8. Change $|2x+4| < 1$ to $-1 < 2x+4 < 1$ to $-5 < 2x < -3$ to $-\frac{5}{2} < x < -\frac{3}{2}$. Answer: d)

9. $|\frac{x-1}{5}| \leqq 1$ leads to $|x-1| \leqq 5$, $-5 \leqq x-1 \leqq 5$, $-4 \leqq x \leqq 6$. Answer: i)

10. Change $|\frac{2x+1}{3}| < 1$ to $|2x+1| < 3$ to $-3 < 2x+1 < 3$ to $-4 < 2x < 2$ to $-2 < x < 1$. Answer: a)

11. $|y-2| \leqq 5$. $-5 \leqq y-2 \leqq 5$, $-3 \leqq y \leqq 7$.

12. $|y+3| < 1$. $-1 < y+3 < 1$, $-4 < y < -2$.

13. $|2y-5| < 1$. $-1 < 2y-5 < 1$, $4 < 2y < 6$, $2 < y < 3$.

14. Answer: $-3 < y < -2$

15. Change $|\frac{y}{2}-1| \leqq 1$ to $-1 \leqq \frac{y}{2}-1 \leqq 1$ to $0 \leqq \frac{y}{2} \leqq 2$ to $0 \leqq y \leqq 4$. Answer: $0 \leqq y \leqq 4$

16. Answer: $3 < y < 5$

17. $|2-y| < \frac{1}{5}$. $-\frac{1}{5} < 2-y < \frac{1}{5}$, $-\frac{11}{5} < -y < -\frac{9}{5}$, $\frac{9}{5} < y < \frac{11}{5}$.

18. $|\frac{5-2y}{3}| < 1$. $|5-2y| < 3$, $-3 < 5-2y < 3$, $-8 < -2y < -2$, $1 < y < 4$.

19. The midpoint of the interval is $\frac{1+8}{2} = \frac{9}{2}$ and it has radius $\frac{9}{2} - 1 = \frac{7}{2}$. Answer: $|x - \frac{9}{2}| < \frac{7}{2}$

20. The midpoint of the interval is $\frac{-2+7}{2} = \frac{5}{2}$ and it has radius $7 - \frac{5}{2} = \frac{9}{2}$. Answer: $|x - \frac{5}{2}| < \frac{9}{2}$

21. The midpoint of the interval is $\frac{-4+1}{2} = -\frac{3}{2}$ and it has radius $1 - (-\frac{3}{2}) = \frac{5}{2}$. Answer: $|x + \frac{3}{2}| < \frac{5}{2}$

22. The midpoint of the interval is $\frac{-8-1}{2} = -\frac{9}{2}$ and it has radius $-1 - (-\frac{9}{2}) = \frac{7}{2}$. Answer: $|x + \frac{9}{2}| < \frac{7}{2}$

23. $0.5 < x^2 < 1.5$ yields $\sqrt{0.5} < |x| < \sqrt{1.5}$ or after appropriate rounding $0.71 < |x| < 1.22$. Thus we obtain $-1.22 < x < -0.71$

24. If $0 < x < \frac{\pi}{2}$, $0.3 < \sin x < 0.7$ yields $\sin^{-1} 0.3 < x < \sin^{-1} 0.7$. In the second quadrant we need $\pi - \sin^{-1} 0.7 < x < \pi - \sin^{-1} 0.3$ or, rounding appropriately, $2.37 < x < 2.83$.

25. For x in the interval $[0, \frac{\pi}{2}]$, $\cos x$ is decreasing and so $0.2 < \cos x < 0.6$ yields $\cos^{-1} 0.6 < x < \cos^{-1} 0.2$, or rounding appropriately, $0.93 < x < 1.36$.

74. $y = x^2e^x \to \infty$ as $x \to \infty$. $y \to 0$ as $x \to -\infty$.

75. As θ increases $\sin\theta$ steadily oscillates between the values -1 and $+1$ passing through all intermediate values. No matter how large θ gets, $\sin\theta$ thereafter does not stay arbitrarily close to any fixed value.

76. We assume that the numerator and denominator of the rational function have no common factor of the form $x - a$. Then the number of vertical asymptotes is the number of distinct factors of the denominator of the form $x - a$. For example, $\frac{2x^3-7x}{x^2(x+1)(x-2)^4}$ has the three distinct factors x, $x+1$, $x-2$ in the denominator and has the vertical asymptotes $x = 0$, $x = -1$, $x = 2$. If the degree of the numerator is larger than the degree of the denominator, the value of the function becomes infinite in absolute value as $x \to \pm\infty$ and there is no horizontal asymptote. If numerator and denominator have the same degree, then there is one horizontal asymptote $y = k \neq 0$ as in Example 12. Finally, if the degree of the denominator exceeds that of the numerator, there is one and only one horizontal asymptote $y = 0$ as illustrated in Example 13.

2.5 CONTROLLING FUNCTION OUTPUTS

1. a) $0 < x < 6$. Not equivalent. b) $1 < x - 1 < 7$. Adding 1, we get $2 < x < 8$. Equivalent. c) $1 < \frac{x}{2} < 4$. Multiplying by 2, we get $2 < x < 8$. Equivalent. d) $\frac{1}{8} < \frac{1}{x} < \frac{1}{2}$. Taking reciprocals, we get $8 > x > 2$. Equivalent. e) $x > 8$. Not equivalent. f) $|x - 5| < 3$. $-3 < x - 5 < 3$. Adding 5, we get $2 < x < 8$. Equivalent. g) $4 < x < 10$. Not equivalent. h) $-8 < -x < -2$. Multiplying by -1, we get $8 > x > 2$. Equivalent.

