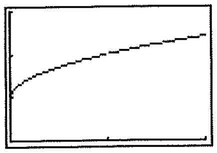
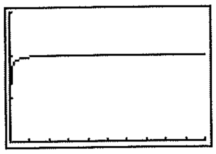
5.



[0, 2] by [0, 3]

As $t \to 1$, $\dfrac{t-1}{\sqrt{t}-1}$ approaches 2.

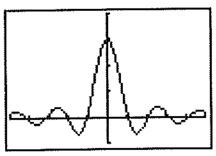
6.



[0, 500] by [0, 3]

As $x \to \infty$, $\dfrac{\sqrt{4x^2+1}}{x+1}$ approaches 2.

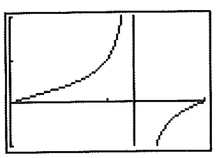
7.



[−5, 5] by [−1, 4]

As $x \to 0$, $\dfrac{\sin 3x}{x}$ approaches 3.

8.



[0, π] by [−1, 2]

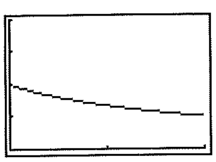
As $\theta \to \dfrac{\pi}{2}$, $\dfrac{\tan \theta}{2 + \tan \theta}$ approaches 1.

9. $y = \dfrac{1}{h} \sin h$

10. $y = (1 + h)^{1/h}$

Section 8.1 Exercises

1.

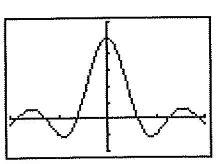


[0, 2] by [0, 1]

From the graph, the limit appears to be $\frac{1}{4}$.

$$\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{1}{2x} = \frac{1}{4}$$

2.

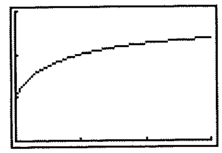


[−2, 2] by [−2, 6]

From the graph, the limit appears to be 5.

$$\lim_{x\to 0} \frac{\sin 5x}{x} = \lim_{x\to 0} \frac{5 \cos 5x}{1} = 5$$

3.



[0, 3] by [0, 3]

From the graph, the limit appears to be 1. The limit leads to the indeterminate form ∞^0.

$$\ln\left(1 + \frac{1}{x}\right)^x = x \ln\left(1 + \frac{1}{x}\right) = \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}}$$

$$\lim_{x\to 0^+} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{1 + 1/x}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

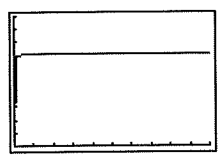
$$= \lim_{x\to 0^+} \frac{1}{1 + \frac{1}{x}}$$

$$= \lim_{x\to 0^+} \frac{x}{x+1} = 0$$

Therefore,

$$\lim_{x\to 0^+} \left(1 + \frac{1}{x}\right)^x = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^0 = 1.$$

4.

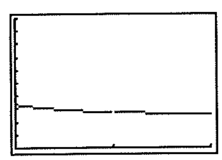


[0, 1000] by [0, 1]

From the graph, the limit appears to be about 0.714.

$$\lim_{x\to\infty} \frac{5x^2 - 3x}{7x^2 + 1} = \lim_{x\to\infty} \frac{10x - 3}{14x} = \lim_{x\to\infty} \frac{10}{14} = \frac{5}{7} \approx 0.71429$$

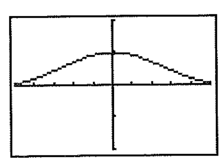
5. $\displaystyle\lim_{x\to 1} \frac{x^3 - 1}{4x^3 - x - 3} = \lim_{x\to 1} \frac{3x^2}{12x^2 - 1} = \frac{3}{11}$



[0, 2] by [0, 1]

The graph supports the answer.

6. $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$



[−5, 5] by [−1, 1]

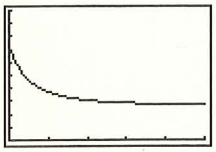
The graph supports the answer.

7. The limit leads to the indeterminate form 1^∞.

Let $\ln f(x) = \ln(e^x + x)^{1/x} = \dfrac{\ln(e^x + x)}{x}$.

$$\lim_{x\to 0^+} \frac{\ln(e^x + x)}{x} = \lim_{x\to 0^+} \frac{\frac{e^x + 1}{e^x + x}}{1} = \lim_{x\to 0^+} \frac{e^x + 1}{e^x + x} = \frac{2}{1} = 2$$

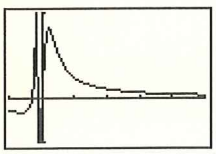
$$\lim_{x\to 0^+} (e^x + x)^{1/x} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^2$$



[0, 5] by [0, 10]

The graph supports the answer.

