initial condition $y(0) = 1$, you have $\frac{1}{2} - 0 = -\frac{1}{2} + C$, which implies that $C = 1$. So, the particular solution has the implicit form

$$\begin{aligned} \frac{y^2}{2} - \ln |y| &= -\frac{1}{2}e^{x^2} + 1 \\ y^2 - \ln y^2 + e^{x^2} &= 2. \end{aligned}$$

You can check this by differentiating and rewriting to get the original equation.

EXAMPLE 3 Finding a Particular Solution Curve

Find the equation of the curve that passes through the point $(1, 3)$ and has a slope of y/x^2 at any point (x, y) .

Solution Because the slope of the curve is given by y/x^2 , you have

$$\frac{dy}{dx} = \frac{y}{x^2}$$

with the initial condition $y(1) = 3$. Separating variables and integrating produces

$$\begin{aligned} \int \frac{dy}{y} &= \int \frac{dx}{x^2}, \quad y \neq 0 \\ \ln |y| &= -\frac{1}{x} + C_1 \\ y &= e^{-(1/x) + C_1} = Ce^{-1/x}. \end{aligned}$$

Because $y = 3$ when $x = 1$, it follows that $3 = Ce^{-1}$ and $C = 3e$. So, the equation of the specified curve is

$$y = (3e)e^{-1/x} = 3e^{(x-1)/x}, \quad x > 0.$$

Because the solution is not defined at $x = 0$ and the initial condition is given at $x = 1$, x is restricted to positive values. See Figure 6.12. ■

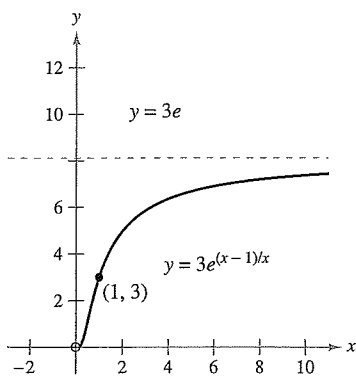


Figure 6.12

Homogeneous Differential Equations

Some differential equations that are not separable in x and y can be made separable by a change of variables. This is true for differential equations of the form $y' = f(x, y)$, where f is a **homogeneous function**. The function given by $f(x, y)$ is **homogeneous of degree n** if

$$f(tx, ty) = t^n f(x, y) \quad \text{Homogeneous function of degree } n$$

where n is an integer.

EXAMPLE 4 Verifying Homogeneous Functions

a. $f(x, y) = x^2y - 4x^3 + 3xy^2$ is a homogeneous function of degree 3 because

$$\begin{aligned} f(tx, ty) &= (tx)^2(ty) - 4(tx)^3 + 3(tx)(ty)^2 \\ &= t^3(x^2y) - t^3(4x^3) + t^3(3xy^2) \\ &= t^3(x^2y - 4x^3 + 3xy^2) \\ &= t^3f(x, y). \end{aligned}$$

b. $f(x, y) = xe^{x/y} + y \sin(y/x)$ is a homogeneous function of degree 1 because

$$\begin{aligned} f(tx, ty) &= txe^{tx/ty} + ty \sin \frac{ty}{tx} \\ &= t \left(xe^{x/y} + y \sin \frac{y}{x} \right) \\ &= tf(x, y). \end{aligned}$$

c. $f(x, y) = x + y^2$ is *not* a homogeneous function because

$$f(tx, ty) = tx + t^2y^2 = t(x + ty^2) \neq t^n(x + y^2).$$

d. $f(x, y) = x/y$ is a homogeneous function of degree 0 because

$$f(tx, ty) = \frac{tx}{ty} = t^0 \frac{x}{y}.$$

DEFINITION OF HOMOGENEOUS DIFFERENTIAL EQUATION

A **homogeneous differential equation** is an equation of the form

$$M(x, y) dx + N(x, y) dy = 0$$

where M and N are homogeneous functions of the same degree.

EXAMPLE 5 Testing for Homogeneous Differential Equations

- $(x^2 + xy) dx + y^2 dy = 0$ is homogeneous of degree 2.
- $x^3 dx = y^3 dy$ is homogeneous of degree 3.
- $(x^2 + 1) dx + y^2 dy = 0$ is *not* a homogeneous differential equation.