2. All are equivalent except c), d) and h).

3. Change $|x + 3| < 1$ to $-1 < x + 3 < 1$ to $-4 < x < -2$. Answer: g). Equivalently, we can read $|x+3| < 1$ as $|x - (-3)| < 1$, that is, the distance between x and -3 is less than 1 and thus $-4 < x < -1$.

4. Change $|x - 5| < 2$ to $3 < x < 7$. Answer: c)

5. Change $|\frac{x}{2}| < 1$ to $-1 < \frac{x}{2} < 1$ to $-2 < x < 2$. Answer: e)

6. $|1 - x| < 2$ or $|x - 1| < 2$ yields $-1 < x < 3$. Answer: b)

7. Change $|2x - 5| \leqq 1$ to $-1 \leqq 2x - 5 \leqq 1$ to $4 \leqq 2x \leqq 6$ to $2 \leqq x \leqq 3$. Answer: h)

26. $1.98 < \tan x < 2.2$ yields $\tan^{-1} 1.98 < x < \tan^{-1} 2.2$ (since $\tan x$ is an increasing function on $[0, \frac{\pi}{2})$) or, rounding appropriately, $1.11 < x < 1.14$.

27. The graph of $y = \cos x$ is symmetric with respect to the y-axis. We need only reflect the corresponding interval in $[0, \frac{\pi}{2}]$ found in Exercise 25 over the y-axis. We obtain $-1.36 < x < -0.93$.

28. By the symmetry of the graph of $y = \tan x$ with respect to the origin, we need only take the reflection of the interval found in Exercise 26: $-1.14 < x < -1.11$.

29. $99.9 < x^2 < 100.1$ yields $\sqrt{99.9} < x < \sqrt{100.1}$, or rounding to thousandths appropriately, $9.995 < x < 10.004$.

30. From Exercise 29 we obtain $-10.004 < x < -9.995$.

31. Change $3.9 < \sqrt{x-7} < 4.1$ to $3.9^2 < x - 7 < 4.1^2$ to $22.21 < x < 23.81$.

32. Change $2 < \sqrt{19-x} < 4$ to $4 < 19 - x < 16$ to $-15 < -x < -3$ to $3 < x < 15$.

33. Change $4 < \frac{120}{x} < 6$ to $\frac{1}{4} > \frac{x}{120} > \frac{1}{6}$ to $30 > x > 20$ or $20 < x < 30$.

34. Change $\frac{1}{2} < \frac{1}{4x} < \frac{3}{2}$ to $2 > 4x > \frac{2}{3}$ to $\frac{1}{2} > x > \frac{1}{6}$ or $\frac{1}{6} < x < \frac{1}{2}$.

35. The graph of $y = \frac{3-2x}{x-1}$ is steadily falling as it passes through the horizontal channel between $y = -3.1$ and $y = -2.9$. Next we solve for x in terms of y: $(x-1)y = 3-2x$, $xy - y = 3 - 2x$, $xy + 2x = y + 3$, $x(y+2) = y+3$, $x = (y+3)/(y+2)$. Thus when $y = -2.9$, $x = (-2.9+3)/(-2.9+2) = -\frac{0.1}{0.9} = -\frac{1}{9}$ and, similarly, when $y = -3.1$, $x = \frac{1}{11}$. Answer: $-\frac{1}{9} < x < \frac{1}{11}$

36. $y = \frac{3x+8}{x+2}$ yields $(x+2)y = 3x + 8$, $xy + 2y = 3x + 8$, $x(y-3) = 8 - 2y$, $x = \frac{8-2y}{y-3}$. When $y = 0.9$, we get $x = -62/21$. When $y = 1.1$, we get $x = -58/19$. Thus $-58/19 < x < -62/21$, or rounding to hundredths appropriately, $-3.05 < x < -2.96$.

37. $10.5 < x^2 - 5 < 11.5$ leads to $15.5 < x^2 < 16.5$, $\sqrt{15.5} < x < \sqrt{16.5}$ since $x_0 > 0$. Thus $3.94 < x < 4.06$ rounding to hundredths appropriately.

38. Since the graph is symmetric with respect to the y-axis, we obtain, using Exercise 37, $-4.06 < x < -3.94$.

39. We graph $y = 4.8$, $y = 5.2$ and $y = x^3 - 9x$ in the same viewing rectangle, $[-4, 5]$ by $[-10, 10]$, for example. We then zoom in on the two points of intersection near $x = -3$. Rounding to hundredths appropriately, we obtain $-2.68 < x < -2.66$.

40. In the same viewing rectangle we graph $y = -5.2$, $y = -4.8$ and $y = x^4 - 10x^2$. Then zoom in on the two points of intersection near $x = 1$. We obtain, rounding to hundredths appropriately, $0.72 < x < 0.74$. In this example it is also possible to solve algebraically. $y + 25 = x^4 - 10x^2 + 25$, $y + 25 = (x^2 - 5)^2$, and since x is near 1, $x = \sqrt{5 - \sqrt{y + 25}}$. We can now find the endpoints of the desired interval using $y = -4.8$ and $y = -5.2$.

41. $0.4 < e^x < 0.6$ leads to $\ell n\, 0.4 < \ell n\, e^x < \ell n\, 0.6$ since $\ell n\, x$ is an increasing function. Because $\ell n\, e^x = x$, we obtain after rounding appropriately to hundredths $-0.91 < x < -0.52$.

42. $1.9 < \ell n\, x < 2.1$ yields $e^{1.9} < x < e^{2.1}$ or $6.69 < x < 8.16$ rounding to hundredths appropriately.

43. $|f(x) - y_0| < E = 0.5$ is equivalent to $|x + 1 - 4| < 0.5$ or $|x - 3| < 0.5$ which is the desired inequality.

44. $|f(x) - y_0| = |2x - 1 + 5| = 2|x + 2| < 1$ is equivalent to $|x + 2| < \frac{1}{2}$ which is the desired inequality.

