Semester 1 Final Review Topics – AP Calculus Cheat Sheet

Important Information

Chapter 1 Limits

1.2 Existence of a Limit

A limit exists if and only if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$

Examples of Existing Limits

Limits that Fail to Exist

Jumps

Asymptotes

Oscillating Behavior

1.3 Evaluating Limits

Canceling Out Technique

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}.$$

Rationalizing

$$\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x}.$$

Common Fractions

$$\lim_{x \to 0} \frac{[1/(x + 4)] - (1/4)}{x}$$

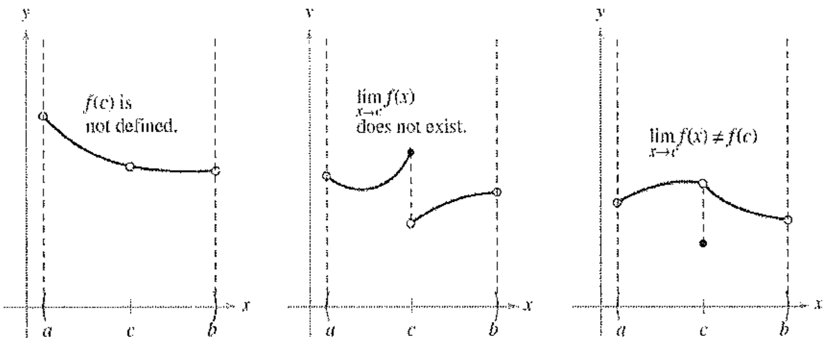
Special Trig Limits **1.** $\lim_{x \to 0} \frac{\sin x}{x} = 1$ **2.** $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

1.4 Continuity at a point

A function is continuous at $x = c$ if an only if

1. $\lim_{x \to c} f(x)$ exists
2. $f(c)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

Conditions in which Continuity Fails



Three conditions exist for which the graph of f is not continuous at $x = c$.

THEOREM 1.13 Intermediate Value Theorem

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

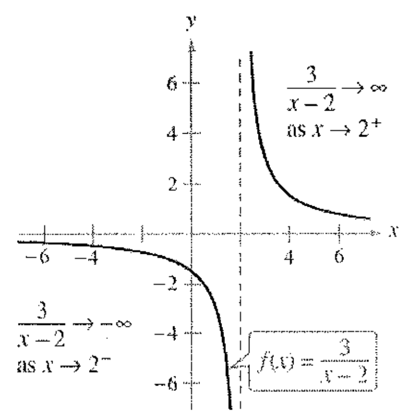
Use the Intermediate Value Theorem to show that the polynomial function

$$f(x) = x^3 + 2x - 1$$

has a zero in the interval $[0, 1]$.

1.5 Infinite Limits

$$\lim_{x\to c^{+/-}} f(x) = \pm\infty \qquad \text{(Vertical Asymptotes)}$$



This behavior is denoted as

$$\lim_{x\to 2^-} \frac{3}{x-2} = -\infty$$

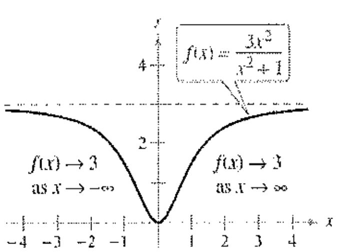
$f(x)$ decreases without bound as x approaches 2 from the left.

and

$$\lim_{x\to 2^+} \frac{3}{x-2} = \infty.$$

$f(x)$ increases without bound as x approaches 2 from the right.

3.5 Horizontal Asymptotes and Limits at Infinity



The limit of $f(x)$ as x approaches $-\infty$ or ∞ is 3.

Definition of a Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** of the graph of f when

$$\lim_{x\to -\infty} f(x) = L \quad \text{or} \quad \lim_{x\to \infty} f(x) = L.$$

GUIDELINES FOR FINDING LIMITS AT $\pm\infty$ OF RATIONAL FUNCTIONS

1. If the degree of the numerator is *less than* the degree of the denominator, then the limit of the rational function is 0.
2. If the degree of the numerator is *equal to* the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.
3. If the degree of the numerator is *greater than* the degree of the denominator, then the limit of the rational function does not exist.