NOTE The notation $f(x, y)$ is used to denote a function of two variables in much the same way as $f(x)$ denotes a function of one variable. You will study functions of two variables in detail in Chapter 13.

To solve a homogeneous differential equation by the method of separation of variables, use the following change of variables theorem.

THEOREM 6.2 CHANGE OF VARIABLES FOR HOMOGENEOUS EQUATIONS

If $M(x, y) dx + N(x, y) dy = 0$ is homogeneous, then it can be transformed into a differential equation whose variables are separable by the substitution

$$y = vx$$

where v is a differentiable function of x .

EXAMPLE 6 Solving a Homogeneous Differential Equation

Find the general solution of

$$(x^2 - y^2) dx + 3xy dy = 0.$$

STUDY TIP The substitution $y = vx$ will yield a differential equation that is separable with respect to the variables x and v . You must write your final solution, however, in terms of x and y .

Solution Because $(x^2 - y^2)$ and $3xy$ are both homogeneous of degree 2, let $y = vx$ to obtain $dy = x dv + v dx$. Then, by substitution, you have

$$\begin{aligned} (x^2 - v^2x^2) dx + 3x(vx)(x dv + v dx) &= 0 \\ (x^2 + 2v^2x^2) dx + 3x^3v dv &= 0 \\ x^2(1 + 2v^2) dx + x^2(3vx) dv &= 0. \end{aligned}$$

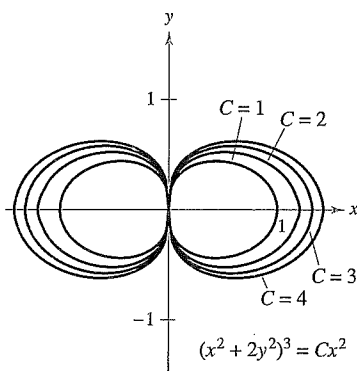
Dividing by x^2 and separating variables produces

$$\begin{aligned} (1 + 2v^2) dx &= -3vx dv \\ \int \frac{dx}{x} &= \int \frac{-3v}{1 + 2v^2} dv \\ \ln|x| &= -\frac{3}{4} \ln(1 + 2v^2) + C_1 \\ 4 \ln|x| &= -3 \ln(1 + 2v^2) + \ln|C| \\ \ln x^4 &= \ln|C(1 + 2v^2)^{-3}| \\ x^4 &= C(1 + 2v^2)^{-3}. \end{aligned}$$

Substituting for v produces the following general solution.

$$\begin{aligned} x^4 &= C \left[1 + 2 \left(\frac{y}{x} \right)^2 \right]^{-3} \\ \left(1 + \frac{2y^2}{x^2} \right)^3 x^4 &= C \\ (x^2 + 2y^2)^3 &= Cx^2 \end{aligned} \quad \text{General solution}$$

You can check this by differentiating and rewriting to get the original equation. ■



General solution of
 $(x^2 - y^2) dx + 3xy dy = 0$

Figure 6.13

TECHNOLOGY If you have access to a graphing utility, try using it to graph several solutions of the equation in Example 6. For instance, Figure 6.13 shows the graphs of

$$(x^2 + 2y^2)^3 = Cx^2$$

for $C = 1, 2, 3,$ and 4 .

Applications

EXAMPLE 7 Wildlife Population

The rate of change of the number of coyotes $N(t)$ in a population is directly proportional to $650 - N(t)$, where t is the time in years. When $t = 0$, the population is 300, and when $t = 2$, the population has increased to 500. Find the population when $t = 3$.

Solution Because the rate of change of the population is proportional to $650 - N(t)$, you can write the following differential equation.

$$\frac{dN}{dt} = k(650 - N)$$

You can solve this equation using separation of variables.

$$dN = k(650 - N) dt \quad \text{Differential form}$$

$$\frac{dN}{650 - N} = k dt \quad \text{Separate variables.}$$

$$-\ln|650 - N| = kt + C_1 \quad \text{Integrate.}$$

$$\ln|650 - N| = -kt - C_1$$

$$650 - N = e^{-kt - C_1} \quad \text{Assume } N < 650.$$

$$N = 650 - Ce^{-kt} \quad \text{General solution}$$

Using $N = 300$ when $t = 0$, you can conclude that $C = 350$, which produces

$$N = 650 - 350e^{-kt}.$$

Then, using $N = 500$ when $t = 2$, it follows that

$$500 = 650 - 350e^{-2k} \quad \Leftrightarrow \quad e^{-2k} = \frac{3}{7} \quad \Leftrightarrow \quad k \approx 0.4236.$$