45. In the same viewing retangle we graph $y = 2.8$, $y = 3.2$ and $y = 2x^2 + 1$. The latter curve is rising near $x = 1$ and we zoom in on its points of intersection with the horizontal lines near $x = 1$. The curve is in the channel for $0.95 < x < 1.04$ after appropriate rounding. Thus we may take $|x - 1| < 0.04$.

46. $1.98 < \sqrt{2x - 3} < 2.2$ yields $1.98^2 < 2x - 3 < 2.2^2$ and $\frac{3 + 1.98^2}{2} < x < \frac{3 + 2.2^2}{2}$. After rounding to hundredths appropriately, we obtain $3.47 < x < 3.92$. Thus we may take $|x - 3.5| <$ ~~0.03~~. .38

47. $\lim_{x\to 1} x^2 = 1$, $\lim_{x\to 1} x^2 = 1$, $\lim_{x\to\pi/6} \sin x = 0.5$, $\lim_{x\to 3} \frac{x+1}{x-2} = 4$, respectively.

48. $\lim_{x\to -1} x^2 = 1$, $\lim_{x\to 5\pi/6} \sin x = 0.5$, $\lim_{x\to x_0} \cos x = 0.4$ where $x_0 = \cos^{-1} 0.4$, $\lim_{x\to x_0} \tan x = 2$ where $x_0 = \tan^{-1} 2$, $\lim_{x\to x_0} \cos x = 0.4$ where $x_0 = \cos^{-1} 0.4$, $\lim_{x\to x_0} \tan x = -2$ where $x_0 = \tan^{-1}(-2)$, respectively.

49. $\lim_{x\to 10} x^2 = 100$, $\lim_{x\to -10} x^2 = 100$, $\lim_{x\to 23}\sqrt{x-7} = 4$, $\lim_{x\to 10}\sqrt{19-x} = 3$, $\lim_{x\to 24}(\frac{120}{x}) = 5$, $\lim_{x\to 1/4}(\frac{1}{4x}) = 1$, $\lim_{x\to 0}\frac{3-2x}{x-1} = -3$, $\lim_{x\to -3}\frac{3x+8}{x+2} = 1$, respectively.

50. $\lim_{x\to 4}(x^2-5) = 11$, $\lim_{x\to -4}(x^2-5) = 11$, $\lim_{x\to x_0}(x^3-9x) = 5$ where x_0 is a solution of $x^3 - 9x = 5$ near -3, $\lim_{x\to x_0}(x^4 - 10x^2) = -5$ where x_0 is the solution of $x^4 - 10x^2 = -5$ near 1, $\lim_{x\to x_0} e^x = 0.5$ where $x_0 = \ell n\, 0.5$, $\lim_{x\to e^2} \ell n\, x = 2$, $\lim_{x\to 3}(x+1) = 4$, $\lim_{x\to -2}(2x-1) = -5$, $\lim_{x\to 1}(2x^2+1) = 3$, $\lim_{x\to 3.5}\sqrt{2x-3} = 2$, respectively.

51. $8.99 < \pi(x/2)^2 < 9.01$ leads to $\frac{8.99}{\pi} < (\frac{x}{2})^2 < \frac{9.01}{\pi}$, $\frac{4(8.99)}{\pi} < x^2 < \frac{4(9.01)}{\pi}$ and to $\sqrt{\frac{4(8.99)}{\pi}} < x < \sqrt{\frac{4(9.01)}{\pi}}$. Rounding appropriately to thousandths, we obtain $3.384 < x < 3.387$ or, in symmetric form, $|x - x_0| < 0.001$.

52. $4.9 < I < 5.1$ or $4.9 < \frac{V}{R} < 5.1$ leads to $\frac{1}{4.9} > \frac{R}{V} > \frac{1}{5.1}$ and to $\frac{120}{4.9} > R > \frac{120}{5.1}$. Rounding to tenths appropriately, we obtain $23.6 < R < 24.4$ or $|R - 24| < 0.4$.

53. $f(x) = \frac{3x+1}{x-2} = 3 + \frac{7}{x-2} \to 3$ as $x \to \infty$ (the equality can be obtained by long division). The same equality (or a graph) shows that $f(x) > 3$ for $x > 2$. Thus we want to solve $3 + \frac{7}{x-2} < 3.01$, $x > 2$. This leads to $\frac{7}{x-2} < 0.01$, $x - 2 > \frac{7}{0.01}$ (since $x - 2 > 0$), $x > 702$.

54. $f(x) = \frac{2x+5}{5x-7} = \frac{2}{5} + \frac{39}{5(5x-7)}$. Thus $f(x) \to \frac{2}{5}$ as $x \to \infty$ and $f(x) > \frac{2}{5}$ if $x > \frac{7}{5}$. $\frac{2x+5}{5x-7} < 0.41$ leads to $2x+5 < 0.41(5x-7)$ (since we may take $x > \frac{7}{5}$), $2x + 5 < 2.05x - 2.87$, $0.05x > 7.87$, $x > 157.4$.