Examples

15. (a) $\lim_{x\to\infty} \frac{x^2+2}{x^3-1}$

(b) $\lim_{x\to\infty} \frac{x^2+2}{x^2-1}$

(c) $\lim_{x\to\infty} \frac{x^2+2}{x-1}$

16. (a) $\lim_{x\to\infty} \frac{3-2x}{3x^3-1}$

(b) $\lim_{x\to\infty} \frac{3-2x}{3x-1}$

(c) $\lim_{x\to\infty} \frac{3-2x^2}{3x-1}$

Chapter 2 – Derivatives

2.1 Limit Definition of a Derivative

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

Example

$$\lim_{h\to 0} \frac{(x+h)^3-x^3}{h} = 3x^2$$

$$\lim_{h\to 0} \frac{\sqrt{3+h}-\sqrt{3}}{h} = \frac{1}{2\sqrt{3}}$$

2.2 Power Rule and Other Rules of Derivatives

$$\frac{d}{dx}(x^n) = n\, x^{n-1}$$

2.3 Product and Quotient Rule

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + g(x)f'(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$$

2.4 Chain Rule

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

2.5 Implicit Differentiation

Example: Find the derivative of $2x - 3y^2 + 4x^2y = 6x$

2.6 Related Rates

Example

21. Moving Ladder A ladder 25 feet long is leaning against the wall of a house (see figure). The base of the ladder is pulled away from the wall at a rate of 2 feet per second.

(a) How fast is the top of the ladder moving down the wall when its base is 7 feet, 15 feet, and 24 feet from the wall?

(b) Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 feet from the wall.

(c) Find the rate at which the angle between the ladder and the wall of the house is changing when the base of the ladder is 7 feet from the wall.

17. Volume At a sand and gravel plant, sand is falling off a conveyor and onto a conical pile at a rate of 10 cubic feet per minute. The diameter of the base of the cone is approximately three times the altitude. At what rate is the height of the pile changing when the pile is 15 feet high? (*Hint:* The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$.)

18. Depth A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. Water is flowing into the tank at a rate of 10 cubic feet per minute. Find the rate of change of the depth of the water when the water is 8 feet deep.

Chapter 3

Increasing – A function $f(x)$ is increasing if $f'(x) > 0$ Decreasing – A function $f(x)$ is decreasing if $f'(x) < 0$

Relative Max: f has a local max at $x = c$ if f' changes from + to – at $x = c$.

Relative Min: f has a local min at $x = c$ if f' changes from – to + at $x = c$.

Concave Up – A function is concave up if $f''(x) > 0$ (on a graph of f', f' is increasing)

Concave Down – A function is concave down if $f''(x) < 0$ (on a graph of f', f' is decreasing)

Point of Inflection - $f(x)$ has a point of inflection at $(c, f(c))$ if $f''(c) = 0$ *or und* and $f''(x)$ changes from (+ to –) or (– to +)

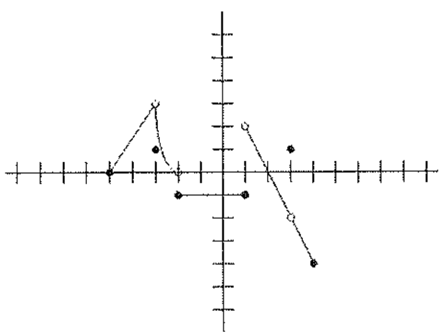
On a graph of f', f' changes direction or attains a local extrema.

Mean Value Theorem: If $f(x)$ is differentiable on $(a, b) \rightarrow$ Continuous on $[a, b]$ and $\frac{f(b)-f(a)}{b-a} = m$, then there exists some $x = c$ contained on the interval $a < c < b$ *or* (a, b) such that $f'(c) = m = \frac{f(b)-f(a)}{b-a}$ by MVT

Rolle's Theorem: If $f(x)$ is differentiable on $(a, b) \rightarrow$ Continuous on $[a, b]$ and $f(a) = f(b)$, then there exists some $x = c$ contained on the interval $a < c < b$ *or* (a, b) such that $f'(c) = 0$ by Rolle's Theorem.