So, the model for the coyote population is

$$N = 650 - 350e^{-0.4236t}. \quad \text{Model for population}$$

When $t = 3$, you can approximate the population to be

$$N = 650 - 350e^{-0.4236(3)} \approx 552 \text{ coyotes.}$$

The model for the population is shown in Figure 6.14. Note that $N = 650$ is the horizontal asymptote of the graph and is the *carrying capacity* of the model. You will learn more about carrying capacity later in this section.

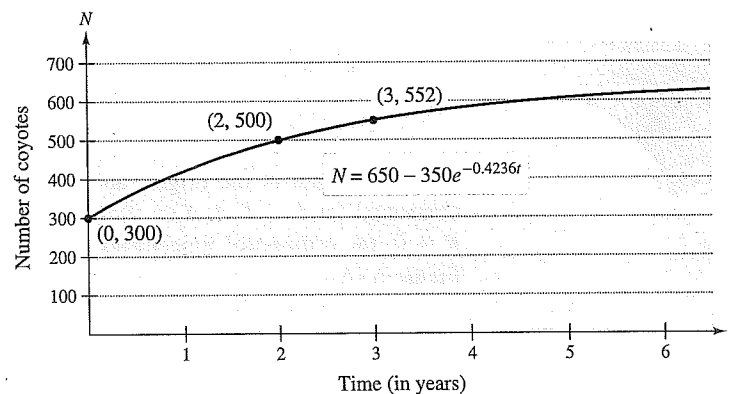
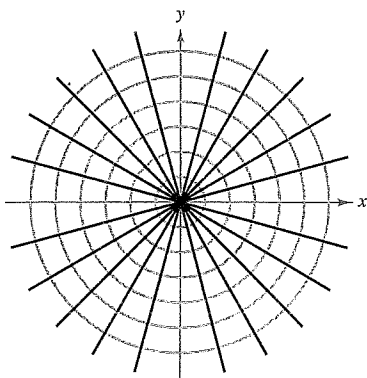


Figure 6.14





Each line $y = Kx$ is an orthogonal trajectory of the family of circles.

Figure 6.15

A common problem in electrostatics, thermodynamics, and hydrodynamics involves finding a family of curves, each of which is orthogonal to all members of a given family of curves. For example, Figure 6.15 shows a family of circles

$$x^2 + y^2 = C \quad \text{Family of circles}$$

each of which intersects the lines in the family

$$y = Kx \quad \text{Family of lines}$$

at right angles. Two such families of curves are said to be **mutually orthogonal**, and each curve in one of the families is called an **orthogonal trajectory** of the other family. In electrostatics, lines of force are orthogonal to the *equipotential curves*. In thermodynamics, the flow of heat across a plane surface is orthogonal to the *isothermal curves*. In hydrodynamics, the flow (stream) lines are orthogonal trajectories of the *velocity potential curves*.

EXAMPLE 8 Finding Orthogonal Trajectories

Describe the orthogonal trajectories for the family of curves given by

$$y = \frac{C}{x}$$

for $C \neq 0$. Sketch several members of each family.

Solution First, solve the given equation for C and write $xy = C$. Then, by differentiating implicitly with respect to x , you obtain the differential equation

$$xy' + y = 0 \quad \text{Differential equation}$$

$$x \frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x} \quad \text{Slope of given family}$$

Because y' represents the slope of the given family of curves at (x, y) , it follows that the orthogonal family has the negative reciprocal slope x/y . So,

$$\frac{dy}{dx} = \frac{x}{y} \quad \text{Slope of orthogonal family}$$

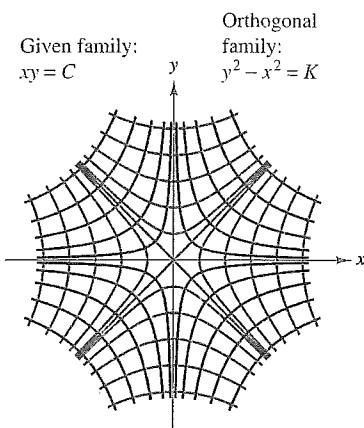
Now you can find the orthogonal family by separating variables and integrating.