55. By long division $f(x) = \frac{2x^2-x+2}{x^2-4} = 2 + \frac{10-x}{x^2-4}$. This shows $f(x) < 2$ for $x > 10$ and $f(x) = 2 + \frac{(10/x^2)-(1/x^2)}{1-(4/x^2)} \to 2 + 0 = 2$ as $x \to \infty$. $f(x) > 1.99$, $x > 10$ leads to $2x^2 - x + 2 > 1.99(x^2-4)$, $0.01x^2 - x + 9.96 > 0$, $x^2 - 100x + 996 > 0$. By the quadratic formula, the larger root of the quadratic is $50+\sqrt{1504}$. Thus we require $x > 50 + \sqrt{1504} \approx$ ~~88.781~~. 109.194

56. By long division $f(x) = \frac{3x^3-x+1}{2x^3+5} = \frac{3}{2} - \frac{2x+13}{2(2x^3+5)}$. Thus $f(x) \to \frac{3}{2}$ as $x \to \infty$ and $f(x) < \frac{3}{2}$ for $x > 0$. $f(x) > 1.5 - 0.01 = 1.49$ leads to $3x^3 - x + 1 > 1.49(2x^3+5)$ (since $2x^3 + 5 > 0$), $0.02x^3 - x - 6.45 > 0$, $2x^3 - 100x - 645 > 0$. By ZOOM-IN, the cubic has the unique root $x = 9.2186...$ Hence we require $x > 9.2186...$

57. In the interval $[0, 2\pi]$, $\sin x = \frac{\sqrt{2}}{2}$ has two solutions $\frac{\pi}{4}$ in the first quadrant and $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$ in the second quadrant, recalling that $\sin(\pi - x) = \sin x$ for all x. We first solve the problem in the first quadrant. We use the result to solve the problem in the second quadrant and then use the periodicity of the sine function to give the complete solution set. Graph $y_1 = \sin x$, $y_2 = \frac{\sqrt{2}}{2} - 0.1$ and $y_3 = \frac{\sqrt{2}}{2} + 0.1$ in $[0.6, 1]$ by $[0.5, 1]$. We then zoom in to the two points of intersection and find that they occur at $x_1 = 0.65241449628$ and $x_2 = 0.93923517764$. Thus in the first quadrant we must have $x_1 < x < x_2$. In the second quadrant we must have $\pi - x_2 < x < \pi - x_1$. The complete solution set is $\{x \mid x_1 + 2n\pi < x < x_2 + 2n\pi \text{ or } (2n+1)\pi - x_2 < x < (2n+1)\pi - x_1\}$.

58. Let h be the width of a stripe in cm. Let ΔV be the volume of liquid in the cylinder at the level of the entire stripe. Then ΔV is the volume of a small right circular cylinder of base radius 5cm and height $h$cm. $\Delta V = \pi 5^2 h$ gives ΔV in cubic centimeters. Since 1 liter = 1000cm^3, we need $\Delta V = \frac{25\pi h}{1000} = \frac{\pi h}{40}$ to have ΔV represented in liters. Thus we must have $\frac{\pi h}{40} < \frac{0.01}{2} = 0.005$ since the middle of the stripe corresponds to exactly one liter. $h < \frac{40(0.005)}{\pi} \approx 0.06$cm.

59. By long division $f(x) = \frac{x-3}{2x+1} = \frac{1}{2} - \frac{7}{2(2x+1)} > \frac{1}{2}$ for $x < -\frac{1}{2}$, and $f(x) \to \frac{1}{2}$ as $x \to -\infty$. $f(x) < 0.51$ leads to $\frac{x-3}{2x+1} < 0.51$, $x - 3 > 0.51(2x+1)$ (since $2x + 1 < 0$), $x - 3 > 1.02x + 0.51$, $-3.51 > .02x$, $x < -175.5$.

60. $f(x) = \frac{3x+4}{2x-1} = \frac{3}{2} + \frac{11}{2(2x-1)} < \frac{3}{2}$ for $x < \frac{1}{2}$ and $f(x) \to \frac{3}{2}$ as $x \to -\infty$. $\frac{3x+4}{2x-1} > 1.49$ leads to $3x + 4 < 1.49(2x-1)$ (since $2x - 1 < 0$), $0.02x < -5.49$, $x < -274.5$.

61. By long division $f(x) = \frac{x-4}{1-3x} = -\frac{1}{3} - \frac{11}{3(1-3x)} < -\frac{1}{3}$. $\frac{x-4}{1-3x} \to -\frac{1}{3}$ as $x \to -\infty$. $\frac{x-4}{1-3x} > -\frac{1}{3} - \frac{1}{100} = -\frac{103}{300}$ leads to $300(x-4) > -103(1-3x)$ (since $1 - 3x > 0$ for $x < \frac{1}{3}$), $9x < -1200 + 103 = -1097$, $x < -\frac{1097}{9}$.

62. $f(x) = \frac{5-2x}{x+1} = -2 + \frac{7}{x+1} < -2$ when $x < -1$. $f(x) \to -2$ as $x \to -\infty$ and $\frac{5-2x}{x+1} > -2.01$ leads to $5 - 2x < -2.01x - 2.01$ (since $x + 1 < 0$ for $x < -1$), $0.01x < -7.01$, $x < -701$.

2.6 DEFINING LIMITS FORMALLY WITH EPSILONS AND DELTAS

1. $x_0 - a = 5 - 1 = 4$ and $b - x_0 = 7 - 5 = 2$. Since $2 < 4$, we choose $\delta = 2$. Then $|x - x_0| < 2$ or $|x - 5| < 2$ implies $3 < x < 7$ and so $1 < x < 7$.

2. $\delta = 1$

3. $x_0 - a = -3 + \frac{7}{2} = \frac{1}{2}$ and $b - x_0 = -\frac{1}{2} + 3 = \frac{5}{2}$. Since $\frac{1}{2} < \frac{5}{2}$, $\delta = \frac{1}{2}$.

4. $\delta = 1$

5. From the graph we see that if x is between 4.9 and 5.1 on the x-axis, then y on the curve (corresponding to x) is in the range $5.8 < y < 6.2$. Therefore we can take $\delta = 0.1$ because $0 < |x - 5| < \delta = 0.1$ does imply $|f(x) - L| = |y - 6| < 0.2 = \varepsilon$.