8. $\displaystyle\lim_{x\to\infty} \frac{2x^2 + 3x}{x^3 + x + 1} = \lim_{x\to\infty} \frac{4x + 3}{3x^2 + 1} = \lim_{x\to\infty} \frac{4}{6x} = 0$



[−5, 25] by [−1, 2]

The graph supports the answer.

9. (a)

x	10	10^2	10^3	10^4	10^5
$f(x)$	1.1513	0.2303	0.0345	0.00461	0.00058

Estimate the limit to be 0.

(b) $\displaystyle\lim_{x\to\infty} \frac{\ln x^5}{x} = \lim_{x\to\infty} \frac{5 \ln x}{x} = \lim_{x\to\infty} \frac{5/x}{1} = \frac{0}{1} = 0$

10. (a)

x	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$f(x)$	0.1585	0.1666	0.1667	0.1667	0.1667

Estimate the limit to be $\dfrac{1}{6}$.

(b)
$$\begin{aligned}\lim_{x\to 0^+} \frac{x - \sin x}{x^3} &= \lim_{x\to 0^+} \frac{1 - \cos x}{3x^2} \\ &= \lim_{x\to 0^+} \frac{\sin x}{6x} \\ &= \lim_{x\to 0^+} \frac{\cos x}{6} \\ &= \frac{1}{6}\end{aligned}$$

11. Let $f(\theta) = \dfrac{\sin 3\theta}{\sin 4\theta}$.

θ	$\pm 10^0$	$\pm 10^{-1}$	$\pm 10^{-2}$	$\pm 10^{-3}$	$\pm 10^{-4}$
$f(\theta)$	−0.1865	0.7589	0.7501	0.7500	0.7500

Estimate the limit to be $\dfrac{3}{4}$.

$$\lim_{\theta\to 0} \frac{\sin 3\theta}{\sin 4\theta} = \lim_{\theta\to 0} \frac{3 \cos 3\theta}{4 \cos 4\theta} = \frac{3}{4}$$

12. Let $f(t) = \dfrac{1}{\sin t} - \dfrac{1}{t} = \dfrac{t - \sin t}{t \sin t}$.

t	$\pm 10^0$	$\pm 10^{-1}$	$\pm 10^{-2}$	$\pm 10^{-3}$
$f(t)$	±0.1884	±0.0167	±0.0017	±0.00017

Estimate the limit to be 0.

$$\begin{aligned}\lim_{t\to 0} \left(\frac{1}{\sin t} - \frac{1}{t}\right) &= \lim_{t\to 0} \frac{t - \sin t}{t \sin t} \\ &= \lim_{t\to 0} \frac{1 - \cos t}{t \cos t + \sin t} \\ &= \lim_{t\to 0} \frac{\sin t}{-t \sin t + \cos t + \cos t} = 0\end{aligned}$$

13. Let $f(x) = (1 + x)^{1/x}$.

x	10	10^2	10^3	10^4	10^5
$f(x)$	1.2710	1.0472	1.0069	1.0009	1.0001

Estimate the limit to be 1.

$$\ln f(x) = \frac{\ln (1 + x)}{x}$$

$$\lim_{x\to\infty} \frac{\ln (1 + x)}{x} = \lim_{x\to\infty} \frac{\frac{1}{1 + x}}{1} = \frac{0}{1} = 0$$

$$\lim_{x\to\infty} (1 + x)^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1$$

14. Let $f(x) = \dfrac{x - 2x^2}{3x^2 + 5x}$.

x	10	10^2	10^3	10^4	10^5
$f(x)$	−0.5429	−0.6525	−0.6652	−0.6665	−0.6667

Estimate the limit to be $-\dfrac{2}{3}$.

$$\lim_{x\to\infty} \frac{x - 2x^2}{3x^2 + 5x} = \lim_{x\to\infty} \frac{1 - 4x}{6x + 5} = \lim_{x\to\infty} -\frac{4}{6} = -\frac{2}{3}$$

15. $\displaystyle\lim_{\theta\to 0} \frac{\sin \theta^2}{\theta} = \lim_{\theta\to 0} \frac{2\theta \cos \theta^2}{1} = (2)(0) \cos (0)^2 = 0$

16.
$$\begin{aligned}\lim_{\theta\to\pi/2} \frac{1 - \sin \theta}{1 + \cos 2\theta} &= \lim_{\theta\to\pi/2} \frac{-\cos \theta}{-2 \sin 2\theta} \\ &= \lim_{\theta\to\pi/2} \frac{\sin \theta}{-4 \cos 2\theta} \\ &= \frac{\sin \pi/2}{-4 \cos \pi} \\ &= \frac{1}{4}\end{aligned}$$

17. $\displaystyle\lim_{t\to 0} \frac{\cos t - 1}{e^t - t - 1} = \lim_{t\to 0} \frac{-\sin t}{e^t - 1} = \lim_{t\to 0} \frac{-\cos t}{e^t} = -1$