$$\int y \, dy = \int x \, dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C_1$$

$$y^2 - x^2 = K$$

The centers are at the origin, and the transverse axes are vertical for $K > 0$ and horizontal for $K < 0$. If $K = 0$, the orthogonal trajectories are the lines $y = \pm x$. If $K \neq 0$, the orthogonal trajectories are hyperbolas. Several trajectories are shown in Figure 6.16. ■



Orthogonal trajectories
Figure 6.16

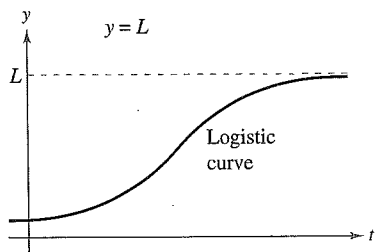
Logistic Differential Equation

In Section 6.2, the exponential growth model was derived from the fact that the rate of change of a variable y is proportional to the value of y . You observed that the differential equation $dy/dt = ky$ has the general solution $y = Ce^{kt}$. Exponential growth is unlimited, but when describing a population, there often exists some upper limit L past which growth cannot occur. This upper limit L is called the **carrying capacity**, which is the maximum population $y(t)$ that can be sustained or supported as time t increases. A model that is often used to describe this type of growth is the **logistic differential equation**

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \quad \text{Logistic differential equation}$$

where k and L are positive constants. A population that satisfies this equation does not grow without bound, but approaches the carrying capacity L as t increases.

From the equation, you can see that if y is between 0 and the carrying capacity L , then $dy/dt > 0$, and the population increases. If y is greater than L , then $dy/dt < 0$, and the population decreases. The graph of the function y is called the *logistic curve*, as shown in Figure 6.17.



Note that as $t \rightarrow \infty$, $y \rightarrow L$.

Figure 6.17

EXAMPLE 9 Deriving the General Solution

Solve the logistic differential equation $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$.

Solution Begin by separating variables.

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \quad \text{Write differential equation.}$$

$$\frac{1}{y(1 - y/L)} dy = k dt \quad \text{Separate variables.}$$

$$\int \frac{1}{y(1 - y/L)} dy = \int k dt \quad \text{Integrate each side.}$$

$$\int \left(\frac{1}{y} + \frac{1}{L - y}\right) dy = \int k dt \quad \text{Rewrite left side using partial fractions.}$$

$$\ln|y| - \ln|L - y| = kt + C \quad \text{Find antiderivative of each side.}$$

$$\ln\left|\frac{L - y}{y}\right| = -kt - C \quad \text{Multiply each side by } -1 \text{ and simplify.}$$

$$\left|\frac{L - y}{y}\right| = e^{-kt - C} = e^{-C}e^{-kt} \quad \text{Exponentiate each side.}$$

$$\frac{L - y}{y} = be^{-kt} \quad \text{Let } \pm e^{-C} = b.$$

Solving this equation for y produces $y = \frac{L}{1 + be^{-kt}}$.

From Example 9, you can conclude that all solutions of the logistic differential equation are of the general form

$$y = \frac{L}{1 + be^{-kt}}$$

EXPLORATION

Use a graphing utility to investigate the effects of the values of L , b , and k on the graph of

$$y = \frac{L}{1 + be^{-kt}}$$

Include some examples to support your results.

EXAMPLE 10 Solving a Logistic Differential Equation

A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the environment can support no more than 4000 elk. The growth rate of the elk population p is

$$\frac{dp}{dt} = kp \left(1 - \frac{p}{4000} \right), \quad 40 \leq p \leq 4000$$

where t is the number of years.

- Write a model for the elk population in terms of t .
- Graph the slope field for the differential equation and the solution that passes through the point $(0, 40)$.
- Use the model to estimate the elk population after 15 years.
- Find the limit of the model as $t \rightarrow \infty$.