6. $\delta = 0.1$

7. From the graph we see that $\sqrt{3} < x < \sqrt{5}$ implies that $|f(x) - L| < \varepsilon$. Rounding the first inequality appropriately to hundredths, we get $1.74 < x < 2.23$. Since 2 is closer to 2.23 than to 1.74, we take $\delta = 2.23 - 2 = 0.23$.

8. $\delta = \min\{-1 + \frac{\sqrt{5}}{2}, -\frac{\sqrt{3}}{2} + 1\} = 0.11$ rounded to hundredths appropriately.

9. From the graph we see that $\frac{9}{16} < x < \frac{25}{16}$ implies $|f(x) - L| < \varepsilon = \frac{1}{4}$. $1 - \frac{9}{16} = \frac{7}{16}$ and $\frac{25}{16} - 1 = \frac{9}{16}$. Thus 1 is closer to $\frac{9}{16}$. We take $\delta = \frac{7}{16}$. Note that $|x - x_0| < \delta$ is the same as $|x - 1| < \frac{7}{16}$ which is equivalent to $\frac{9}{16} < x < \frac{23}{16}$. So $\delta = \frac{7}{16}$ works.

10. $\delta = \min\{3 - 2.61, 3.41 - 3\} = 0.39$

11. $\lim_{x\to 1}(2x + 3) = 5$. $|f(x) - L| = |2x + 3 - 5| = |2x - 2| = |2(x - 1)| = |2||x - 1| = 2|x - 1|$. Thus $|f(x) - L| < 0.01$ is $2|x - 1| < 0.01$ which is equivalent to $|x - 1| < \frac{0.01}{2} = 0.005$. Thus $\delta = 0.005$ will do because $0 < |x - 1| < 0.005 \Rightarrow |f(x) - L| < 0.01 = \varepsilon$.

12. $\lim_{x\to 3}(3 - 2x) = -3$. $|3 - 2x + 3| = |2(3 - x)| = 2|x - 3| < 0.02$ if and only if $|x - 3| < 0.01$. Thus $\delta = 0.01$ will do.

13. $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x + 2) = 4$. If $x \neq 2$, $\frac{x^2-4}{x-2} = x + 2$ and we may assume this here because x is never equal to 2 as $x \to 2$. Thus $|f(x) - L| = |x + 2 - 4| = |x - 2| < \varepsilon = 0.05$ is equivalent to $0 < |x - 2| < 0.05$ and we may take $\delta = 0.05$.

14. $\lim_{x\to -5} \frac{x^2+6x+5}{x+5} = \lim_{x\to -5} \frac{(x+5)(x+1)}{x+5} = -4$. If $x \neq -5$, $|f(x) - L| = |(x + 1) + 4| = |x + 5|$. So we may take $\delta = \varepsilon = 0.05$.

15. $L = \lim_{x\to 11}\sqrt{x-7} = \sqrt{11-7} = 2$. The inequality $|\sqrt{x-7}-2| < 0.01$ leads successively to $-0.01 < \sqrt{x-7} - 2 < 0.01$, $1.99 < \sqrt{x-7} < 2.01$, $1.99^2 < x - 7 < 2.01^2$, $7 + 1.99^2 < x < 7 + 2.01^2$, $10.9601 < x < 11.0401$. The distance, 0.0399, between $x_0 = 11$ and the left endpoint is less than the distance between x_0 and the right endpoint. Thus we may take $\delta = 0.0399$.

16. $L = \lim_{x\to -3}\sqrt{1-5x} = \sqrt{1+15} = 4$. The inequality $|\sqrt{1-5x} - 4| < 0.5$ leads to $-0.5 < \sqrt{1-5x} - 4 < 0.5$, $3.5 < \sqrt{1-5x} < 4.5$, $3.5^2 < 1 - 5x < 4.5^2$, $-4.5^2 < 5x - 1 < -3.5^2$, $\frac{1-4.5^2}{5} < x < \frac{1-3.5^2}{5}$, $-3.85 < x - 2.25$. The distance, 0.75, between $x_0 = -3$ and the right endpoint is smaller than the distance between x_0 and the left endpoint. Thus we may take $\delta = 0.75$ or any other smaller positive number.

17. $\lim_{x\to 2}\frac{4}{x} = 2$. $|f(x) - L| < \varepsilon$ is equivalent to each of the following: $|\frac{4}{x} - 2| < 0.4 = \frac{2}{5}$, $2 - \frac{2}{5} < \frac{4}{x} < 2 + \frac{2}{5}$, $\frac{8}{5} < \frac{4}{x} < \frac{12}{5}$, $\frac{5}{12} < \frac{x}{4} < \frac{5}{8}$, $\frac{5}{3} < x < \frac{5}{2}$, $2 - \frac{1}{3} < x < 2 + \frac{1}{2}$. The last inequality is satisfied if $|x - 2| < \frac{1}{3}$. We may therefore take $\delta = \frac{1}{3}$.

18. $\lim_{x\to \frac{1}{2}}\frac{4}{x} = 8$. $|f(x) - L| < \varepsilon$ is equivalent to each of the following: $|\frac{4}{x} - 8| < 0.04 = \frac{1}{25}$, $8 - \frac{1}{25} < \frac{4}{x} < 8 + \frac{1}{25}$, $\frac{199}{25} < \frac{4}{x} < \frac{201}{25}$, $\frac{25}{201} < \frac{x}{4} < \frac{25}{199}$, $\frac{100}{201} < x < \frac{100}{199}$, $\frac{1}{2} - \frac{1}{402} < x < \frac{1}{2} + \frac{1}{398}$. Thus we may take $\delta = \frac{1}{402}$ or $\delta = 0.0024$ rounding to ten thousandths appropriately.

19. $|f(x) - 5| = |9 - x - 5| = |4 - x| = |x - 4|$. Thus $|f(x) - 5| < \varepsilon$ is equivalent to $|x - 4| < \varepsilon$. Thus in each case $\delta = \varepsilon$.