18. $$\lim_{t\to 1}\frac{t-1}{\ln t-\sin \pi t}=\lim_{t\to 1}\frac{1}{\frac{1}{t}-\pi\cos \pi t}$$

$$=\frac{1}{1-\pi(-1)}$$

$$=\frac{1}{\pi+1}$$

19. $$\lim_{x\to\infty}\frac{\ln(x+1)}{\log_2 x}=\lim_{x\to\infty}\frac{\frac{1}{x+1}}{\frac{1}{x\ln 2}}$$

$$=\lim_{x\to\infty}\frac{x\ln 2}{x+1}$$

$$=\lim_{x\to\infty}\ln 2$$

$$=\ln 2$$

20. $$\lim_{x\to\infty}\frac{\log_2 x}{\log_3 (x+3)}=\lim_{x\to\infty}\frac{\frac{1}{x\ln 2}}{\frac{1}{(x+3)\ln 3}}$$

$$=\lim_{x\to\infty}\frac{(x+3)\ln 3}{x\ln 2}$$

$$=\lim_{x\to\infty}\frac{x\ln 3+3\ln 3}{x\ln 2}$$

$$=\lim_{x\to\infty}\frac{\ln 3}{\ln 2}$$

$$=\frac{\ln 3}{\ln 2}$$

21. $$\lim_{y\to 0^+}\frac{\ln(y^2+2y)}{\ln y}=\lim_{y\to 0^+}\frac{\frac{2y+2}{y^2+2y}}{\frac{1}{y}}$$

$$=\lim_{y\to 0^+}\frac{y(2y+2)}{y^2+2y}$$

$$=\lim_{y\to 0^+}\frac{2y^2+2y}{y^2+2y}$$

$$=\lim_{y\to 0^+}\frac{4y+2}{2y+2}$$

$$=\frac{4(0)+2}{2(0)+2}=\frac{2}{2}=1$$

22. $$\lim_{y\to\pi/2}\left(\frac{\pi}{2}-y\right)\tan y=\lim_{y\to\pi/2}\frac{\left(\frac{\pi}{2}-y\right)\sin y}{\cos y}$$

$$=\lim_{y\to\pi/2}\frac{\left(\frac{\pi}{2}-y\right)\cos y+(-1)\sin y}{-\sin y}$$

$$=\frac{\left(\frac{\pi}{2}-\frac{\pi}{2}\right)\cos\frac{\pi}{2}+(-1)\sin\frac{\pi}{2}}{-\sin\frac{\pi}{2}}$$

$$=\frac{(-1)(1)}{-(1)}=1$$

23. $$\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}$$

$$=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$=\lim_{x\to 0^+}\frac{-x^2}{x}$$

$$=\lim_{x\to 0^+}-x=0$$

24. $$\lim_{x\to\infty}x\tan\frac{1}{x}=\lim_{x\to\infty}\frac{\tan\frac{1}{x}}{\frac{1}{x}}$$

$$=\lim_{x\to\infty}\frac{-\frac{1}{x^2}\sec^2\frac{1}{x}}{-\frac{1}{x^2}}$$

$$=\lim_{x\to\infty}\sec^2\frac{1}{x}$$

$$=\sec^2 0=1$$

25. $$\lim_{x\to 0^+}(\csc x-\cot x+\cos x)$$

$$=\lim_{x\to 0^+}\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}+\cos x\right)$$

$$=\lim_{x\to 0^+}\frac{1-\cos x+\cos x\sin x}{\sin x}$$

$$=\lim_{x\to 0^+}\frac{\sin x+\cos x\cos x-\sin x\sin x}{\cos x}=1$$

26. $\lim_{x\to\infty} (\ln 2x - \ln (x+1)) = \lim_{x\to\infty} \ln \left(\frac{2x}{x+1}\right)$

Let $f(x) = \frac{2x}{x+1}$.

$$\lim_{x\to\infty} \frac{2x}{x+1} = \lim_{x\to\infty} \frac{2}{1} = 2$$

Therefore,

$$\lim_{x\to\infty} (\ln 2x - \ln (x+1)) = \lim_{x\to\infty} \ln f(x) = \ln 2$$

27. $\lim_{x\to 0^+} (\ln x - \ln \sin x) = \lim_{x\to 0^+} \ln \frac{x}{\sin x}$

Let $f(x) = \frac{x}{\sin x}$.

$$\lim_{x\to 0^+} \frac{x}{\sin x} = \lim_{x\to 0^+} \frac{1}{\cos x} = 1$$

Therefore,

$$\lim_{x\to 0^+} (\ln x - \ln \sin x) = \lim_{x\to 0^+} \ln f(x) = \ln 1 = 0$$

28. $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sqrt{x}}\right) = \lim_{x\to 0^+} \frac{1-\sqrt{x}}{x} = \infty$