Solution

- a. You know that $L = 4000$. So, the solution of the equation is of the form

$$p = \frac{4000}{1 + be^{-kt}}$$

Because $p(0) = 40$, you can solve for b as follows.

$$40 = \frac{4000}{1 + be^{-k(0)}}$$

$$40 = \frac{4000}{1 + b} \quad \Rightarrow \quad b = 99$$

Then, because $p = 104$ when $t = 5$, you can solve for k .

$$104 = \frac{4000}{1 + 99e^{-k(5)}} \quad \Rightarrow \quad k \approx 0.194$$

So, a model for the elk population is given by $p = \frac{4000}{1 + 99e^{-0.194t}}$.

- b. Using a graphing utility, you can graph the slope field for

$$\frac{dp}{dt} = 0.194p \left(1 - \frac{p}{4000} \right)$$

and the solution that passes through $(0, 40)$, as shown in Figure 6.18.

- c. To estimate the elk population after 15 years, substitute 15 for t in the model.

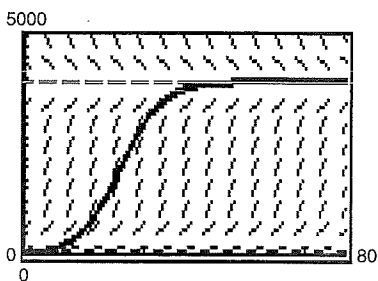
$$\begin{aligned} p &= \frac{4000}{1 + 99e^{-0.194(15)}} && \text{Substitute 15 for } t. \\ &= \frac{4000}{1 + 99e^{-2.91}} \approx 626 && \text{Simplify.} \end{aligned}$$

- d. As t increases without bound, the denominator of $\frac{4000}{1 + 99e^{-0.194t}}$ gets closer and closer to 1.

$$\text{So, } \lim_{t \rightarrow \infty} \frac{4000}{1 + 99e^{-0.194t}} = 4000. \quad \blacksquare$$

EXPLORATION

Explain what happens if $p(0) = L$.



Slope field for

$$\frac{dp}{dt} = 0.194p \left(1 - \frac{p}{4000} \right)$$

and the solution passing through $(0, 40)$

Figure 6.18

6.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–14, find the general solution of the differential equation.

1. $\frac{dy}{dx} = \frac{x}{y}$

3. $x^2 + 5y \frac{dy}{dx} = 0$

5. $\frac{dr}{ds} = 0.75r$

7. $(2 + x)y' = 3y$

9. $yy' = 4 \sin x$

11. $\sqrt{1 - 4x^2}y' = x$

13. $y \ln x - xy' = 0$

2. $\frac{dy}{dx} = \frac{3x^2}{y^2}$

4. $\frac{dy}{dx} = \frac{x^2 - 3}{6y^2}$

6. $\frac{dr}{ds} = 0.75s$

8. $xy' = y$

10. $yy' = -8 \cos(\pi x)$

12. $\sqrt{x^2 - 16}y' = 11x$

14. $12yy' - 7e^x = 0$

35. $f(x, y) = 2 \ln xy$

37. $f(x, y) = 2 \ln \frac{x}{y}$

36. $f(x, y) = \tan(x + y)$

38. $f(x, y) = \tan \frac{y}{x}$

In Exercises 39–44, solve the homogeneous differential equation.

39. $y' = \frac{x + y}{2x}$

41. $y' = \frac{x - y}{x + y}$

43. $y' = \frac{xy}{x^2 - y^2}$

40. $y' = \frac{x^3 + y^3}{xy^2}$

42. $y' = \frac{x^2 + y^2}{2xy}$

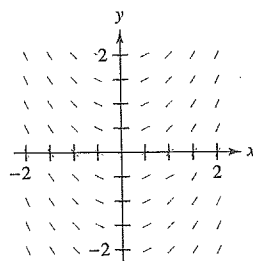
44. $y' = \frac{2x + 3y}{x}$

In Exercises 45–48, find the particular solution that satisfies the initial condition.