20. $|f(x) - 5| = |3x - 12| = 3|x - 4| < \varepsilon$ if and only if $|x - 4| < \varepsilon/3$. Thus $\delta = 0.001,\ 0.0001,\ \varepsilon/3$, respectively.

21. Let $f(x) = \frac{x+2}{x+1} = \frac{(x+1)+1}{x+1} = 1 + \frac{1}{x+1}$. $\lim_{x\to\infty}(1 + \frac{1}{x+1}) = 1$. Let $\varepsilon > 0$ be given. Then $|f(x) - 1| = \frac{1}{x+1}$ (for $x > -1$) $< \varepsilon$ if and only if $x + 1 > \frac{1}{\varepsilon}$ or $x > \frac{1}{\varepsilon} - 1$. Thus if $N = \frac{1}{\varepsilon} - 1$, $x > N$ implies $|f(x) - 1| < \varepsilon$.

22. Let $f(x) = \frac{x^2}{2x^2-1}$. By long division $f(x) = \frac{1}{2} + \frac{1}{2(2x^2-1)}$. We can see that $\lim_{x\to\infty} f(x) = \frac{1}{2}$. Let $\varepsilon > 0$ be given. $|f(x) - \frac{1}{2}| = \frac{1}{2(2x^2-1)}$ (for $x > \frac{1}{\sqrt{2}}$) $< \varepsilon$ is equivalent to $2x^2 - 1 > \frac{1}{2\varepsilon}$, $2x^2 > 1 + \frac{1}{2\varepsilon}$, $x > \sqrt{\frac{1}{2} + \frac{1}{4\varepsilon}}$. Thus if $N = \sqrt{\frac{1}{2} + \frac{1}{4\varepsilon}}$, $x > N$ implies $|f(x) - \frac{1}{2}| < \varepsilon$.

23. Let $f(x) = \frac{x+2}{x+1} = 1 + \frac{1}{x+1}$. Here since $x \to -1^+$, $x > -1$ or $x + 1 > 0$. Let $N > 1$ be given, $f(x) > N$ is equivalent to $1 + \frac{1}{x+1} > N$, $\frac{1}{x+1} > N - 1$, $x + 1 < \frac{1}{N-1}$. Let $\delta = \frac{1}{N-1}$. Then $-1 < x < -1 + \delta$ implies $f(x) > N$. This proves $\lim_{x \to -1^+} f(x) = \infty$.

24. Let $N > 2$ be given. Here $2x^2 - 1 > 0$ because $x > \frac{\sqrt{2}}{2}$. $f(x) = \frac{x^2}{2x^2-1} > N$ leads to $x^2 > (2x^2 - 1)N$, $x^2(1 - 2N) > -N$, $x^2(2N - 1) < N$, $x^2 < \frac{N}{2N-1}$, $x < \sqrt{\frac{N}{2N-1}}$, $x - \frac{\sqrt{2}}{2} < \sqrt{\frac{N}{2N-1}} - \frac{\sqrt{2}}{2}$. So we choose $\delta = \sqrt{\frac{N}{2N-1}} - \frac{\sqrt{2}}{2}$. Then $\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2} + \delta$ implies $f(x) > N$. This proves $\lim_{x \to \frac{\sqrt{2}}{2}^+} \frac{x^2}{2x^2-1} = \infty$.

25. $\lim_{x \to 1} \sin x = 0.84 (= \sin 1)$ rounded to hundredths. In the viewing rectangle $[0.99, 1.01]$ by $[0.83, 0.85]$ we graph $y = 0.84$ and $y = \sin x$ (which appears as a straight line). Using the endpoints of $y = \sin x$ in this rectangle, we calculate the slope $m = \frac{0.8468 - 0.8360}{1.01 - 0.99} = 0.54$. As in Example 6, $\delta = \varepsilon/|m| = \varepsilon/0.54 = 1.85\varepsilon$ rounding δ down to be safe. , 018

26. $\lim_{x \to 1} \tan x = \tan 1 = 1.56$ after rounding. In the viewing rectangle $[0.99, 1.01]$ by $[1.55, 1.57]$ we graph $y = 1.56$ and $y = \tan x$ (which appears as a straight line). Using the endpoints of $y = \tan x$ in the rectangle, we find $m = 3.44$ (rounding up to be safe). $\delta = \frac{\varepsilon}{|m|} = 0.29\varepsilon$ rounding down to be safe.

27. $\lim_{x \to 1} \cos x = \cos 1 = 0.54$ after rounding. In the viewing rectangle $[0.99, 1.01]$ by $[0.53, 0.55]$ we graph $y = 0.54$ and $y = \cos x$. Using two points on the graph of $y = \cos x$, we get for the estimate of m, $m = -0.85$. Thus $\delta = \varepsilon/0.85 = 1.17\varepsilon$. .012

28. $\lim_{x \to 4} \sec x = \sec 4 = -1.53$ after rounding. We graph $y = -1.53$ and $y = \sec x$ in the viewing rectangle $[3.99, 4.0]$ by $[-1.54, -1.52]$. Our estimate of m is $m = -1.78$. Hence $\delta = \varepsilon/1.78 = 0.56\varepsilon$.

29. $\lim_{x \to 0.5}(x^3 - 4x) = (0.5)^3 - 4(0.5) = -1.88$ after rounding. In the viewing rectangle $[0.49, 0.51]$ by $[-1.89, 1.87]$ we graph $y = -1.88$ and $y = x^3 - 4x$. Using the endpoints to estimate the slope, we get $m = -3.25$. Therefore $\delta = \varepsilon/3.25 = 0.30\varepsilon$ rounding down to be safe. , 0031

30. $\lim_{x \to 2.5} 9x - x^3 = 6.875$. In the viewing rectangle $[2.49, 2.51]$ by $[6.874, 6.876]$ we graph $y = 6.875$ and $y = 9x - x^3$. In this viewing rectangle, the latter graph appears as a vertical line and we are unable to calculate a slope. In several smaller viewing rectangles we obtain slope estimates near -10. To be safe we round $|m|$ to be 11. Thus $\delta = \varepsilon/11$ is our estimate.