Sem 1 Final Review

Limits – Graphically



1.) $\lim_{x \to 3} g(x)$

2.) $\lim_{x \to 0} g(x)$

3.) $\lim_{x \to -3} g(x)$

4.) $\lim_{x \to 1^+} g(x)$

5.) $\lim_{x \to 1^-} g(x)$

6.) $\lim_{x \to 1} g(x)$

7.) $\lim_{x \to -2^-} g(x)$

8.) $\lim_{x \to 4} g(x)$

9.) $\lim_{x \to 2} g(x)$

10.) $\lim_{x \to -2^+} g(x)$

Limit – Evaluating Algebraically

7) $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$

8) $\lim_{x \to 5} \dfrac{x^2 - 5x}{x - 5}$

9) $\lim_{x \to 2} \dfrac{x^2 - x - 2}{x - 2}$

10) $\lim_{x \to -5} \dfrac{x^2 + 3x - 10}{x + 5}$

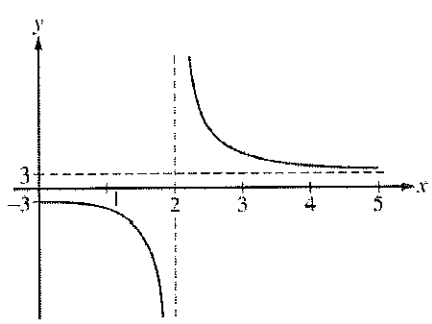
11) $\lim_{x \to 0} \dfrac{\dfrac{1}{-4 + x} + \dfrac{1}{4}}{x}$

12) $\lim_{x \to -3} \dfrac{x}{\dfrac{1}{3 + x} - \dfrac{1}{3}}$

13) $\lim_{x \to 5} \dfrac{x - 5}{\sqrt{x + 4} - 3}$

14) $\lim_{x \to 3} \dfrac{\sqrt{x + 6} - 3}{x - 3}$

Limits – Horizontal and Vertical Asymptotes



10. The function f is given by $f(x) = \dfrac{ax^2 + 12}{x^2 + b}$. The figure above shows a portion of the graph of f. Which of the following could be the values of the constants a and b?

(A) $a = -3,\ b = 2$

(B) $a = 2,\ b = -3$

(C) $a = 2,\ b = -2$

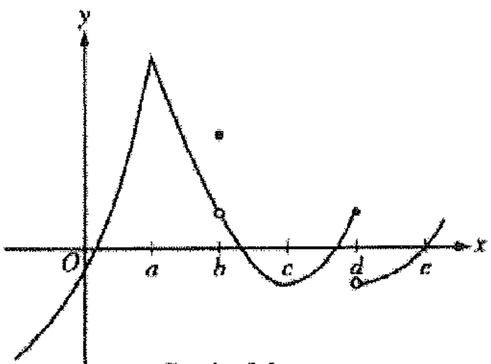
(D) $a = 3,\ b = -4$

(E) $a = 3,\ b = 4$

Continuity

21. Find the constant(s) c for which the function $f(x)$ is continuous on $(-\infty,\infty)$.

$$f(x)=\begin{cases} c^2-x^2 & \text{if } x<2 \\ 2(x+c) & \text{if } x\geq 2 \end{cases}$$



Graph of f

4.

The graph of a function f is shown above. At which value of x is f continuous, but not differentiable?

a. a
b. b
c. c
d. d
e. e

5. $\lim\limits_{x\to\infty}\dfrac{x^3-2x^2+3x-4}{4x^3-3x^2+2x-1}=$

a. 4
b. 1
c. $\frac{1}{4}$
d. 0
e. -1

6. Let f be the function defined by

$$f(x)=\begin{cases} \sqrt{x+1} & \text{for } 0\leq x\leq 3 \\ 5-x & \text{for } 3<x\leq 5. \end{cases}$$

(a) Is f continuous at $x=3$? Explain why or why not.

(b) Find the average value of $f(x)$ on the closed interval $0\leq x\leq 5$.

(c) Suppose the function g is defined by

$$g(x)=\begin{cases} k\sqrt{x+1} & \text{for } 0\leq x\leq 3 \\ mx+2 & \text{for } 3<x\leq 5, \end{cases}$$

where k and m are constants. If g is differentiable at $x=3$, what are the values of k and m ?

Chapter 2 – Derivatives

Rules of Derivatives

Examples

1. If $y = \left(x^3 + 1\right)^2$, then $\dfrac{dy}{dx} =$

   a. $\left(3x^2\right)^2$
   b. $2\left(x^3 + 1\right)$
   c. $2\left(3x^2 + 1\right)$
   d. $3x^2\left(x^3 + 1\right)$
   e. $6x^2\left(x^3 + 1\right)$

6. If $f(x) = x\ \ 2x - 3$, then $f'(x) =$

   a. $\dfrac{3x - 3}{2x - 3}$
   b. $\dfrac{x}{2x - 3}$
   c. $\dfrac{1}{2x - 3}$
   d. $\dfrac{-x + 3}{2x - 3}$
   e. $\dfrac{5x - 6}{2\ \ 2x - 3}$

12.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
−1	6	5	3	−2
1	3	−3	−1	2
3	1	−2	2	3

The table above gives values of f, f', g, and g' at selected values of x. If $h(x) = f(g(x))$, then $h'(1) =$

a. 5
b. 6
c. 9
d. 10
e. 12

Implicit Differentiation

4. Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

(a) Show that $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$.