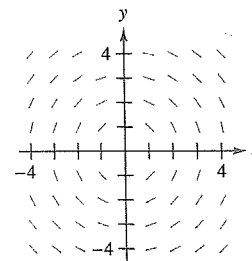
Differential Equation	Initial Condition
45. $x dy - (2xe^{-y/x} + y) dx = 0$	$y(1) = 0$
46. $-y^2 dx + x(x + y) dy = 0$	$y(1) = 1$
47. $(x \sec \frac{y}{x} + y) dx - x dy = 0$	$y(1) = 0$
48. $(2x^2 + y^2) dx + xy dy = 0$	$y(1) = 0$

Slope Fields In Exercises 49–52, sketch a few solutions of the differential equation on the slope field and then find the general solution analytically. To print an enlarged copy of the graph, go to the website www.mathgraphs.com.

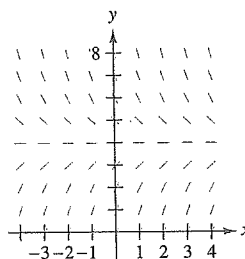
49. $\frac{dy}{dx} = x$



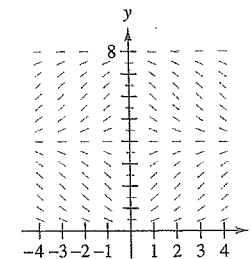
50. $\frac{dy}{dx} = -\frac{x}{y}$



51. $\frac{dy}{dx} = 4 - y$



52. $\frac{dy}{dx} = 0.25x(4 - y)$



In Exercises 15–24, find the particular solution that satisfies the initial condition.

Differential Equation	Initial Condition
15. $yy' - 2e^x = 0$	$y(0) = 3$
16. $\sqrt{x} + \sqrt{y}y' = 0$	$y(1) = 9$
17. $y(x + 1) + y' = 0$	$y(-2) = 1$
18. $2xy' - \ln x^2 = 0$	$y(1) = 2$
19. $y(1 + x^2)y' - x(1 + y^2) = 0$	$y(0) = \sqrt{3}$
20. $y\sqrt{1 - x^2}y' - x\sqrt{1 - y^2} = 0$	$y(0) = 1$
21. $\frac{du}{dv} = uv \sin v^2$	$u(0) = 1$
22. $\frac{dr}{ds} = e^{r-2s}$	$r(0) = 0$
23. $dP - kP dt = 0$	$P(0) = P_0$
24. $dT + k(T - 70) dt = 0$	$T(0) = 140$

In Exercises 25–28, find an equation of the graph that passes through the point and has the given slope.

25. $(0, 2), y' = \frac{x}{4y}$

26. $(1, 1), y' = -\frac{9x}{16y}$

27. $(9, 1), y' = \frac{y}{2x}$

28. $(8, 2), y' = \frac{2y}{3x}$

In Exercises 29 and 30, find all functions f having the indicated property.

29. The tangent to the graph of f at the point (x, y) intersects the x -axis at $(x + 2, 0)$.

30. All tangents to the graph of f pass through the origin.

In Exercises 31–38, determine whether the function is homogeneous, and if it is, determine its degree.

31. $f(x, y) = x^3 - 4xy^2 + y^3$

32. $f(x, y) = x^3 + 3x^2y^2 - 2y^2$

33. $f(x, y) = \frac{x^2y^2}{\sqrt{x^2 + y^2}}$

34. $f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}$

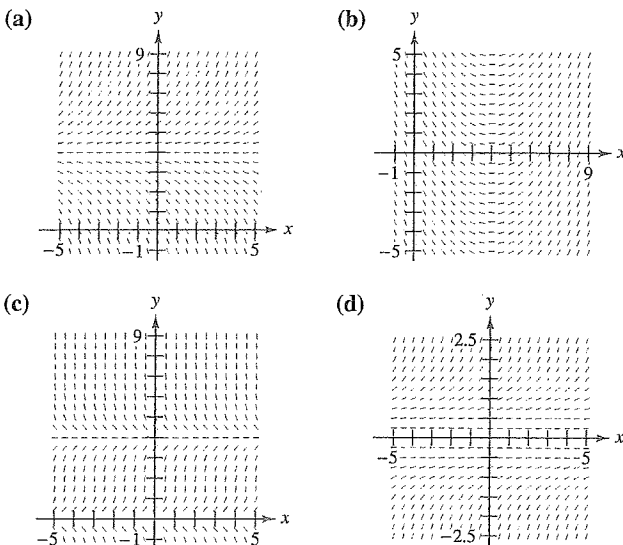
Euler's Method In Exercises 53–56, (a) use Euler's Method with a step size of $h = 0.1$ to approximate the particular solution of the initial value problem at the given x -value, (b) find the exact solution of the differential equation analytically, and (c) compare the solutions at the given x -value.