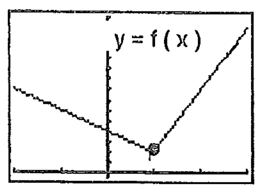
31. $\lim_{x\to -1} \frac{x}{x^2-4} = \frac{1}{3}$. In the viewing rectangle $[-1.01, -0.99]$ by $[0.32, 0.34]$ we graph $y = 0.33$ and $y = \frac{x}{x^2-4}$. Using the endpoints of the latter graph, we get our estimate of the slope, $m = -0.56$. Thus $\delta = \varepsilon/0.56 = 1.78\varepsilon$ rounding appropriately. .008 .015

32. $\lim_{x\to -1} \frac{2x}{5-x^2} = -\frac{1}{2}$. In the viewing rectangle $[-1.01, -0.99]$ by $[-0.51, -0.49]$ we graph $y = -0.5$ and $y = \frac{2x}{5-x^2}$. Using the endpoints of the latter graph, we get our estimate of the slope, $m = 0.75$. Hence our estimate is $\delta = \frac{\varepsilon}{|m|} = \frac{4\varepsilon}{3}$.

33. $\sqrt{x-5} < \varepsilon$ is equivalent to $x - 5 < \varepsilon^2$ or $x < 5 + \varepsilon^2$ since $x \geqq 5$. Thus $I = (5, 5+\varepsilon^2)$. Since $5 < x < 5 + \varepsilon^2$ implies $|\sqrt{x-5} - 0| < \varepsilon$, this verifies that $\lim_{x\to 5^+} \sqrt{x-5} = 0$.

34. $\sqrt{4-x} < \varepsilon$ is equivalent to $4 - x < \varepsilon^2$ or $4 - \varepsilon^2 < x$ since $x \leqq 4$. Thus $I = (4 - \varepsilon^2, 4)$. This verifies $\lim_{x\to 4^-} \sqrt{4-x} = 0$.

35.



$[-2, 3]$ by $[-1, 16]$

Since $y \geqq 2$, we need only be concerned that $y < 2 + \varepsilon$. For $x < 1$, this is equivalent to $4 - 2x < 2 + \varepsilon$ which leads to $2 < 2x + \varepsilon$ and to $x > 1 - \varepsilon/2$. For $x \geqq 1$, $y < 2+\varepsilon$ is $6x - 4 < 2 + \varepsilon$ which leads to $x < 1 + \varepsilon/6$. The largest δ can be is the smaller of $\varepsilon/2$, $\varepsilon/6$ which is $\varepsilon/6$ and $I = (1 - \varepsilon/6, 1 + \varepsilon/6)$.

36. $f(x) = -1$ for $x < 5$ and $f(x) = 1$ for $x > 5$ and f is not defined for $x = 5$. If $\varepsilon = 4$, $1 - \varepsilon < f(x) < 1 + \varepsilon$ becomes $-3 < f(x) < 4$ which is true for all $x \neq 5$. If $\varepsilon = 2$, $-1 < f(x) < 3$ is true for all $x > 5$. If $\varepsilon = 1$, $0 < f(x) < 2$ is true for all $x > 5$. If $\varepsilon = \frac{1}{2}$, $\frac{1}{2} < f(x) < \frac{3}{2}$ is true for all $x > 5$.

37. $\lim_{x\to 2} f(x) = 5$ means corresponding to any radius $\varepsilon > 0$ about 5, there exists a radius $\delta > 0$ about 2 such that $0 < |x-2| < \delta$ implies $|f(x) - 5| < \varepsilon$.

38. $\lim_{x\to 0} g(x) = k$ means corresponding to each radius $\varepsilon > 0$ about k, there exists a radius $\delta > 0$ about 0 such that $0 < |x-0| < \delta$ implies $|g(x)-k| < \varepsilon$, x in the domain of g.

39. Since we need $|x^2-4| < \varepsilon < 4$, x cannot be 0 and so $\delta < 2$, $0 < 2-\delta < x$, i.e., $x > 0$. Now $|x^2-4| < \varepsilon$ is equivalent to each of the following: $-\varepsilon < x^2-4 < \varepsilon$, $4-\varepsilon < x^2 < 4+\varepsilon$, $\sqrt{4-\varepsilon} < |x| < \sqrt{4+\varepsilon}$, $\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$ since $x > 0$. The distance from $x = 2$ to the left endpoint is $2-\sqrt{4-\varepsilon} = (2-\sqrt{4-\varepsilon})\frac{(2+\sqrt{4-\varepsilon})}{2+\sqrt{4-\varepsilon}} = \frac{\varepsilon}{2+\sqrt{4-\varepsilon}}$, and the distance from $x = 2$ to the right endpoint is $\sqrt{4+\varepsilon}-2 = (\sqrt{4+\varepsilon}-2)\frac{(\sqrt{4+\varepsilon}+2)}{\sqrt{4+\varepsilon}+2} = \frac{\varepsilon}{\sqrt{4+\varepsilon}+2}$. Since the second distance has a larger denominator, it is smaller and $\delta = \sqrt{4+\varepsilon}-2$. This verifies $\lim_{x\to 2} x^2 = 4$ or $\lim_{x\to 2}(x^2-4) = 0$. $\delta \to 0$ as $\varepsilon \to 0$. The graph of δ as a function of ε can be viewed by graphing of δ as a function of ε can be viewed by graphing $y = \sqrt{4+x}-2$ in the rectangle $[0,4]$ by $[0,1]$. The endpoints are not included.