(b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

(c) Find the value of $\frac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P? Justify your answer.

Related Rates

5.) A conical tank (with vertex down) is 12 feet across and 10 feet deep. If water is flowing into the tank at a rate of 5 cubic feet per minute, find the rate of change of the depth of the water when the water is 6 feet deep?

6.) A rocket, rising vertically, is tracked by a radar station that is on the ground 3000 feet from the launching pad. At what rate is the angle of elevation changing when the rocket is 4000 feet up and rising vertically at 5000 feet per second?

Chapter 3 – Applications of the Derivative

First derivative Analysis

Second Derivative Analysis

Examples – Find local extrema, POIs, inc/dec intervals and concavity.

701. $f(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x - 1$

702. $g(x) = x^3 - 5x^2 - 8x$

703. $h(x) = x + \dfrac{4}{x}$

704. $p(x) = \sqrt[3]{x} + \dfrac{1}{\sqrt[3]{x}}$

705. $h(x) = (2 - x)^2(x + 3)^3$

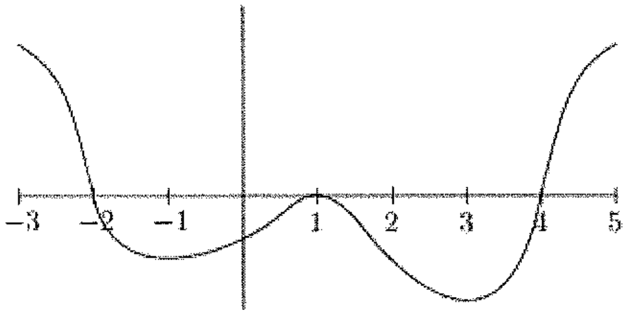
706. $m(x) = 3x\sqrt{5 - x}$

707. $f(x) = x^{2/3}(x - 5)^{-1/3}$

708. $h(x) = \frac{1}{7}x^{7/3} - x^{4/3}$

Graphs of f'

813 (1996AB). The figure below shows the graph of f', the derivative of a function f. The domain of f is the set of all real numbers x such that $-3 < x < 5$.



a) For what values of x does f have a relative maximum? Why?

b) For what values of x does f have a relative minimum? Why?

c) On what intervals is the graph of f concave upward? Use f' to justify your answer.

d) Suppose that $f(1) = 0$. Draw a sketch of f that shows the general shape of the graph on the open interval $0 < x < 2$.

Motion Applications

1. A particle moves along a line so that at any time t its position is given by $x(t) = 2\pi t + \cos 2\pi t$.

a) Find the velocity at time t.

b) Find the acceleration at time t.

c) What are all values of t, for $0 \leq t \leq 3$, for which the particle is at rest?

d) What is the maximum velocity?