Differential Equation	Initial Condition	x -value
53. $\frac{dy}{dx} = -6xy$	(0, 5)	$x = 1$
54. $\frac{dy}{dx} + 6xy^2 = 0$	(0, 3)	$x = 1$
55. $\frac{dy}{dx} = \frac{2x + 12}{3y^2 - 4}$	(1, 2)	$x = 2$
56. $\frac{dy}{dx} = 2x(1 + y^2)$	(1, 0)	$x = 1.5$

57. **Radioactive Decay** The rate of decomposition of radioactive radium is proportional to the amount present at any time. The half-life of radioactive radium is 1599 years. What percent of a present amount will remain after 50 years?

58. **Chemical Reaction** In a chemical reaction, a certain compound changes into another compound at a rate proportional to the unchanged amount. If initially there is 40 grams of the original compound, and there is 35 grams after 1 hour, when will 75 percent of the compound be changed?

Slope Fields In Exercises 59–62, (a) write a differential equation for the statement, (b) match the differential equation with a possible slope field, and (c) verify your result by using a graphing utility to graph a slope field for the differential equation. [The slope fields are labeled (a), (b), (c), and (d).] To print an enlarged copy of the graph, go to the website www.mathgraphs.com.



59. The rate of change of y with respect to x is proportional to the difference between y and 4.

60. The rate of change of y with respect to x is proportional to the difference between x and 4.

61. The rate of change of y with respect to x is proportional to the product of y and the difference between y and 4.

62. The rate of change of y with respect to x is proportional to y^2 .

CAS 63. **Weight Gain** A calf that weighs 60 pounds at birth gains weight at the rate $dw/dt = k(1200 - w)$, where w is weight in pounds and t is time in years. Solve the differential equation.

(a) Use a computer algebra system to solve the differential equation for $k = 0.8, 0.9,$ and 1 . Graph the three solutions.

(b) If the animal is sold when its weight reaches 800 pounds, find the time of sale for each of the models in part (a).

(c) What is the maximum weight of the animal for each of the models?

64. **Weight Gain** A calf that weighs w_0 pounds at birth gains weight at the rate $dw/dt = 1200 - w$, where w is weight in pounds and t is time in years. Solve the differential equation.

Graphing Utility In Exercises 65–70, find the orthogonal trajectories of the family. Use a graphing utility to graph several members of each family.

65. $x^2 + y^2 = C$

66. $x^2 - 2y^2 = C$

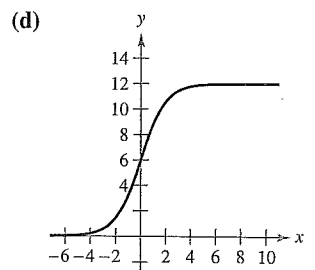
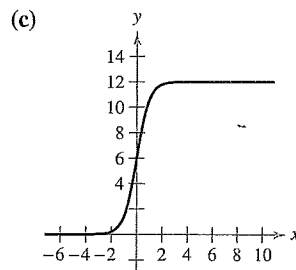
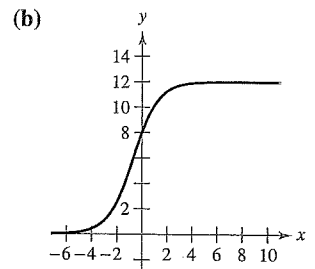
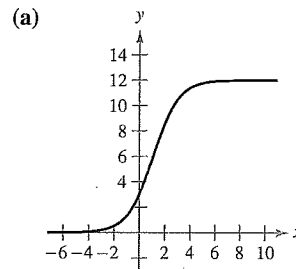
67. $x^2 = Cy$

68. $y^2 = 2Cx$

69. $y^2 = Cx^3$

70. $y = Ce^x$

In Exercises 71–74, match the logistic equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]



71. $y = \frac{12}{1 + e^{-x}}$

72. $y = \frac{12}{1 + 3e^{-x}}$

73. $y = \frac{12}{1 + \frac{1}{2}e^{-x}}$

74. $y = \frac{12}{1 + e^{-2x}}$

In Exercises 75 and 76, the logistic equation models the growth of a population. Use the equation to (a) find the value of k , (b) find the carrying capacity, (c) find the initial population, (d) determine when the population will reach 50% of its carrying capacity, and (e) write a logistic differential equation that has the solution $P(t)$.