40. $|\frac{2}{x-1}-1| < \varepsilon$ is equivalent to the following: $-\varepsilon < \frac{2}{x-1}-1 < \varepsilon$, $1-\varepsilon < \frac{2}{x-1} < 1+\varepsilon$, $\frac{1}{1+\varepsilon} < \frac{x-1}{2} < \frac{1}{1-\varepsilon}$ (since $\varepsilon < 1$), $\frac{2}{1+\varepsilon} < x-1 < \frac{2}{1-\varepsilon}$, $1+\frac{2}{1+\varepsilon} < x < 1+\frac{2}{1-\varepsilon}$. The distance between 3 and the left endpoint is $3-(1+\frac{2}{1+\varepsilon}) = 2(1-\frac{1}{1+\varepsilon}) = \frac{2\varepsilon}{1+\varepsilon}$. The distance between 3 and the right endpoint is $1+\frac{2}{1-\varepsilon}-3 = 2(\frac{1}{1-\varepsilon}-1) = \frac{2\varepsilon}{1-\varepsilon}$. We must use the shorter distance so $\delta = 2\varepsilon/(1+\varepsilon)$. This verifies $\lim_{x\to 3}\frac{2}{x-1} = 1$ or $\lim_{x\to 3}(\frac{2}{x-1}-1) = 0$. $\delta = 2\varepsilon/(1+\varepsilon) \to 0$ as $\varepsilon \to 0$. The graph of δ as a function of ε may be viewed by graphing $y = 2x/(1+x)$ in the rectangle $[0,1]$ by $[0,1]$ and excluding the endpoints since $0 < \varepsilon < 1$. This verifies $\lim_{x\to 3}\frac{2}{x-1} = 1$ or $\lim_{x\to 3}(\frac{2}{x-1}-1) = 0$.

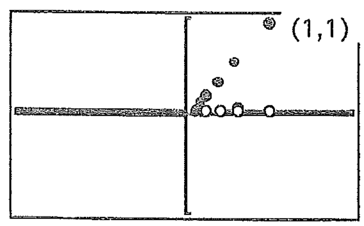
41. One need only reverse the order of the steps in Example 3. Each line implies the preceding line in that list of steps.

42. a) The inequality is supported by graphing $y_1 = \frac{1}{2+x}$, $y_2 = \frac{1}{2}$, $y_3 = \frac{1}{2-x}$ in $[0,2]$ by $[-1,3]$. b) If $\varepsilon > 2$, $\frac{1}{2} < \frac{1}{2-\varepsilon}$ fails because $\frac{1}{2-\varepsilon}$ is negative.

43. Suppose $\varepsilon < 2$. Then $\delta = \frac{\varepsilon}{2(2+\varepsilon)}$. $|x-0.5| < \delta$ implies $-\frac{\varepsilon}{2(2+\varepsilon)} < x-\frac{1}{2} < \frac{\varepsilon}{2(2+\varepsilon)} < \frac{\varepsilon}{2(2-\varepsilon)}$. We may now use the steps in Example 4 in reverse order to obtain $|f(x)-2| < \varepsilon$. Now suppose $\varepsilon \geqq 2$, $\delta = \frac{1}{4}$. $|x-0.5| < \delta = \frac{1}{4}$ implies $\frac{1}{4} < x < \frac{3}{4}$, $\frac{4}{3} < \frac{1}{x} < 4$, $-\frac{2}{3} < \frac{1}{x}-2 < 2$ which implies $|\frac{1}{x}-2| < 2 \leqq \varepsilon$.

44. a) Graph $y_1 = x^3 + 1.001 + 0\sqrt{x}$ and $y_2 = x^3 + 0.009 + 0\sqrt{-x}$ in $[-1, 1]$ by $[-0.9, 1.6]$. b) Since $\lim_{x\to 0^+} f(x) = 1.001 \neq 0.009 = \lim_{x\to 0^-} f(x)$, $\lim_{x\to 0} f(x)$ does not exist. c) If x_1 is negative and x_2 is positive, then $|f(x_2) - f(x_1)| > 1.001 - 0.009 = 0.992$. For $\varepsilon = 0.4$ the implication fails no matter what $\delta > 0$ is. Because if it did hold, and if x_1 is negative and x_2 is positive, both within δ of $x_0 = 0$, we would have $|f(x_2) - f(x_1)| = |f(x_2) - L + L - f(x_1)| \leqq |f(x_2) - L| + |L - f(x_1)| < \varepsilon + \varepsilon = 2\varepsilon = 0.8$, a contradiction.

45. a)



$[-2, 2]$ by $[-1, 1]$

b) Since $|f(x) - 0| = |f(x)| \leqq |x| = |x - 0|$, we may take $\delta = \varepsilon$ in the definition of limit to prove $\lim_{x\to 0} f(x) = 0$.

46. Let $f(x) = \frac{|x+1|}{x+1}$. Then $f(x) = \begin{cases} -1, & x < -1 \\ 1, & x > -1 \end{cases}$. For any $\delta > 0$ let $x_0 = -1 - \frac{\delta}{2}$. Then $0 < |x_0 + 1| = \frac{\delta}{2} < \delta$, but $|f(x_0) - 1| = |-1 - 1| = 2$. Thus given ε, $0 < \varepsilon \leqq 2$, there is no corresponding δ as in the definition of limit. Hence the limit is not 1 and it therefore does not exist.

PRACTICE EXERCISES, CHAPTER 2

1. Exists 2. Exists 3. Exists

4. Does not exist. The right-hand and left-hand limits exist but are not equal.

5. Exists 6. Exists 7. Continuous at $x = a$