Sem Review FRQs

6. Consider the closed curve in the xy-plane given by

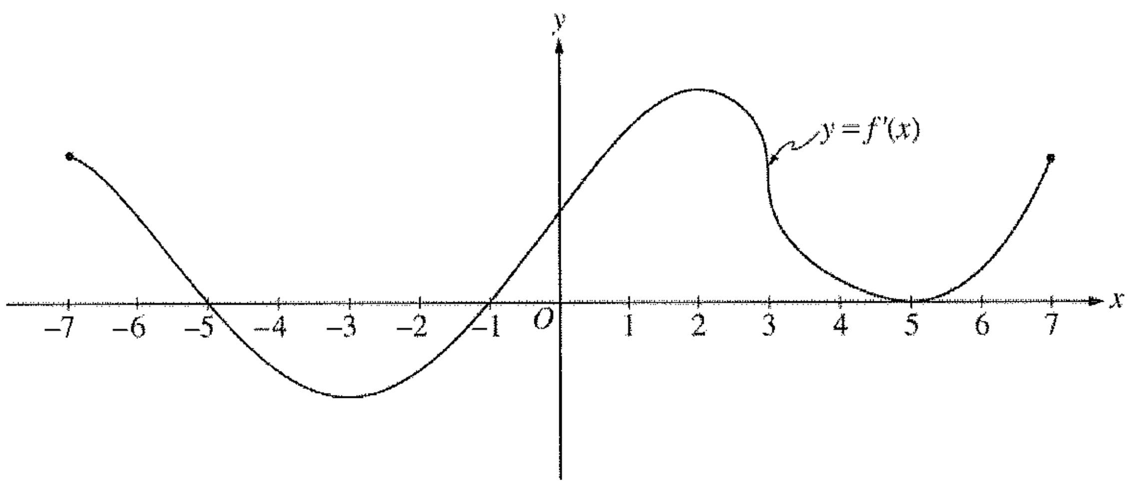
$$x^2 + 2x + y^4 + 4y = 5.$$

(a) Show that $\dfrac{dy}{dx} = \dfrac{-(x+1)}{2\left(y^3+1\right)}$.

(b) Write an equation for the line tangent to the curve at the point $(-2, 1)$.

(c) Find the coordinates of the two points on the curve where the line tangent to the curve is vertical.

(d) Is it possible for this curve to have a horizontal tangent at points where it intersects the x-axis? Explain your reasoning.

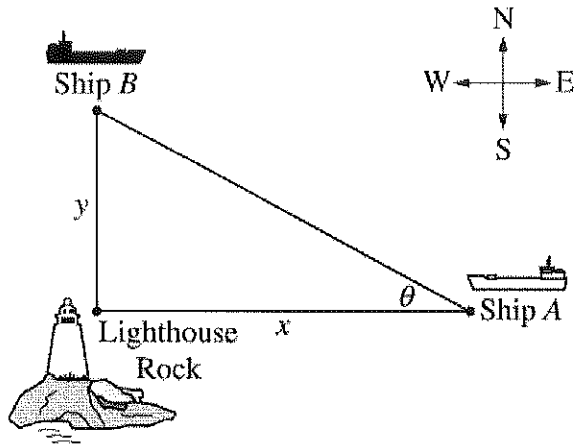


3. The figure above shows the graph of f', the derivative of the function f, for $-7 \le x \le 7$. The graph of f' has horizontal tangent lines at $x = -3$, $x = 2$, and $x = 5$, and a vertical tangent line at $x = 3$.

(a) Find all values of x, for $-7 < x < 7$, at which f attains a relative minimum. Justify your answer.

(b) Find all values of x, for $-7 < x < 7$, at which f attains a relative maximum. Justify your answer.

(c) Find all values of x, for $-7 < x < 7$, at which $f''(x) < 0$.

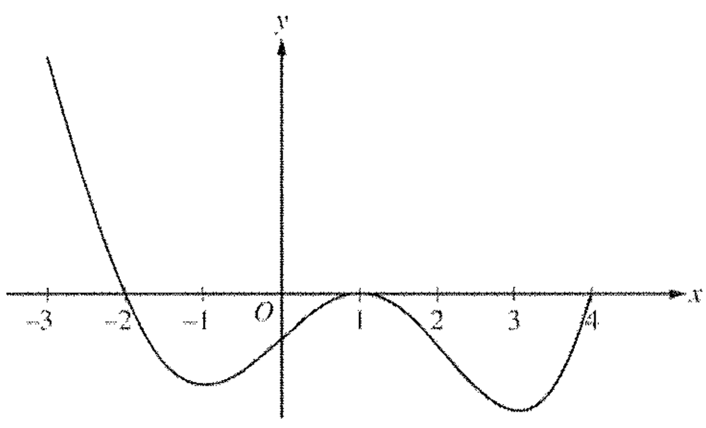


6. Ship A is traveling due west toward Lighthouse Rock at a speed of 15 kilometers per hour (km/hr). Ship B is traveling due north away from Lighthouse Rock at a speed of 10 km/hr. Let x be the distance between Ship A and Lighthouse Rock at time t, and let y be the distance between Ship B and Lighthouse Rock at time t, as shown in the figure above.

(a) Find the distance, in kilometers, between Ship A and Ship B when $x = 4$ km and $y = 3$ km.

(b) Find the rate of change, in km/hr, of the distance between the two ships when $x = 4$ km and $y = 3$ km.

(c) Let θ be the angle shown in the figure. Find the rate of change of θ, in radians per hour, when $x = 4$ km and $y = 3$ km.



Graph of f'

5. The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x-axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.

(a) Find all x-coordinates at which f has a relative maximum. Give a reason for your answer.

(b) On what open intervals contained in $-3 < x < 4$ is the graph of f both concave down and decreasing? Give a reason for your answer.

(c) Find the x-coordinates of all points of inflection for the graph of f. Give a reason for your answer.