29. The limit leads to the indeterminate form 1^∞.

Let $f(x) = (e^x + x)^{1/x}$.

$$\ln (e^x + x)^{1/x} = \frac{\ln (e^x + x)}{x}$$

$$\lim_{x\to 0} \frac{\ln (e^x + x)}{x} = \lim_{x\to 0} \frac{\frac{e^x + 1}{e^x + x}}{1} = 2$$

$$\lim_{x\to 0} (e^x + x)^{1/x} = \lim_{x\to 0} e^{\ln f(x)} = e^2$$

30. The limit leads to the indeterminate form ∞^0.

Let $f(x) = \left(\frac{1}{x^2}\right)^x$.

$$\ln \left(\frac{1}{x^2}\right)^x = x \ln \left(\frac{1}{x^2}\right) = \frac{\ln \left(\frac{1}{x^2}\right)}{\frac{1}{x}}$$

$$\lim_{x\to 0} \frac{\ln \left(\frac{1}{x^2}\right)}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{-2/x^3}{1/x^2}}{-1/x^2} = \lim_{x\to 0} 2x = 0$$

$$\lim_{x\to 0} \left(\frac{1}{x^2}\right)^x = \lim_{x\to 0} e^{\ln f(x)} = e^0 = 1$$

31. $\lim_{x\to\pm\infty} \frac{3x-5}{2x^2 - x + 2} = \lim_{x\to\pm\infty} \frac{3}{4x-1} = 0$

32. $\lim_{x\to 0} \frac{\sin 7x}{\tan 11x} = \lim_{x\to 0} \frac{7 \cos 7x}{11 \sec^2 11x} = \frac{7}{11}$

33. The limit leads to the indeterminate form ∞^0.

Let $f(x) = (\ln x)^{1/x}$.

$$\ln (\ln x)^{1/x} = \frac{\ln (\ln x)}{x}$$

$$\lim_{x\to\infty} \frac{\ln (\ln x)}{x} = \lim_{x\to\infty} \frac{\frac{1/x}{\ln x}}{1} = \lim_{x\to\infty} \frac{1}{x \ln x} = 0$$

$$\lim_{x\to\infty} (\ln x)^{1/x} = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1$$

34. The limit leads to the indeterminate form ∞^0.

Let $f(x) = (1 + 2x)^{1/(2 \ln x)}$.

$$\ln (1+2x)^{1/(2 \ln x)} = \frac{\ln (1+2x)}{2 \ln x}$$

$$\lim_{x\to\infty} \frac{\ln (1+2x)}{2 \ln x} = \lim_{x\to\infty} \frac{\frac{2}{1+2x}}{\frac{2}{x}} = \lim_{x\to\infty} \frac{x}{1+2x} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}$$

$$\lim_{x\to\infty} (1+2x)^{1/(2 \ln x)} = \lim_{x\to\infty} e^{\ln f(x)} = e^{1/2} = \sqrt{e}$$

35. The limit leads to the indeterminate form 0^0.

Let $f(x) = (x^2 - 2x + 1)^{x-1}$

$$\ln (x^2 - 2x + 1)^{x-1} = (x-1) \ln (x^2 - 2x + 1)$$

$$= \frac{\ln (x^2 - 2x + 1)}{\frac{1}{x-1}}$$

$$\lim_{x\to 1} \frac{\ln (x^2 - 2x + 1)}{\frac{1}{x-1}} = \lim_{x\to 1} \frac{\frac{2x-2}{x^2 - 2x + 1}}{-\frac{1}{(x-1)^2}}$$

$$= \lim_{x\to 1} \frac{\frac{2(x-1)}{(x-1)^2}}{-\frac{1}{(x-1)^2}}$$

$$= \lim_{x\to 1} -2(x-1) = 0$$

$$\lim_{x\to 1} (x^2 - 2x + 1)^{x-1} = \lim_{x\to 1} e^{\ln f(x)} = e^0 = 1$$

36. The limit leads to the indeterminate form 0^0.

Let $f(x) = (\cos x)^{\cos x}$.

$$\ln (\cos x)^{\cos x} = (\cos x) \ln (\cos x) = \frac{\ln (\cos x)}{\sec x}$$

$$\lim_{x\to \pi/2^-} \frac{\ln (\cos x)}{\sec x} = \lim_{x\to \pi/2^-} \frac{\frac{-\sin x}{\cos x}}{\sec x \tan x}$$

$$= \lim_{x\to \pi/2^-} \frac{-\tan x}{\sec x \tan x}$$

$$= \lim_{x\to \pi/2^-} -\cos x = 0$$

$$\lim_{x\to \pi/2^-} (\cos x)^{\cos x} = \lim_{x\to \pi/2^-} e^{\ln f(x)} = e^0 = 1$$