75.
$$P(t) = \frac{2100}{1 + 29e^{-0.75t}}$$

76.
$$P(t) = \frac{5000}{1 + 39e^{-0.2t}}$$

CAS In Exercises 77 and 78, the logistic differential equation models the growth rate of a population. Use the equation to (a) find the value of k , (b) find the carrying capacity, (c) graph a slope field using a computer algebra system, and (d) determine the value of P at which the population growth rate is the greatest.

77.
$$\frac{dP}{dt} = 3P \left(1 - \frac{P}{100} \right)$$

78.
$$\frac{dP}{dt} = 0.1P - 0.0004P^2$$

In Exercises 79–82, find the logistic equation that satisfies the initial condition.

<i>Logistic Differential Equation</i>	<i>Initial Condition</i>
79. $\frac{dy}{dt} = y \left(1 - \frac{y}{36} \right)$	(0, 4)
80. $\frac{dy}{dt} = 2.8y \left(1 - \frac{y}{10} \right)$	(0, 7)
81. $\frac{dy}{dt} = \frac{4y}{5} - \frac{y^2}{150}$	(0, 8)
82. $\frac{dy}{dt} = \frac{3y}{20} - \frac{y^2}{1600}$	(0, 15)

83. **Endangered Species** A conservation organization releases 25 Florida panthers into a game preserve. After 2 years, there are 39 panthers in the preserve. The Florida preserve has a carrying capacity of 200 panthers.

- Write a logistic equation that models the population of panthers in the preserve.
- Find the population after 5 years.
- When will the population reach 100?
- Write a logistic differential equation that models the growth rate of the panther population. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answers.
- At what time is the panther population growing most rapidly? Explain.

84. **Bacteria Growth** At time $t = 0$, a bacterial culture weighs 1 gram. Two hours later, the culture weighs 4 grams. The maximum weight of the culture is 20 grams.

- Write a logistic equation that models the weight of the bacterial culture.
- Find the culture's weight after 5 hours.
- When will the culture's weight reach 18 grams?

(d) Write a logistic differential equation that models the growth rate of the culture's weight. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answers.

(e) At what time is the culture's weight increasing most rapidly? Explain.

WRITING ABOUT CONCEPTS

- In your own words, describe how to recognize and solve differential equations that can be solved by separation of variables.
- State the test for determining if a differential equation is homogeneous. Give an example.
- In your own words, describe the relationship between two families of curves that are mutually orthogonal.

CAPSTONE

88. Suppose the growth of a population is modeled by a logistic equation. As the population increases, its rate of growth decreases. What do you think causes this to occur in real-life situations such as animal or human populations?

89. Show that if $y = \frac{1}{1 + be^{-kt}}$, then $\frac{dy}{dt} = ky(1 - y)$.

90. **Sailing** Ignoring resistance, a sailboat starting from rest accelerates (dv/dt) at a rate proportional to the difference between the velocities of the wind and the boat.

- The wind is blowing at 20 knots, and after 1 half-hour the boat is moving at 10 knots. Write the velocity v as a function of time t .
- Use the result of part (a) to write the distance traveled by the boat as a function of time.

True or False? In Exercises 91–94, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

- The function $y = 0$ is always a solution of a differential equation that can be solved by separation of variables.
- The differential equation $y' = xy - 2y + x - 2$ can be written in separated variables form.
- The function $f(x, y) = x^2 - 4xy + 6y^2 + 1$ is homogeneous.
- The families $x^2 + y^2 = 2Cy$ and $x^2 + y^2 = 2Kx$ are mutually orthogonal.

PUTNAM EXAM CHALLENGE

95. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b) .

This problem was composed by the Committee on the Putnam Prize Competition.
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